J. Math. Soc. Japan
Vol. 39, No. 4, 1987

Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited

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(Received March 25, 1986)

1. Introduction.

The theory of viscosity solutions of first order scalar equations of the form

$$F(x, u, Du) = 0 \quad \text{in } \Omega\,, \tag{E}$$

where Ω is an open set in $\boldsymbol{R}^n$, $u:\Omega\to\boldsymbol{R}$ is continuous (i.e., $u\in C(\Omega)$), $F\in C(\Omega\times\boldsymbol{R}\times\boldsymbol{R}^n)$ enjoys some monotonicity in u and Du denotes the gradient of u, has undergone a rapid and intensive development. The original uniqueness proofs of M.G. Crandall and P.L. Lions [**4**], [**5**] were recast in M.G. Crandall, L.C. Evans and P.L. Lions [**3**] and then further improvements were made by H. Ishii [**11**], [**12**], [**13**] and others. These and other improvements, however, did not alter the basic structure of the original uniqueness argument in the sense that each significant step in the original argument had its parallel in the modified ones. As we think of it today, if u and v are solutions of (E) in the viscosity sense, then the idea was to consider maximum points of $\Phi(x, y)=u(x)-v(y)-|x-y|^2/\varepsilon$ (or an analogue) over $\bar{\Omega}\times\bar{\Omega}$ and then to reduce to the case that the maximum is an interior point and to use the equation satisfied in the viscosity sense to estimate $u-v$ at such a maximum. Of course, if Ω is unbounded the possibility of the maximum not being attained must be disposed of by adding appropriate terms to Φ, and in all cases suitable use must be made of structure conditions on the equation and so on. In all but exceptionally simple cases, careful estimates on terms corresponding to $(x-y)/\varepsilon$ at the maximum point were made and used in the course of argument.

In this note we change the point of view a bit and emphasize a way of thinking that has evolved in recent papers. The result is simpler proofs of greater generality. This carries over, as well, into other aspects of the theory

1) Sponsored in part by the United States Army under Contract No. DAAG29-80-C-0041.
2) Supported in part by the National Science Foundation under Grant No. DMS-8505531.
3) Supported in part by Grant-in-Aid for Scientific Research (No. 60540120), Ministry of Education, Science and Culture.

like error estimation and estimating moduli of continuity. The price we will pay for this generality is that the "natural assumptions" become somewhat awkward and lengthy relative to the proofs. We will tolerate this, for the assumptions are indeed natural and one can choose elegant special cases to present if desired, as will be indicated.

We will consider two particular cases of (E) in some detail; the model stationary problem

(SP) $$u+H(x, u, Du)=0 \quad \text{in } \boldsymbol{R}^n$$

and the Cauchy problem

(CP) $$\begin{aligned} &u_t+H(x, t, u, Du)=0 && \text{in } \boldsymbol{R}^n\times(0, T)\,, \\ &u(x, 0)=\varphi(x) && \text{in } \boldsymbol{R}^n \end{aligned}$$

in which H is nondecreasing in u. In both cases, Du stands for the spatial gradient $D_x u$ of u.

The program in the case of (SP) begins with:

THE FIRST STEP. Observe—as is standard—that if u and v are, respectively, viscosity sub- and supersolutions of (SP), then $z(x, y)=u(x)-v(y)$ is a viscosity solution of

(SP)′ $$z+\hat{H}(x, y, D_x z, D_y z)\leqq 0\,,$$

where the Hamiltonian $\hat{H}$ is defined on $\boldsymbol{R}^{2n}\times\boldsymbol{R}\times\boldsymbol{R}^{2n}$ by

(1) $$\hat{H}(x, y, z, p, q)=H(x, u(x), p)-H(y, v(y), -q)\,.$$

One then continues with the next step.

THE SECOND STEP. Prove a comparison theorem which allows one to conclude that suitable everywhere differentiable solutions $w(x, y)$ of

(2) $$w+\hat{H}(x, y, D_x w, D_y w)\geqq 0$$

on the subset Δ' of $\Delta=\{(x, y)\in\boldsymbol{R}^n\times\boldsymbol{R}^n : |x-y|<1\}$ on which $u(x)\geqq v(y)$ which satisfy $z(x, y)\leqq w(x, y)$ on $\partial\Delta$ will satisfy $w\geqq z$ on Δ.

The definition of "suitable" in the second step is determined by the solutions of (2) one constructs in:

THE THIRD STEP. For each $\varepsilon>0$, construct a nonnegative everywhere differentiable function w on Δ satisfying (2) on Δ' such that $w\geqq z$ on $\partial\Delta$, $w(x, x)\leqq\varepsilon$ for $x\in\boldsymbol{R}^n$ and the second step implies $w\geqq z$ on Δ.

After the third step we are done, for we have that $z(x, x)=u(x)-v(x)\leqq w(x, x)\leqq\varepsilon$ on $\boldsymbol{R}^n$, and ε is arbitrary. The program in the case of (CP) is entirely parallel. Moreover, using $u=v$, it is clear how this program can be used to estimate moduli of continuity (as was first done by Ishii [**11**] in the precursor of the setting we have in mind here). Of course, one does not want to interpret this program too rigidly, and there are probably situations where it is better not to separate out a "Step 2" for technical reasons, but it is still useful to think of this outline. Moreover, as we will see, there may be an auxiliary step involved and we may have to prove separately that $u(x)-v(y)$ is bounded on Δ if this is not assumed in advance.

