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Fourier integral operators with weighted symbols and micro-local resolvent estimates

Dedicated to the memory of Hitoshi Kumano-go

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§ 0. Introduction.

In this paper, we investigate a calculus of Fourier integral operators with phase functions and symbols belonging to certain classes of weighted functions. As an application we give another proof of the micro-local resolvent estimates established in [3] and [4].

The phase function $\varphi(x, \xi)$ we consider satisfies

$$(0.1) \qquad |\partial_x^\alpha \partial_\xi^\beta(\varphi(x, \xi)-x\xi)| \leqq C_{\alpha\beta}\langle x\rangle^{\sigma-|\alpha|}$$

for some $0 \leqq \sigma \leqq 1$ and

$$(0.2) \qquad \sum_{|\alpha+\beta|\leq l} \sup_{x,\xi} |\partial_x^\alpha \partial_\xi^\beta \vec{\nabla}_x \nabla_\xi(\varphi(x, \xi)-x\xi)| \leqq \tau$$

for some integer $l \geqq 0$ and $0 \leqq \tau < 1$. Namely $\varphi(x, \xi)$ is in a "neighborhood" of $x\xi=\sum_{j=1}^n x_j\xi_j$ in this sense. The symbol $p(x, \xi)$ satisfies

$$(0.3) \qquad \max_{|\alpha+\beta|\leq k} \sup_{x,\xi} \{\langle x\rangle^{-l+|\alpha|}\langle\xi\rangle^{-m}|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)|\} < \infty$$

for some $l, m \in \boldsymbol{R}^1$ and an integer $k \geqq 0$. The essential feature of $\varphi(x, \xi)$ and $p(x, \xi)$ is that the decay order in x increases as the order of their derivatives with respect to x increases, which makes the asymptotic expansion of symbols with respect to x and the calculus possible. Our calculus is a version of that of families of Fourier integral operators involving the parameter $0<h<1$, which has been discussed in Kitada-Kumano-go [6]. Schematically, we can write "$\langle x\rangle^{-1}=h$". However, the details have to be studied separately.

Our main result of the calculus is the following. Let $\varphi(x, \xi)$ satisfy (0.1) and (0.2) for some τ small enough and some l large enough, and let $a(x, \xi)$ satisfy (0.3) with $l=m=0$ and be close enough to 1 with respect to the semi-norms defined through (0.3) with $l=m=0$. Then the Fourier integral operator defined by

$$A_{\varphi}f(x)=\text{Os-}\iint e^{i(\varphi(x,\xi)-y\xi)}a(x,\xi)f(y)dyd\xi \tag{0.4}$$

has an inverse in the class of conjugate Fourier integral operators with the same phase function: namely for some symbol $b(\xi, y)$ satisfying (0.3) with $l=m=0$

$$A_{\varphi}^{-1}f(x)=\text{Os-}\iint e^{i(x\xi-\varphi(y,\xi))}b(\xi,y)f(y)dyd\xi \tag{0.5}$$

(cf. Theorem 3.4 below).

This result is useful in proving the micro-local estimates for the resolvent $R(z)=(H-z)^{-1}$, where

$$H=-\Delta/2+V(x) \tag{0.6}$$

with a real-valued smooth $V(x)$ satisfying

$$|\partial_x^{\alpha}V(x)|\leqq C_{\alpha}\langle x\rangle^{-|\alpha|-\varepsilon} \tag{0.7}$$

for all α and some $0<\varepsilon<1$.

The property (0.7) of $V(x)$ suggests that the use of our weighted classes of phase and symbol functions is quite natural. In fact, in our previous works [**3**] and [**4**], we have partly discussed some calculus of such Fourier integral operators with weighted symbols. However, from the standpoint of the theory of Fourier integral operators, the calculus has not been developed fully there. The purpose of this paper is firstly to supplement this point, and secondly to give a new proof of the micro-local resolvent estimates, which heavily leans on the calculus thus obtained.

The plan of the paper is as follows. In section 1, we introduce the classes of symbols and phase functions and give a definition of pseudodifferential and Fourier integral operators. In section 2, we discuss a calculus of pseudodifferential operators and in section 3 that of Fourier integral operators. The L^2-boundedness theorems are summarized in section 4. Then in section 5 the micro-local estimates for the free resolvent will be summarized and in section 6 we give a proof of those for the perturbed resolvent $R(z)$ using the results of calculus and section 5. In section 7 we add a theorem on the micro-local approximation of resolvent.

§1. Definitions of pseudodifferential and Fourier integral operators.