The goal of this paper is to illustrate the manner of looking at uniqueness just outlined by proving results which generalize some basic theorems already in the literature. However, we want to emphasize the point of view more than the precise hypotheses, as there are many other types of uniqueness questions and there are always variants of any result in this subject. Precise statements and proofs are given in the next section, which then ends with some remarks concerning some basic extensions of the setting we have chosen for the presentation.

2. Uniqueness results.

We begin by formulating the conditions on the Hamiltonian H which we will use. B_R will denote the ball of radius R and center 0 in $\boldsymbol{R}^n$ and $|\ |$ will be both the norm on $\boldsymbol{R}^n$ and the absolute value on $\boldsymbol{R}$. The following assumptions, which have evolved in the papers [7]-[**13**] will be employed. We assume that $H: \boldsymbol{R}^n\times[0, T]\times\boldsymbol{R}\times\boldsymbol{R}^n\rightarrow\boldsymbol{R}$ satisfies:

(H0) H is continuous.

(H1) The map $r\rightarrow H(x, t, r, p)$ is nondecreasing for each $(x, t, p)\in\boldsymbol{R}^n\times[0, T]\times\boldsymbol{R}$.

(H2) There is a Lipschitz continuous everywhere differentiable function $\mu: \boldsymbol{R}^n\rightarrow[0, \infty)$ and a continuous function $\sigma: [0, \infty)\times[0, \infty)\rightarrow[0, \infty)$ which is nondecreasing in both of its arguments, satisfies $\sigma(0, R)=0$ for $R>0$, and

$$H(x, t, r, p)-H(x, t, r, p+\lambda D\mu(x))\leqq\sigma(\lambda, |p|)$$

for $x\in\Omega$, $r\in\boldsymbol{R}$, $p\in\boldsymbol{R}^n$ and $0\leqq\lambda\leqq 1$ and

$$\lim_{|x|\to\infty}\mu(x)=+\infty.$$

(H3) There is a Lipschitz continuous everywhere differentiable function $\nu: \boldsymbol{R}^n\rightarrow[0, \infty)$ and for each $R>0$ a constant C_R such that

$$H(x, t, r, p)-H(x, t, r, p+\lambda D\nu(x))\leqq C_R$$

for $x\in\boldsymbol{R}^n$, $t\in[0, T]$, $p\in B_R$ and $0\leqq\lambda\leqq R$. Moreover

$$\nu(x)\geqq|x| \qquad \text{for large } |x|.$$

Each of these assumptions is sensible for (SP) as well as (CP), as (SP) just corresponds to H independent of t. The role of (H0) and (H1) is clear and familiar. The above formulations of (H2) and (H3) evolved from Ishii [**11**] and Crandall and Lions [**7**] to the current formulation in Crandall and Lions [**9**]. We will recall examples to illustrate the significance of the conditions later. The next assumption we make is not in a form immediately recognizable in earlier works, but it is implied by the assumptions used before, as we will show. It requires a different formulation for (SP) than (CP). For (SP) we will use:

(H4) There is an $r_0>0$ and for each $\varepsilon>0$ a continuous function $w_\varepsilon:\bar{\Delta}\to[0,\infty)$ where

$$\Delta=\{(x,y)\in\boldsymbol{R}^n\times\boldsymbol{R}^n: \ |x-y|<r_0\}$$

which is differentiable at each point of Δ, Lipschitz continuous and satisfies

$$w_\varepsilon(x,y)+H(x,r,D_x w_\varepsilon(x,y))-H(y,r,-D_y w_\varepsilon(x,y))\geqq 0$$

for $r\in\boldsymbol{R}$ and $(x,y)\in\Delta$. Moreover

$$w_\varepsilon(x,x)\leqq\varepsilon \quad \text{for } x\in\boldsymbol{R}^n \qquad \text{and} \quad w_\varepsilon(x,y)\geqq 1/\varepsilon \quad \text{for } (x,y)\in\partial\Delta.$$

In the case of (CP) we assume

(H4)′ There is an $r_0>0$ and for each $\varepsilon>0$ a continuous function $w_\varepsilon:\bar{\Delta}\times[0,T]\to[0,\infty)$ which is Lipschitz continuous and differentiable at each point of $\Delta\times[0,T]$ and satisfies

$$w_{\varepsilon t}(x,y,t)+H(x,t,r,D_x w_\varepsilon(x,y))-H(y,t,r,-D_y w_\varepsilon(x,y))\geqq 0$$

for $r\in\boldsymbol{R}$ and $(x,y,t)\in\Delta\times[0,T]$ and

$$w_\varepsilon(x,x,t)\leqq\varepsilon \quad \text{for } x\in\boldsymbol{R}^n \qquad \text{and } w_\varepsilon(x,y,t)\geqq 1/\varepsilon \quad \text{for } (x,y)\in\partial\Delta\times[0,T].$$

Moreover

$$\liminf_{\varepsilon\downarrow 0}\{w_\varepsilon(x,y,0): |x-y|\geqq r\}=+\infty \qquad \text{for } r_0\geqq r>0.$$

We now formulate our main results.