We denote by $(x_1, \cdots, x_n)$ a point of $\boldsymbol{R}^n$, and by $\alpha=(\alpha_1, \cdots, \alpha_n)$ a multi-index whose components α_j are non-negative integers. We use the usual notations:

$$(1.1)\qquad \begin{aligned} &|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!,\\ &x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n},\\ &D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad \partial_{x_j} = \partial/\partial x_j, \quad D_{x_j} = -i\partial_{x_j},\\ &\langle x\rangle = (1+|x|^2)^{1/2}, \quad \nabla_x = {}^t(\partial_{x_1}, \cdots, \partial_{x_n}), \quad \vec{\nabla}_x = {}^t\nabla_x. \end{aligned}$$

$\mathcal{S}=\mathcal{S}(\boldsymbol{R}^n)$ denotes the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}^n$.

DEFINITION 1.1. A C^∞ function $p(x, \xi, x', \xi', x'')$ is said to belong to the symbol class $B_{l_1, l_2, l_3}^{m_1; m_2}$ for $m_1, m_2, l_1, l_2, l_3 \in \boldsymbol{R}^1$ if p satisfies for any integer $k \geqq 0$

$$(1.2)\qquad \begin{aligned} &|p|_k^{(m_1, m_2; l_1, l_2, l_3)}\\ &\equiv \max_{|\alpha+\beta+\alpha'+\beta'+\alpha''| \leq k} \sup_{\substack{x, \xi\\ x', \xi', x''}} \{\langle x\rangle^{-l_1+|\alpha|}\langle \xi\rangle^{-m_1}\langle x'\rangle^{-l_2+|\alpha'|}\langle \xi'\rangle^{-m_2}\langle x''\rangle^{-l_3+|\alpha''|}\\ &\qquad \times |\partial_x^\alpha \partial_\xi^\beta \partial_{x'}^{\alpha'} \partial_{\xi'}^{\beta'} \partial_{x''}^{\alpha''} p(x, \xi, x', \xi', x'')|\} < \infty. \end{aligned}$$

In particular, when p is independent of (ξ', x'') [resp. (x', ξ', x'') or (x, ξ', x'')], we write $p \in B_{l_1, l_2}^{m_1}$ [resp. $p \in B_{l_1}^{m_1}$ or $p \in B_{l_2}^{m_1}$] and denote its norm by $|p|_k^{(m_1; l_1, l_2)}$ [resp. $|p|_k^{(m_1; l_1)}$ or $|p|_k^{(m_1; l_2)}$].

REMARK 1°. The space $B_{l_1, l_2, l_3}^{m_1; m_2}$ forms a Fréchet space with semi-norms above.

2°. It is easy to extend our results in the following to the symbols $p(x, \xi, x', \xi', x'')$ satisfying

$$\begin{aligned} &|\partial_x^\alpha \partial_\xi^\beta \partial_{x'}^{\alpha'} \partial_{\xi'}^{\beta'} \partial_{x''}^{\alpha''} p(x, \xi, x', \xi', x'')|\\ &\leqq C_{\alpha\beta\alpha'\beta'\alpha''} \langle x\rangle^{l_1-\rho|\alpha|+\delta|\beta|}\langle \xi\rangle^{m_1}\langle x'\rangle^{l_2-\rho|\alpha'|+\delta|\beta'|}\langle \xi'\rangle^{m_2}\langle x''\rangle^{l_3-\rho|\alpha''|} \end{aligned}$$

for some $0 \leqq \delta < \rho \leqq 1$.

DEFINITION 1.2. A real-valued C^∞ function $\varphi(x, \xi)$ is said to belong to the phase class $P_\sigma(\tau; l)$ for $0 \leqq \sigma \leqq 1$, $0 \leqq \tau < 1$ and $l = 0, 1, 2, \cdots$, if φ satisfies

$$(1.3)\qquad \begin{cases} \text{i)} & |\partial_x^\alpha \partial_\xi^\beta (\varphi(x, \xi) - x\xi)| \leqq C_{\alpha\beta} \langle x\rangle^{\sigma-|\alpha|},\\ \text{ii)} & |\varphi|_{2, l} = \sum_{|\alpha+\beta| \leq l} \sup_{x, \xi} |\partial_x^\alpha \partial_\xi^\beta \vec{\nabla}_x \nabla_\xi (\varphi(x, \xi) - x\xi)| \leqq \tau. \end{cases}$$

In particular, when $l=0$ we write $P_\sigma(\tau; 0)$ simply as $P_\sigma(\tau)$.

We define the pseudodifferential operator $P = p(X, D_x, X', D_{x'}, X'')$ with the symbol $p(x, \xi, x', \xi', x'') \in B_{l_1, l_2, l_3}^{m_1; m_2}$ by

$$(1.4)\qquad \begin{aligned} Pf(x) = \text{Os-}\iiiint e^{-i(y^1\eta^1+y^2\eta^2)} p(x, \eta^1, x+y^1, \eta^2, x+y^1+y^2)\\ \times f(x+y^1+y^2) dy^1 dy^2 đ\eta^1 đ\eta^2 \end{aligned}$$

for $f \in \mathcal{S}$, where $đ\eta = (2\pi)^{-n} d\eta$ and $\text{Os-}\iiiint \cdots dy^1 dy^2 đ\eta^1 đ\eta^2$ means the usual oscillatory integral (cf. e.g. [6]).

We next define the Fourier integral operators which include the pseudo-differential operators (cf. Proposition 2.1 below).

DEFINITION 1.3. i) Let $p(x, \xi)\in B_l^m$ and $\varphi(x, \xi)\in P_\sigma(\tau)$. We define the Fourier integral operator $P_\varphi=p_\varphi(X, D)$ with the symbol $p(x, \xi)$ and the phase function $\varphi(x, \xi)$ by

$$P_\varphi f(x)=\text{Os-}\iint e^{i(x-y)\xi}[e^{i(\varphi(x,\xi)-x\xi)}p(x, \xi)f(y)]dyd\xi \tag{1.5}$$

for $f\in\mathcal{S}$. We write this oscillatory integral simply as

$$P_\varphi f(x)=\text{Os-}\iint e^{i(\varphi(x,\xi)-y\xi)}p(x, \xi)f(y)dyd\xi. \tag{1.5$'$}$$

ii) Let $q(\xi, y)\in B_l^m$ and $\varphi(x, \xi)\in P_\sigma(\tau)$. We define the conjugate Fourier integral operator $Q_{\varphi^*}=q_{\varphi^*}(D, Y)$ by

$$Q_{\varphi^*}f(x)=\text{Os-}\iint e^{i(x-y)\xi}[e^{i(y\xi-\varphi(y,\xi))}q(\xi, y)f(y)]dyd\xi. \tag{1.6}$$

Or simply we write

$$Q_{\varphi^*}f(x)=\text{Os-}\iint e^{i(x\xi-\varphi(y,\xi))}q(\xi, y)f(y)dyd\xi. \tag{1.6$'$}$$

§ 2. Pseudodifferential operators.

We consider fundamental properties of pseudodifferential operators in this section.

For $j=0, 1, 2, \cdots$, let $p_j(x, \xi, x')\in B_{l_j,l}^m$ with $l_j\to-\infty$ $(j\to\infty)$, and let $\chi\in C_0^\infty(\boldsymbol{R}^n)$ satisfy

$$\chi(\theta)=\begin{cases}1 & |\theta|\leqq 1/2,\\ 0 & |\theta|\geqq 1.\end{cases} \tag{2.1}$$

Then it is easily seen that there exists a positive sequence $\{\varepsilon_j\}$ tending to zero as $j\to\infty$ such that the series

$$p(x, \xi, x')=\sum_{j=1}^\infty \chi(\varepsilon_j^{-1}\langle x\rangle^{-1})p_j(x, \xi, x') \tag{2.2}$$

converges absolutely in $B_{l_0,l}^m$ and

$$p(x, \xi, x')-\sum_{j=1}^{N-1}p_j(x, \xi, x')\in B_{l_N,l}^m \tag{2.3}$$

for all $N\geqq 1$ (cf. Theorem 1.3 in [6]). We write (2.3) as

$$p(x, \xi, x')\sim\sum_{j=0}^\infty p_j(x, \xi, x') \tag{2.4}$$

and call it an asymptotic expansion of $p(x, \xi, x')$ with respect to x. The asymptotic expansion with respect to x' can be defined similarly.

PROPOSITION 2.1. *For* $p(x, \xi, x', \xi', x'') \in B^{m_1; m_2}_{l_1, l_2, l_3}$, *we set*

$$(2.5)\quad \begin{cases} \text{i)} & p_{\mathrm{LL}}(x, \xi) = \mathrm{Os}\text{-}\iint e^{-iy\eta} p_{\mathrm{L}}(x, \xi+\eta, x+y) dy d\eta, \\ & p_{\mathrm{L}}(x, \xi', x'') = \mathrm{Os}\text{-}\iint e^{-iy\eta} p(x, \xi'+\eta, x+y, \xi', x'') dy d\eta, \\ \text{ii)} & p_{\mathrm{RR}}(\xi, x') = \mathrm{Os}\text{-}\iint e^{-iy\eta} p_{\mathrm{R}}(x'+y, \xi-\eta, x') dy d\eta, \\ & p_{\mathrm{R}}(x, \xi, x'') = \mathrm{Os}\text{-}\iint e^{-iy\eta} p(x, \xi, x''+y, \xi-\eta, x'') dy d\eta. \end{cases}$$

Then $p_{\mathrm{L}} \in B^{m_1+m_2}_{l_1+l_2, l_3}$, $p_{\mathrm{R}} \in B^{m_1+m_2}_{l_1, l_2+l_3}$, $p_{\mathrm{LL}}, p_{\mathrm{RR}} \in B^{m_1+m_2}_{l_1+l_2+l_3}$, *and we have*

$$(2.6)\qquad p(X, D_x, Y, D_y, Z) = p_{\mathrm{LL}}(X, D) = p_{\mathrm{RR}}(D, Y).$$