THEOREM 1 (Uniqueness for (SP)). *Let* (H0), (H1), (H2), *and* (H4) *hold. Let* $u, v\in C(\boldsymbol{R}^n)$ *be, respectively, viscosity sub- and supersolutions of* (SP) *and* C *be a constant such that*

(3) $$|u(x)-u(y)|\leqq C \qquad or \quad |v(x)-v(y)|\leqq C$$

holds for all $(x,y)\in\Delta$ *(where* Δ *is from* (H4)). *If also*

(4) $$\sup_{R^n}(u-v)<\infty,$$

then $u\leqq v$ *on* $\boldsymbol{R}^n$. *In addition, if* (H3) *holds and for some* $K>0$

(5) $$u(x)-v(x)\leqq K(1+|x|)\qquad \text{for } x\in\boldsymbol{R}^n,$$

then $u\leqq v$ *on* $\boldsymbol{R}^n$.

THEOREM 2 (Uniqueness for (CP)). *Let* (H0), (H1), (H2) *and* (H4) *hold. Let* $u, v\in C(\boldsymbol{R}^n\times[0, T])$ *be, respectively, viscosity sub- and supersolutions of the equation in* (CP) *on* $\boldsymbol{R}^n\times(0, T]$. *Assume that* C *is a constant such that either*

(6) $$|u(x, t)-u(y, t)|\leqq C \qquad or \qquad |v(x, t)-v(y, t)|\leqq C$$

for $(x, y)\in\Delta$ *and* $0\leqq t\leqq T$ *and that*

(7) *either* $u(x, 0)$ *or* $v(x, 0)$ *is uniformly continuous.*

If also

(8) $$\sup_{R^n\times[0, T]}(u-v)<\infty,$$

then

(9) $$\sup_{R^n\times[0, T]}(u-v)^+\leqq\sup_{R^n\times\{0\}}(u-v)^+.$$

In addition, if (H3) *holds and there is a constant* K *such that*

(10) $$u(x, t)-v(x, t)\leqq K(1+|x|),$$

then (9) *holds*

REMARK 1. Observe that if u or v is the sum of a bounded and a uniformly continuous function, then (3) holds. Even in the special case of Theorem 1 in which both u and v are bounded, the theorem is more general than earlier analogues, for it does not require either u or v to be uniformly continuous. A result of this sort was noted in [7] where it was pointed out that the existence of uniformly continuous solutions (which was established in a more restricted generality—see also [**9**], [**13**]) implied the uniqueness of more general solutions. However, there is an example in [**9**] which shows that the current assumptions do not imply the existence of uniformly continuous solutions even in the case $n=1$. Remarks similar to the above apply to Theorem 2 as well, except that the gap between existence and uniqueness remarked above does not exist for (CP). There are, of course, new existence theorems which can be proved by using the modulus of continuity estimates which may be proved in the spirit of the uniqueness proofs given below—that is, we may relax the sort of assumptions used in the current existence theory in the place of (H4), (H4)′ by (H4) and (H4)′ (see below), but we will not stop to formulate such results here. All of the papers in the bibliography are relevant for the development of the current existence theory. During the preparation of this manuscript, significant simplifications of part of the existence theory which rely in part on the results of this paper have been obtained by H. Ishii [**15**].

We preface the proofs of Theorems 1 and 2 with some discussion of the assumptions and a review of some examples. The reader may, if he prefers, skip this discussion for the moment and proceed directly to the proofs.

Let us describe the typical structure which enables one to verify that (H4) or (H4)′ holds. We assume for simplicity in writing that $H(x, t, r, p)=H(x, t, p)$ is independent of r (although this doesn't matter) and that we can find a function $d: \boldsymbol{R}^n \times \boldsymbol{R}^n \to [0, \infty)$ and a function $F: [0, \infty)\times[0, \infty)\to[0, \infty)$ such that

(11) d is a Lipschitz continuous function, $d(x, y)$ is differentiable in x and y separately off the diagonal $y=x$, $d(x, x)=0$ and $d(x, y)\geqq |x-y|$.

Moreover, for some $r_0>0$

(12) $$H(x, t, \lambda D_x d(x, y))-H(y, t, -\lambda D_y d(x, y))\geqq -F(\lambda, d(x, y)) \quad \text{for } \lambda\geqq 0 \text{ and } x, y\in\boldsymbol{R}^n,\ x\neq y \text{ and } d(x, y)\leqq r_0.$$

Finally we assume, in the case of (SP), that

(13) F is nondecreasing in both arguments and for all $\eta>0$ there is a nondecreasing continuous function $G_\eta: [0, r_0]\to[0, \infty)$ which is continuously differentiable on $(0, r_0]$ and satisfies $G_\eta(r)\geqq F(G'_\eta(r), r)$ for $r\in(0, r_0]$, $G_\eta(0)\leqq\eta$, and

$$\liminf_{\eta\downarrow 0}\{G_\eta(r): r_1\leqq r\leqq r_0\} = +\infty \qquad \text{for } 0<r_1<r_0$$

If (11)-(13) hold then so does (H4) with the choice

$$w_\varepsilon(x, y) = G_\eta((\delta^2+d(x, y)^2)^{1/2})$$

provide η and δ are well chosen. Indeed, we have

$$D_x w_\varepsilon(x, y) = G'_\eta((\delta^2+d(x, y)^2)^{1/2})(\delta^2+d(x, y)^2)^{-1/2}d(x, y)d_x(x, y)$$