Further, for any $k \geqq 0$ *and for an arbitrarily fixed even* n_0 $(>n+|l_1|+|l_2|+|l_3| +|m_1|+|m_2|+1)$

$$(2.7)\quad \begin{cases} \text{i)} & |p_{\mathrm{LL}}|_k^{(m_1+m_2; l_1+l_2+l_3)} \leqq C_k |p_{\mathrm{L}}|_{2k+2n_0}^{(m_1+m_2; l_1+l_2, l_3)} \\ & \qquad\qquad\qquad\qquad \leqq C_k |p|_{3k+4n_0}^{(m_1, m_2; l_1, l_2, l_3)} \\ \text{ii)} & |p_{\mathrm{RR}}|_k^{(m_1+m_2; l_1+l_2+l_3)} \leqq C_k |p_{\mathrm{R}}|_{2k+2n_0}^{(m_1+m_2; l_1, l_2+l_3)} \\ & \qquad\qquad\qquad\qquad \leqq C_k |p|_{3k+4n_0}^{(m_1, m_2; l_1, l_2, l_3)} \end{cases}$$

for some constant $C_k > 0$, *and*

$$(2.8)\quad \begin{cases} \text{i)} & p_{\mathrm{LL}}(x, \xi) \sim \sum_\alpha (\alpha!)^{-1} (\partial_\xi^\alpha D_{x'}^\alpha p_{\mathrm{L}})(x, \xi, x), \\ & p_{\mathrm{L}}(x, \xi, x') \sim \sum_\alpha (\alpha!)^{-1} (\partial_\xi^\alpha D_{x'}^\alpha p)(x, \xi, x, \xi, x'), \\ \text{ii)} & p_{\mathrm{RR}}(\xi, x') \sim \sum_\alpha (-1)^{|\alpha|} (\alpha!)^{-1} (\partial_\xi^\alpha D_x^\alpha p_{\mathrm{R}})(x', \xi, x'), \\ & p_{\mathrm{R}}(x, \xi, x') \sim \sum_\alpha (-1)^{|\alpha|} (\alpha!)^{-1} (\partial_\xi^\alpha D_{x'}^\alpha p)(x, \xi, x', \xi, x'). \end{cases}$$

PROOF. (2.6) is shown by virtue of Fourier's inversion formula. By Taylor's expansions and integration by parts, we have

$$(2.9)\quad \begin{aligned} p_{\mathrm{L}}(x, \xi', x'') = & \sum_{|\alpha|<N} (\alpha!)^{-1} (\partial_\xi^\alpha D_{x'}^\alpha p_0)(x, \xi', x, \xi', x'') \\ & + N \sum_{|\gamma|=N} (\gamma!)^{-1} \int_0^1 (1-\theta)^{N-1} (\partial_\xi^\gamma D_{x'}^\gamma p_\theta)(x, \xi', x, \xi', x'') d\theta, \end{aligned}$$

where $N \geqq 1$ and for $0 \leqq \theta \leqq 1$

$$(2.10)\quad p_\theta(x, \xi, x', \xi', x'') = \mathrm{Os}\text{-}\iint e^{-iy\eta} p(x, \xi+\theta\eta, x'+y, \xi', x'') dy d\eta.$$

For a fixed even n_0' $(>n+|l_1|+|l_2|+|l_3|+|m_1|+|m_2|)$, we have by integration by parts

$$(2.11)\quad p_\theta(x, \xi, x', \xi', x'') = \text{Os-}\iint e^{-iy\eta}(1+|\eta|^{n_0'})^{-1}(1+(-\Delta_y)^{n_0'/2}) \times\{(1+|y|^{n_0'})^{-1}(1+(-\Delta_\eta)^{n_0'/2})p(x, \xi+\theta\eta, x'+y, \xi', x'')\}dyd\eta.$$

Then, by $p\in B^{m_1;m_2}_{l_1,l_2,l_3}$, we have for $\theta\in[0, 1]$ and an even $n_1\geqq|\alpha'|$

$$(2.12)\quad |D_x^\alpha\partial_\xi^\beta D_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}D_{x''}^{\alpha''}p_\theta(x, \xi, x', \xi', x'')| \leqq C_{\alpha\beta\alpha'\beta'\alpha''}|p|^{(m_1,m_2;l_1,l_2,l_3)}_{2n_0'+|\alpha+\beta+\alpha'+\beta'+\alpha''|+n_1} \times\langle x\rangle^{l_1-|\alpha|}\langle\xi\rangle^{m_1}\langle x'\rangle^{l_2-|\alpha'|}\langle\xi'\rangle^{m_2}\langle x''\rangle^{l_3-|\alpha''|},$$

by a further integration by parts with respect to η in (2.11), where $C_{\alpha\beta\alpha'\beta'\alpha''}>0$ is independent of θ, x, ξ, x', ξ', x''. Setting $\theta=1$, $x'=x$ and $\xi=\xi'$, we obtain the second inequality in (2.7)-i) from (2.5)-i), (2.10) and (2.12). The first inequality in (2.7)-i) is a special case of the second one. (2.8)-i) follows from (2.9), $p_0(x, \xi, x', \xi', x'')=p(x, \xi, x', \xi', x'')$ (by Fourier's inversion formula), and (2.12). (2.8)-ii) can be proved similarly. □

THEOREM 2.2. *For $p_j(x, \xi)\in B^{m_j}_{l_j}$ ($j=1, \cdots, \nu+1$, $\nu\geqq1$), define $q_{\nu+1}(x, \xi)$ by*

$$(2.13)\quad q_{\nu+1}(x, \xi) = \text{Os-}\overbrace{\int\cdots\int}^{2\nu}\exp\Big(-i\sum_{j=1}^{\nu}y^j\eta^j\Big)\prod_{j=1}^{\nu}p_j(x+\overline{y^{j-1}}, \xi+\eta^j) \times p_{\nu+1}(x+\overline{y^\nu}, \xi)dy^1\cdots dy^\nu d\eta^1\cdots d\eta^\nu,$$

where $\overline{y^0}=0$ and $\overline{y^j}=y^1+\cdots+y^j$ ($j=1, \cdots, \nu$). Then we have

$$(2.14)\quad q_{\nu+1}(x, \xi)\in B^{\overline{m_{\nu+1}}}_{\overline{l_{\nu+1}}},$$

where $\overline{m_{\nu+1}}=m_1+\cdots+m_{\nu+1}$, $\overline{l_{\nu+1}}=l_1+\cdots+l_{\nu+1}$, and

$$(2.15)\quad q_{\nu+1}(X, D) = p_1(X, D)\cdots p_{\nu+1}(X, D).$$

Furthermore, for an arbitrarily fixed even n_0 $(>n+|\overline{l_{\nu+1}}|+|\overline{m_{\nu+1}}|+1)$, there exists a constant $C_0>0$ such that

$$(2.16)\quad |q_{\nu+1}|_k^{(\overline{m_{\nu+1}};\overline{l_{\nu+1}})}\leqq C_0^{\nu+1}\sum_{k_1+\cdots+k_{\nu+1}\leqq k}\prod_{j=1}^{\nu+1}|p_j|^{(m_j;l_j)}_{2n_0+2k_j}.$$

PROOF. (2.15) is easy (cf. e.g. [7]). By integration by parts, we have with an even integer $n_0'>n+|\overline{l_{\nu+1}}|+|\overline{m_{\nu+1}}|$,

$$(2.17)\quad q_{\nu+1}(x, \xi) = \text{Os-}\overbrace{\int\cdots\int}^{2\nu}\exp\Big(-i\sum_{j=1}^{\nu}y^j\eta^j\Big)\prod_{j'=1}^{\nu}(1+|y^{j'}|^{n_0'})^{-1}(1+(-\Delta_{\eta^{j'}})^{n_0'/2}) \times\prod_{j=1}^{\nu}p_j(x+\overline{y^{j-1}}, \xi+\eta^j)p_{\nu+1}(x+\overline{y^\nu}, \xi)dy^1\cdots dy^\nu d\eta^1\cdots d\eta^\nu.$$

Making a change of variables

$$z^j = y^1 + \cdots + y^j, \qquad \text{i.e.,} \quad y^j = z^j - z^{j-1}, \quad z^0 = 0 \tag{2.18}$$

for $j=1, \cdots, \nu$, and noting

$$\sum_{j=1}^{\nu} y^j \eta^j = \sum_{k=1}^{\nu} z^k(\eta^k - \eta^{k+1}), \qquad \eta^{\nu+1} = 0, \tag{2.19}$$

we integrate by parts again. Then we have

$$\begin{aligned} q_{\nu+1}(x, \xi) = \text{Os-}\overbrace{\int \cdots \int}^{2\nu} & \exp\Big(-i \sum_{k=1}^{\nu} z^k(\eta^k - \eta^{k+1})\Big) \\ & \times \prod_{k=1}^{\nu} (1+|\eta^k - \eta^{k+1}|^{n_0'})^{-1}(1+(-\Delta_{z^k})^{n_0'/2}) \\ & \times \Big\{ \prod_{j'=1}^{\nu} (1+|z^{j'} - z^{j'-1}|^{n_0'})^{-1}(1+(-\Delta_{\eta^{j'}})^{n_0'/2}) \\ & \times \prod_{j=1}^{\nu} p_j(x+z^{j-1}, \xi+\eta^j) p_{\nu+1}(x+z^{\nu}, \xi)\Big\} dz^1 \cdots dz^{\nu} d\eta^1 \cdots d\eta^{\nu}. \end{aligned} \tag{2.20}$$