(where $d(x, y)d_x(x, y)$ is taken to be 0 if $x=y$) etc., and so, using (12), (13) and the monotonicities of F

$$\begin{aligned} w_\varepsilon(x, y)+H(x, D_x w_\varepsilon(x, y))-H(y, -D_y w_\varepsilon(x, y)) &\geqq G_\eta - F(d(\delta^2+d^2)^{-1/2}G'_\eta, d) \\ &\geqq G_\eta - F(G'_\eta, (\delta^2+d^2)^{1/2}) \geqq 0 \end{aligned}$$

where $d=d(x, y)$ and the arguments of G_η and G'_η are $(\delta^2+d(x, y)^2)^{1/2}$ in each occurrence. We have $w_\varepsilon(x, x)=G_\eta(\delta)$ which can be made as small as desired by choosing η and then δ small, and similarly, $w_\varepsilon(x, y) \geqq G_\eta((\delta^2+d(x, y)^2)^{1/2})$ can be made as large as desired on each set of the form $|x-y|=r_2$ for r_2 sufficiently small by taking δ and η to be small.

In the case of (CP) we modify (13) to ask that $G_\eta \geqq F((1+T)G'_\eta, d)$ (for example) and use $w_\varepsilon(x, y, t)=(1+t)G_\eta((\delta^2+d(x, y)^2)^{1/2})$ in a manner similar to the above.

In previous works the assumptions (12) and (13) were used with

$$F(\lambda, d) = m(\lambda d+d) \tag{14}$$

for some modulus of continuity m. For this F, the inequality $G \geqq F(G', r)$ holds on $(0, 1]$ for $G(r)=A+Br^\gamma$ if

$$A+Br^\gamma \geqq m(\gamma Br^\gamma+r) \qquad \text{for } 0<r<1$$

Let us put $B=1/\eta$ and define $A(\eta, \gamma)$ by

$$A(\eta, \gamma) = \sup_{0 \leq r \leq 1}\Big(m\Big(\frac{\gamma r^\gamma}{\eta}+r\Big)-\frac{r^\gamma}{\eta}\Big),$$

so that $A(\eta, \gamma)+r^\gamma/\eta$ satisfies the desired differential inequality. It is easy to see that $A(\eta, 0+)=0$ for small $\eta>0$ and then that we can choose $\gamma=\gamma(\eta)>0$ so that $G_\eta(r)=A(\eta, \gamma(\eta))+r^{\gamma(\eta)}/\eta$ has the desired properties with $r_0=1$.

In order to illustrate the generality we gain by assuming the existence of a supersolution as in (H4) and (13) as opposed to (14), we consider the case of a linear Hamiltonian

$$H(x, p) = (b(x), p)$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product. The assumption that (12) and (14) hold for some m and $d(x, y)=|x-y|$ amounts to requiring that $x \to b(x)+cx$ be monotone for some c, i.e.

$$(b(x)-b(y), x-y)+c|x-y|^2 \geqq 0 \qquad \text{for } x, y \in \boldsymbol{R}^n .$$

However, if $\phi:(0, \infty) \to (0, \infty)$ is continuous, nondecreasing and

$$(b(x)-b(y), x-y)+|x-y|\phi(|x-y|) \geqq 0 \qquad \text{for } x, y \in \boldsymbol{R}^n \tag{15}$$

and

$$\int_0^1 \frac{1}{\phi(s)}ds = +\infty \tag{16}$$

hold, then we may choose $F(\lambda, d)=\lambda\phi(d)$ and

$$G_\eta(r) = \frac{1}{\eta}\exp\Big(-\int_r^1 \frac{1}{\phi(s)}ds\Big),$$

it is easy to see that this condition on b is strictly weaker than the requirement that $b+cI$ be monotone for some c.

An example in [**5**] shows that uniqueness of viscosity solutions for the linear Hamiltonian is closely associated with the uniqueness of solutions of the backwards Cauchy problem for the characteristic equation $X'=b(X)$, and the conditions (15) and (16) are known to be rather general sufficient conditions for this uniqueness. Indeed, it was proved in [**5**] that if the Cauchy problem $X'=b(X)$, $X(0)=x$ has a solution $X(t, x)$ for $x\in \boldsymbol{R}^n$ with properties associated with uniqueness and global existence, that is $X(t, X(s, x))=X(t+s, x)$ for $t, s\in \boldsymbol{R}$ and $x\in \boldsymbol{R}^n$ and $(t, x)\to(t, X(t, x))$ is a homeomorphism of $\boldsymbol{R}\times\boldsymbol{R}^n$, then the formula associated with the method of characteristics (i.e., $u(X(t, x), t)=\varphi(x)$) defines a viscosity solution of the Cauchy problem for the linear Hamiltonian $H(x, p)=(b(x), p)$. It is known that there are continuous vector fields b for which the uniqueness fails so badly that there are distinct flows $X(t, x)$ and $Y(t, x)$ with the above properties (see [**5**] and its references). In this case, the Cauchy problem admits distinct (compactly supported) viscosity solutions for suitable φ. As we will see below, (H2) and (H3) are associated with growth conditions and are irrelevant for compactly supported solutions. The failure of uniqueness here is due entirely to the failure of H to satisfy (H4)'.