Noting $p_j \in B_{l_j}^{m_j}$ and using $(1+|z^1|)^{-1}(1+|z^2-z^1|)^{-1} \cdots (1+|z^j - z^{j-1}|)^{-1} \leqq C\langle z^j\rangle^{-1}$, we have for a constant $C_1>0$ and an even $n_0''>n$

$$\begin{aligned} |q_{\nu+1}(x, \xi)| \leqq\ & C_1^{\nu+1} \prod_{j=1}^{\nu+1} |p_j|_{2n_0'}^{(m_j;l_j)} \langle x\rangle^{\overline{l_{\nu+1}}} \langle \xi \rangle^{\overline{m_{\nu+1}}} \\ & \times \overbrace{\int \cdots \int}^{2\nu} \prod_{k=1}^{\nu} (1+|\eta^k - \eta^{k+1}|^{n_0'})^{-1} \\ & \times \prod_{j'=1}^{\nu} (1+|z^{j'} - z^{j'-1}|^{n_0'})^{-1} dz^1 \cdots dz^{\nu} d\eta^1 \cdots d\eta^{\nu}. \end{aligned} \tag{2.21}$$

Thus for another constant $C_2>0$ one has

$$|q_{\nu+1}(x, \xi)| \leqq C_2^{\nu+1} \prod_{j=1}^{\nu+1} |p_j|_{2n_0'}^{(m_j;l_j)} \langle x\rangle^{\overline{l_{\nu+1}}} \langle \xi \rangle^{\overline{m_{\nu+1}}}. \tag{2.22}$$

Differentiating (2.20) and estimating similarly to the proof of Proposition 2.1, we obtain (2.16). □

THEOREM 2.3. *Let $n_0>n+1$ be an even integer. Then there exists a constant $c_0>0$ such that, for any $P=p(X, D)$ with $p(x, \xi) \in B_0^0$ and $|p|_{2n_0}^{(0;0)} \leqq c_0$, the operator $I-P$ has an inverse $(I-P)^{-1}$ in the class of pseudodifferential operators with symbols in B_0^0.*

PROOF. For $\nu \geqq 1$ we define $p_{\nu+1}(x, \xi) \in B_0^0$ by (2.13) in Theorem 2.2 with $p_j = p$ for $j=1, \cdots, \nu+1$. Then Theorem 2.2 yields

$$P^{\nu+1} = p_{\nu+1}(X, D) \tag{2.23}$$

and

(2.24) $$|p_{\nu+1}|_k^{(0;0)} \leqq C_0^{\nu+1} \sum_{k_1+\cdots+k_{\nu+1}\leq k} \prod_{j=1}^{\nu+1} |p|_{2n_0+2}^{(0;0)}{}_j$$

Hence, for $\nu+1 \geqq k$, we have

(2.25) $$\begin{aligned}|p_{\nu+1}|_k^{(0;0)} &\leqq C_0^{\nu+1}(|p|_{2n_0}^{(0;0)})^{\nu+1-k} \sum_{k_1+\cdots+k_{\nu+1}\leq k} (|p|_{2n_0+2k}^{(0;0)})^k \\ &\leqq (C_0c_0)^{\nu+1-k}(C_0|p|_{2n_0+2k}^{(0;0)})^k C_{\nu,k},\end{aligned}$$

where $C_{\nu,k}=\sum_{j=0}^k \binom{\nu+j}{j}$. Noting that $C_{\nu,k} \leqq C_k \nu^k$ $(\nu=1, 2, \cdots)$ for a constant $C_k>0$, we see that, if c_0 satisfies $C_0c_0<1$, the series

(2.26) $$1+p+p_2+\cdots+p_{\nu+1}+\cdots$$

converges in B_0^0. The sum then gives the symbol in B_0^0 of the operator $(I-P)^{-1}=I+P+P^2+\cdots+P^{\nu+1}+\cdots$. □

§ 3. Fourier integral operators.

In this section we investigate the fundamental properties of Fourier integral operators which will be useful in later sections.

THEOREM 3.1. *Let* $p(x, \xi)$, $p(\xi, y) \in B_{l_1}^{m_1}$, $q(x, \xi)$, $q(\xi, y) \in B_{l_2}^{m_2}$ *and* $\varphi(x, \xi) \in P_\sigma(\tau)$ $(m_j, l_j \in \boldsymbol{R}^1,\ 0 \leqq \sigma \leqq 1,\ 0 \leqq \tau < 1)$.

i) *There exist symbols* $r(x, \xi)$, $s(x, \xi) \in B_{l_1+l_2}^{m_1+m_2}$ *such that*

(3.1) $$\begin{cases} \text{a)} & p(X, D)q_\varphi(X, D) = r_\varphi(X, D), \\ \text{b)} & q_\varphi(X, D)p(X, D) = s_\varphi(X, D). \end{cases}$$