The role of (H2) and (H3) is illustrated by simple examples as well, and we recall some from [**9**]. If $H(x, p)=(b(x), p)$ and b satisfies (15) we may take

$$\mu(x)=\int_1^{|x|}\frac{1}{1+\psi(r)}dr$$

for $|x|\geqq 1$ and this will satisfy (H2) if

$$\int_1^{\infty}\frac{1}{1+\psi(r)}dr=\infty,$$

which is the case if, e.g., $\psi(r)=cr$. Moreover, $\nu=\mu$ satisfies (H3) if μ happens to grow at the desired rate (as is the case if $\psi(r)=0$). However, if $b(x)=-2x$ then we may take $\psi(r)=2r$ (so (H2) is satisfied) while the problem $u-(b(x), Du)=0$ has the uniformly continuous solutions $u=0$ and $u(x)=|x|^{1/2}$ which both grow less than linearly. Thus we see (H2) is not enough to guarantee the uniqueness of uniformly continuous viscosity solutions of (SP).

It is also the case that (H4) alone is not enough to guarantee the uniqueness of bounded solutions of (SP)—even Lipschitz continuous ones. A simple example of this sort is given by the equation $u-x(|Du|)^{1/2}=0$ in $\boldsymbol{R}$. Taking $d(x, y)=|x-y|$, the Hamiltonian $H(x, p)=-x(|p|)^{1/2}$ satisfies (11), (12), (14) with $m(r)=r$. However, $u=0$ and $u(x)=x/(1+|x|)$ are distinct bounded and Lipschitz continuous solutions of (SP). Finally we observe that (H3) and (H4) do not imply the uniqueness of bounded solutions (and hence do not imply (H2)). Indeed, notice that any bounded Hamiltonian satisfies (H3) with $\nu(x)=(|x|^2+1)^{1/2}$. Next, if $g(r)=$

$\max(\min(r, 1), -1)$ the Hamiltonian $H(x, p)=-g(x(|p|)^{1/2})$ is bounded (and therefore satisfies (H3)) and the same F, m as above may be used. However, the equation $u-g(x(|Du|)^{1/2})=0$ still has the solutions $u=0$ and $u=x/(1+|x|)$.

PROOF OF UNIQUENESS FOR (SP). The second step of the program is contained in the next lemma. The first step is trivial in this case.

LEMMA 1. *Let Ω be an open subset of $\boldsymbol{R}^n$ and let $H: \Omega\times\boldsymbol{R}^n\to\boldsymbol{R}$ satisfy* (H2). *Let u be a viscosity solution of $u+H(x, Du)\leqq 0$ in Ω, and v be everywhere differentiable on Ω and satisfy*

$$v(x)+H(x, Dv(x)) \geqq 0 \quad \text{and} \quad |Dv(x)| \leqq L \quad \text{if } x\in\Omega \text{ and } u(x)>v(x). \tag{17}$$

Assume that

$$\sup_{y\in\partial\Omega} \limsup_{\Omega\ni x\to y} (u(x)-v(x)) < \sup_{\Omega} (u-v)\,. \tag{18}$$

If also

$$\sup_{\Omega} (u-v) < \infty\,, \tag{19}$$

then $u\leqq v$ on Ω. Moreover, if (H3) *holds and*

$$u(x)-v(x) \leqq C(1+|x|) \qquad \text{for some constant } C \text{ and } x\in\Omega\,, \tag{20}$$

then (19) *holds (and so $u\leqq v$).*

Before embarking on the proof, let us note that if (18) is replaced by

$$\limsup_{\Omega\ni x\to y} (u(x)-v(x)) \leqq 0 \qquad \text{for } y\in\partial\Omega\,, \tag{18'}$$

then the conclusions still hold (because then (18) holds unless $u\leqq v$). We also remark that while the case in which (20) holds is not used to prove Theorem 1, it is included for later use (see Remark 5) and the proof will illustrate the process of obtaining an initial bound as was mentioned in the introduction.

PROOF. We begin with the case in which (19) holds. We may assume that

$$\sup_{\Omega} (u-v) > 0 \tag{21}$$

since there is nothing to prove otherwise. Let $0<\beta<1$ and $g_R\in C^1(\boldsymbol{R})$ satisfy

$$0 \leqq g_R' \leqq 1,\ g_R(r)/r \to 1 \quad \text{as } r\to\infty \text{ and } g_R(r)=0 \text{ for } 0\leqq r\leqq R\,. \tag{22}$$

Let $\Phi(x)=u(x)-v(x)-\beta g_R(\mu(x))$. It follows from (H2), (18), (19), (21) and (22) that Φ attains a positive maximum on Ω as soon as $R>0$ is sufficiently large. Let $y\in\Omega$ be this maximum point—then we use that u is a subsolution, (17) and (H2) to conclude that

$$\begin{aligned}(23)\qquad 0 &\geqq u(y)+H(y,\, Dv(y)+\beta g_R'(\mu(y))D\mu(y)) \\ &= u(y)-v(y)+v(y)+H(y,\, Dv(y)) \\ &\quad +H(y,\, Dv(y)+\beta g_R'(\mu(y))D\mu(y))-H(y,\, Dv(y)) \\ &\geqq u(y)-v(y)-\sigma(\beta,\, |Dv(y)|)\end{aligned}$$

and so

$$(24)\qquad u(x)-v(x)-\beta g_R(\mu(x)) = \Phi(x) \leqq \Phi(y) \leqq u(y)-v(y) \leqq \sigma(\beta,\, L)$$

for $x\in\Omega$. We may let $R\to\infty$ and then $\beta\to 0$ to conclude that $u-v \leqq 0$ as desired.