Here r and s are given by

(3.2) $$\begin{cases} r(x, \xi) = \text{Os-}\iint e^{i(x-y)\eta} p(x, \eta+\nabla_x\varphi(x, \xi, y))q(y, \xi)dyd\eta, \\ s(x, \xi) = \text{Os-}\iint e^{-i(\eta-\xi)y} q(x, \eta)p(y+\nabla_\xi\varphi(\eta, x, \xi), \xi)dyd\eta, \end{cases}$$

where

(3.3) $$\begin{cases} \nabla_x\varphi(x, \xi, y) = \int_0^1 \nabla_x\varphi(y+\theta(x-y), \xi)d\theta, \\ \nabla_\xi\varphi(\eta, x, \xi) = \int_0^1 \nabla_\xi\varphi(x, \xi+\theta(\eta-\xi))d\theta. \end{cases}$$

In particular,

(3.4) $$\begin{cases} r(x, \xi) \sim \sum_\alpha (\alpha!)^{-1} D_y^\alpha \{(\partial_\eta^\alpha p)(x, \nabla_x\varphi(x, \xi, y))q(y, \xi)\}|_{y=x}, \\ s(x, \xi) \sim \sum_\alpha (\alpha!)^{-1} \partial_\eta^\alpha \{q(x, \eta)(D_x^\alpha p)(\nabla_\xi\varphi(\eta, x, \xi), \xi)\}|_{\eta=\xi}. \end{cases}$$

ii) *There exist symbols* $r(\xi, y)$ *and* $s(\xi, y)\in B^{m_1+m_2}_{l_1+l_2}$ *such that*

$$(3.5)\qquad \begin{cases} \text{a)} & q_{\varphi*}(D, Y)p(D, Y)=r_{\varphi*}(D, Y),\\ \text{b)} & p(D, Y)q_{\varphi*}(D, Y)=s_{\varphi*}(D, Y).\end{cases}$$

r and s are calculated by noting that (3.5) *are conjugate to* (3.1).

PROOF. i) For $f\in\mathcal{S}$ we have formally

$$(3.6)\qquad p(X, D)q_{\varphi}(X, D)f(x)$$
$$=\iint e^{i(\varphi(x,\eta)-z\eta)}\left\{\iint e^{i\psi}p(x, \xi)q(y, \eta)dyd\xi\right\}f(z)dzd\eta,$$

where

$$(3.7)\qquad \psi=\varphi(y, \eta)-\varphi(x, \eta)+(x-y)\xi=(x-y)(\xi-\nabla_x\varphi(x, \eta, y)).$$

By a change of variables: $\tilde{\eta}=\xi-\nabla_x\varphi(x, \eta, y)$, we obtain (3.1)-a) and (3.2). It is easy to see $r(x, \xi)\in B^{m_1+m_2}_{l_1+l_2}$ and (3.4) by integration by parts and Taylor's expansion using $\varphi\in P_\sigma(\tau)$, $p(x, \xi)\in B^{m_1}_{l_1}$, and $q(y, \xi)\in B^{m_2}_{l_2}$. (3.1)-b) and ii) can be proved similarly. □

THEOREM 3.2. *Let* $p(x, \xi)\in B^{m_1}_{l_1}$, $q(\xi, y)\in B^{m_2}_{l_2}$, *and* $\varphi\in P_\sigma(\tau)$ *with* $0<\tau\ll 1$. *Then there exist symbols* $r(x, \xi)$, $s(x, \xi)\in B^{m_1+m_2}_{l_1+l_2}$ *such that*

$$(3.8)\qquad \begin{cases} \text{i)} & p_\varphi(X, D)q_{\varphi*}(D, Y)=r(X, D),\\ \text{ii)} & q_{\varphi*}(D, Y)p_\varphi(X, D)=s(X, D).\end{cases}$$

Here r and s are given by

$$(3.9)\qquad \begin{cases} \text{i)} & r(x, \xi)=\text{Os-}\iint e^{-iy\eta}\tilde{r}(x, \xi+\eta, x+y)dyd\eta,\\ & \tilde{r}(x, \xi, y)=p(x, \nabla_x\varphi^{-1}(x, \xi, y))q(\nabla_x\varphi^{-1}(x, \xi, y), y)J_\xi(x, \xi, y),\\ \text{ii)} & s(x, \xi)=\text{Os-}\iint e^{-iy\eta}\tilde{s}(\xi+\eta, x+y, \xi)dyd\eta,\\ & \tilde{s}(\xi, y, \eta)=q(\xi, \nabla_\xi\varphi^{-1}(\xi, y, \eta))p(\nabla_\xi\varphi^{-1}(\xi, y, \eta), \eta)J_y(\xi, y, \eta).\end{cases}$$