If (H3) and (20) hold in place of (19), then the arguments which led us to (23) may be repeated with μ replaced by ν provided we choose $\beta > C$. Using (H3) in place of (H2), the conclusion is now

$$u(x)-v(x)-\beta g_R(\nu(x)) \leqq C_{\beta+L}$$

and letting $R\to\infty$ we find that $u-v$ is bounded, reducing to the previous case.

END OF PROOF OF THEOREM 1. To complete the proof of Theorem 1, we begin by assuming that (4) holds and

$$(25)\qquad 0 < \sup_{R^n}(u-v)$$

Now $z(x, y)=u(x)-v(y)$ is a solution of

$$z(x,\, y)+\hat{H}(x,\, y,\, D_x z,\, D_y z) \leqq 0$$

in $\boldsymbol{R}^{2n}=\boldsymbol{R}^n\times\boldsymbol{R}^n$ in the viscosity sense, where $\hat{H}$ is given by (1). Moreover, $\hat{H}$ satisfies (H2) ((H3)) with the function $\hat{\mu}(x, y)=\mu(x)+\mu(y)$ (respectively, $\hat{\nu}(x, y)=\nu(x)+\nu(y)$) if H does with the function μ (respectively, ν). Let Δ and w_ε be as in (H4). Observe that w_ε is a solution of $w_\varepsilon(x, y)+\hat{H}(x, y, D_x w_\varepsilon(x, y), D_y w_\varepsilon(x, y)) \geqq 0$ on the subset of Δ on which $z\geqq w_\varepsilon$, since there $z(x, y)=u(x)-v(y)\geqq w_\varepsilon\geqq 0$ and so

$$\hat{H}(x, y, p, q) = H(x, u(x), p)-H(y, v(y), -q) \geqq H(x, u(x), p)-H(y, u(x), -q)$$

by (H1). In order to conclude that $z\leqq w_\varepsilon$ on Δ we have only to verify the analogue of (18) and apply Lemma 1. Let us assume that it is v that satisfies (3). If (18) fails (for z, w_ε, Δ in place of u, v, Ω) then there is a point $(x, y)\in\partial\Delta$ such that

$$z(x,\, y)-w_\varepsilon(x,\, y) \geqq \sup_{\Delta}(z-w_\varepsilon)-1\,,$$

and, in particular,

$$z(x,\, y)-w_\varepsilon(x,\, y) \geqq z(x,\, x)-w_\varepsilon(x,\, x)-1\,.$$

However, this amounts to $v(x)-v(y)\geqq w_\varepsilon(x, y)-w_\varepsilon(x, x)-1$ and by (3) for v, (H4) and $(x, y)\in\partial\Delta$ we conclude that $C\geqq 1/\varepsilon-\varepsilon-1$. This is false if ε is small enough, so (18) holds for small ε. Now we may apply Lemma 1 to conclude that $z\leqq w_\varepsilon$, and the proof is complete in the case where (4) holds.

It remains to establish that the result holds under the assumptions (H2) and (5) in place of (4). We have a situation similar to that in the final assertion of Lemma 1—we prove that $u-v$ is bounded from above by an argument similar to that used before and reduce to the previous case. Let it be v that satisfies (3), $\beta>K$, $\varepsilon>0$ and consider $\Phi(x, y)=z(x, y)-w_\varepsilon(x, y)-\beta g_R(\nu(x))$. If this function is never positive for small ε and large R, we are done. In the contrary case, it is clear that there is a positive maximum point $(\bar{x}, \bar{y})$ of this function in $\bar{\Delta}$. Since $\Phi(\bar{x}, \bar{y})\geqq\Phi(\bar{x}, \bar{x})$ implies that $v(\bar{x})-v(\bar{y})\geqq w_\varepsilon(\bar{x}, \bar{y})-w_\varepsilon(\bar{x}, \bar{x})$, which is impossible for ε small and $(\bar{x}, \bar{y})\in\partial\Delta$ as above, we conclude that $(\bar{x}, \bar{y})\in\Delta$. Now we use the relations satisfied by z and w_ε to deduce first that

$$u(\bar{x})-v(\bar{y})+H(\bar{x}, u(\bar{x}), D_x w_\varepsilon(\bar{x}, \bar{y})+\beta g'_R(\nu(\bar{x}))D\nu(\bar{x}))-H(\bar{y}, v(\bar{y}), -D_y w_\varepsilon(\bar{x}, \bar{y}))\leqq 0$$

and then

$$u(\bar{x})-v(\bar{y})-w_\varepsilon(\bar{x}, \bar{y})\leqq H(\bar{x}, u(\bar{x}), D_x w_\varepsilon(\bar{x}, \bar{y}))$$
$$-H(\bar{x}, u(\bar{x}), D_x w_\varepsilon(\bar{x}, \bar{y})+\beta g'_R(\nu(x))D\nu(\bar{x}))\,.$$

It now follows from the Lipschitz continuity of w_ε, etc., and (H3), that

$$u(x)-v(x)-w_\varepsilon(x, x)-\beta g_R(\nu(x))\leqq\Phi(\bar{x}, \bar{y})\leqq u(\bar{x})-v(\bar{y})-w_\varepsilon(\bar{x}, \bar{y})\leqq C_M$$

where M is a bound on $|D_x w_\varepsilon|$ and β. Since $w_\varepsilon(x, x)\leqq\varepsilon$, the boundedness of $u-v$ follows upon letting $R\to\infty$.

PROOF OF UNIQUENESS FOR (CP). The outline of the proof is as similar to the one given above. The lemma corresponding to Lemma 1 is

LEMMA 2. *Let Ω be an open subset of $\boldsymbol{R}^n$, $Q_T=\Omega\times(0, T]$ and $\partial_p Q_T=\partial\Omega\times(0, T]\cup\bar{\Omega}\times\{0\}$. Assume $H:\Omega\times[0, T]\times\boldsymbol{R}^n\to\boldsymbol{R}$ satisfies* (H2). *Let $u:Q_T\to\boldsymbol{R}$ be a viscosity solution of $u_t+H(x, t, Du)\leqq 0$ in Q_T and $v:Q_T\to\boldsymbol{R}$ be everywhere differentiable and satisfy*

$$(26)\qquad v_t+H(x, v(x, t), Dv(x, t))\geqq 0 \quad\textit{and}\quad |Dv|\leqq L$$
$$\textit{when } (x, t)\in Q_T \textit{ and } u(x, t)>v(x, t)\,.$$

Assume that

$$(27)\qquad \limsup_{\substack{(x,t)\to(s,y)\\(x,t)\in Q_T}}(u(x, t)-v(x, t))<\sup_{Q_T}(u-v)\qquad \textit{for } (s, y)\in\partial_p Q_T\,.$$

If also

$$(28)\qquad \sup_{Q_T}(u-v)<\infty\,,$$

then $u \leqq v$ *in* Q_T. *Moreover, if*

(29) $$u(x, t)-v(x, t) \leqq C(1+|x|) \quad \text{for some } C \text{ and all } (x, t) \in Q_T$$

and (H3) *holds, then* (28) *holds (and so* $u \leqq v$ *in* Q_T*).*

PROOF. We first assume that (28) holds and

(30) $$\sup_{Q_T}(u-v)>0$$

Let g_R be as in (22), μ as in (H2), $\alpha>0$, $1>\beta>0$ and consider

$$\Phi(x, t)=u(x, t)-v(x, t)-\beta g_R(\mu(x))-\alpha t .$$

By virtue of (27), (28), (30) and (H2), Φ attains a positive maximum at some point $(y, s) \in Q_T$ provided R is sufficiently large and $\alpha>0$ is sufficiently small. By the assumptions

$$\begin{aligned} 0 & \geqq v_t(y, s)+\alpha+H(y, s, Dv(y, s)+\beta g_R'(\mu(y)) D\mu(y)) \\ & \geqq v_t(y, s)+H(y, s, Dv(y, s))+\alpha-\sigma(\beta, L) \geqq \alpha-\sigma(\beta, L) \end{aligned}$$

and we reach a contradiction (to (30)) upon choosing β so that $\alpha>\sigma(\beta, L)$.

If we assume (H3) and (29) in place of (28), we proceed as above with ν in place of μ and $\beta>C$ to conclude that if

$$\sup_{Q_T}(u-v-\alpha t)>0$$

then $\alpha \leqq C_{\beta+L}$. In particular, $u-v \leqq (C_{\beta+L}+1)T$ and (28) holds, completing the proof.

Of course, just as (18) could be replaced by (18)′ in Lemma 1 without changing the conclusions, so may (27) be replaced by a (27)′ in Lemma 2.

The only new point in the remainder of the proof of Theorem 1 comes from the role of the uniform continuity assumed of one of $u(x, 0)$ and $v(x, 0)$. In the usual way, we can assume $u \leqq v$ at $t=0$. Then we want to use the lemma to see that $u(x, t)-v(y, t) \leqq w_\varepsilon(x, y, t)$ and for this it suffices to have either $u(x, 0)-u(y, 0) \leqq w_\varepsilon(x, y, 0)$ or $v(x, 0)-v(y, 0) \leqq w_\varepsilon(x, y, 0)$ for small $|x-y|$. Of course, this cannot be unless one of these initial functions is uniformly continuous and does not necessarily hold even in that case. However, using the uniform continuity, we may use $w_\varepsilon+A_\varepsilon$ in place of w_ε where $A_\varepsilon=\sup\{z(x, y, 0)-w_\varepsilon(x, y, 0):|x-y| \leqq r_0\}$ and still have all the desired properties. We leave the remaining details to the reader.

Further considerations.

We now briefly comment on the arguments of the sort presented above relative to other situations in which one might wish to use them besides the simplest framework we have discussed. These include the situations in which one wants to consider boundary problems in the context of Theorems 1 and 2, the situation in which u and v are not continuous, the situation in which $\boldsymbol{R}^n$ is replaced by an infinite dimensional Banach space V and in constructing solutions.

REMARK 2 (On boundary problems). First, let us remark that if Ω is bounded, then the assertions of Lemmas 1 and 2 are obviously valid without assuming (H2) or (H3). We did not formulate Theorems 1 and 2 for boundary problems, as this seems an awkward thing to do in the spirit in which we have presented this note. However, there is no difficulty in using the basic program in this situation to prove particular theorems. For example, let us replace $\boldsymbol{R}^n$ by an open subset Ω in Theorem 1 and ask that $(u-v)^+$ have the limit 0 at points of $\partial\Omega$. We proceed as before, replacing Δ by $\Delta'=\{(x, y)\in \Omega\times\Omega : |x-y|<r_0\}$. Now we will want, roughly, $(u(x)-v(y))^+-w_\varepsilon(x, y)$ to be nonpositive on $\partial\Delta'$ and then we may continue as before. It is this last condition which we didn't want to formulate in the beginning, although we remarked upon a difficulty of this sort when discussing the comparison at $t=0$ in the Cauchy problem. Similarly, the reader may verify that if $\rho(x, \partial\Omega)$ is the distance from x to $\partial\Omega$,

$$\lim_{r\downarrow 0} \sup\{(u(x)-v(y))^+ : x, y\in\Omega, \min(\rho(x, \partial\Omega), \rho(y, \partial\Omega))\leqq r, \\ |x-y|\leqq r\}=0$$

holds and w_ε is as constructed in the discussion of (H4), then we may succeed by using $w_\varepsilon+C_\varepsilon$ with a suitable small constant C_ε in place of w_ε in the proofs. This can be made more general: The process succeeds here because

$$\lim_{\varepsilon\downarrow 0} \inf\{w_\varepsilon(x, y) : |x-y|\geqq r\} = \infty \quad \text{for } 1>r>0\,,$$

and if we simply assume this we obtain results in general Ω. Analogous remarks hold for the Cauchy problem.

REMARK 3 (On semicontinuous functions). The definition of viscosity subsolutions (resp. supersolutions) generalizes in a straightforward way to upper (resp. lower) semicontinuous functions (see also [3, Remark 1.4]). For example, an upper-semicontinuous function $u: \Omega\to\boldsymbol{R}$ is called a viscosity subsolution of (E) if whenever $\varphi\in C^1(\Omega)$, $y\in\Omega$ and $u-\varphi$ attains a local maximum at y, then $F(y, u(y), D\varphi(y))\leqq 0$. With these extensions of the definition, Theorems 1 and

2 remain true even if the requirement that u and v be continuous is weakened to the requirement that u be upper semicontinuous and v be lower semicontinuous. The proofs completely parallel the above. This observation is powerful when one wants to identify the value functions of differential games with the viscosity solutions of the associated Isaacs equations without knowing the continuity of the value functions (see Ishii [**14**]). Moreover, it is useful to define a locally bounded function u to be a subsolution (supersolution) provided its upper-semicontinuous (respectively, lower-semicontinuous) envelope u^* (respectively, u_*) has the same property. See Ishii [**14**], [**15**]. With this convention, comparison results for discontinuous sub- and supersolutions are immediate.

REMARK 4 (On infinite dimensions). The current lines of argument may be used in theory of Hamilton-Jacobi equations in infinite dimensional spaces developed in [8]—[**10**]. For example, Theorems 1 and 2 remain true if $\boldsymbol{R}^n$ is replaced by a Banach space V with the Radon-Nikodym property. Indeed, we have left the assumption that H be continuous in Theorem 1 so that this remark would hold—it is not needed in finite dimensions.

REMARK 5 (On verifying (3) and (6)). The following situation arises when attempting to prove existence theorems: Suppose, for example, we have a solution u of (SP) which grows at most linearly. We would like to know that $u(x)-u(y)\leqq w_\varepsilon(x, y)$ for $|x-y|\leqq 1$ and small ε in order to establish a modulus of continuity for u. As it stands, we do not know it this holds because we have no bound on $u(x)-u(y)$ for $|x-y|=1$. In applications we obtain such bounds by further comparison arguments. In the spirit of this note, we should assume:

(H5) There is a Lipschitz continuous differentiable function $w: \boldsymbol{R}^n\times\boldsymbol{R}^n\to\boldsymbol{R}$ such that

$$w(x, y)+H(x, r, D_x w(x, y))-H(y, r, -D_y w(x, y)) \geqq 0$$

for $x, y\in\boldsymbol{R}^n$, $r\in\boldsymbol{R}$ and $w(x, x)$ is bounded above on $\boldsymbol{R}^n$.

Then Lemma 1 would imply $u(x)-u(y)\leqq w$. An analogous formulation for (CP) which would allow us to bound the difference of solutions of Cauchy problems is:

(H5)′ For each uniformly continuous φ on $\boldsymbol{R}^n$ there is a Lipschitz continuous differentiable function $w: \boldsymbol{R}^n\times\boldsymbol{R}^n\times[0, T]\to\boldsymbol{R}$ which satisfies

$$w_t(x, y, t)+H(x, t, r, D_x w(x, y, t))-H(y, t, r, -D_y w(x, y, t)) \geqq 0$$

on $\boldsymbol{R}^n\times\boldsymbol{R}^n\times[0, T]$, $w(x, y, 0)\geqq\varphi(x)-\varphi(y)$ and $w(x, x, t)$ is bounded from above.

One can replace $\boldsymbol{R}^n$ by arbitrary open sets and take into account the boundary if desired. Conditions of the sort (11), (12), (13) imply (H5)′ but not (H5). When

(H0)—(H5) ((H0)—(H4)′, (H5)′) hold, we have the uniqueness of viscosity solutions with at most linear growth.

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