Here $\eta=\nabla_x\varphi^{-1}(x, \xi, y)$ *and* $x=\nabla_\xi\varphi^{-1}(\xi, x, \eta)$ *are the inverse mappings of* $\xi=\nabla_x\varphi(x, \eta, y)$ *and* $y=\nabla_\xi\varphi(\xi, x, \eta)$, *respectively, and* $J_\xi(x, \xi, y)$ *and* $J_y(\xi, y, \eta)$ *are the Jacobians of* $\nabla_x\varphi^{-1}(x, \xi, y)$ *and* $\nabla_\xi\varphi^{-1}(\xi, y, \eta)$, *respectively.*

PROOF. i) of (3.8): For $f\in\mathcal{S}$ we have formally

$$(3.10)\qquad p_\varphi(X, D)q_{\varphi*}(D, Y)f(x)=\iint e^{i(x-y)\nabla_x\varphi(x,\xi,y)}p(x, \xi)q(\xi, y)f(y)dyd\xi.$$

By $\varphi\in P_\sigma(\tau)$ and $0<\tau\ll 1$, $\nabla_x\varphi^{-1}(x, \eta, y)$ exists. Making a change of variables $\eta=\nabla_x\varphi(x, \xi, y)$, we then obtain (3.8)-i). Using $p\in B^{m_1}_{l_1}$, $q\in B^{m_2}_{l_2}$ and $\varphi\in P_\sigma(\tau)$

and integrating by parts in (3.9)-i) with respect to η and y as in (2.11), we obtain $r\in B_{l_1+l_2}^{m_1+m_2}$. (3.8)-ii) can be proved similarly. □

THEOREM 3.3. *Let $n_0>n+1$ be an even integer and set $\tilde{l}=6n_0$. Let c_0 be the constant in Theorem 2.3. Let $0<\tilde{\tau}<1$ be small enough such that $\tilde{\tau}\ll c_0$. Then for any $\varphi(x, \xi)\in P_\sigma(\tilde{\tau}; \tilde{l})$, there exist $q(\xi, y)$ and $r(x, \xi)\in B_0^0$ such that*

$$\text{(3.11)}\qquad \begin{cases} \text{i)} & I_\varphi Q_{\varphi*}=Q_{\varphi*}I_\varphi=I, \\ \text{ii)} & I_{\varphi*}R_\varphi=R_\varphi I_{\varphi*}=I. \end{cases}$$

Here $Q_{\varphi*}=q_{\varphi*}(D, Y)$, $R_\varphi=r_\varphi(X, D)$, and I_φ, $I_{\varphi*}$ denote the Fourier and conjugate Fourier integral operators with symbols 1 and phase functions φ, respectively.

PROOF. From Theorem 3.2, there exists a symbol $p(x, \xi)\in B_0^0$ such that

$$\text{(3.12)}\qquad I_\varphi I_{\varphi*}=p(X, D).$$

By Theorem 3.2-(3.9)-i)

$$\text{(3.13)}\qquad \begin{cases} p_0(x, \xi)=p(x, \xi)-1=\text{Os-}\iint e^{-iy\eta}t_0(x, \xi+\eta, x+y)dy\,đ\eta, \\ t_0(x, \xi, y)=J_\xi(x, \xi, y)-1. \end{cases}$$

Thus

$$\text{(3.14)}\qquad I_\varphi I_{\varphi*}=I+p_0(X, D).$$

By Proposition 2.1 and $\varphi\in P_\sigma(\tilde{\tau}; \tilde{l})$

$$\text{(3.15)}\qquad |p_0|_k^{(0;0)}\leqq C_k|t_0|_{2k+2n_0}^{(0;0,0)}\leqq C_k'|\varphi|_{2,2k+2n_0}.$$

Hence by $\varphi\in P_\sigma(\tilde{\tau}; \tilde{l})$ and $\tilde{\tau}\ll c_0$, we have

$$\text{(3.16)}\qquad |p_0|_{2n_0}^{(0;0)}\leqq C_{2n_0}'|\varphi|_{2,6n_0}<c_0.$$

Thus $(I+p_0(x, \xi))^{-1}$ exists in B_0^0 by Theorem 2.3. Therefore $I_\varphi Q_{\varphi*}=I$ holds, if we set

$$\text{(3.17)}\qquad Q_{\varphi*}=I_{\varphi*}(I+p_0(X, D))^{-1}.$$

Hence $Q_{\varphi*}I_\varphi Q_{\varphi*}=Q_{\varphi*}$, which implies

$$\text{(3.18)}\qquad (Q_{\varphi*}I_\varphi-I)I_{\varphi*}=0.$$

We can prove $I_{\varphi*}R_\varphi=I$ similarly to the above using Theorem 3.2-(3.9)-ii), which and (3.18) imply $Q_{\varphi*}I_\varphi=I$. This completes the proof of (3.11)-i). (3.11)-ii) can be proved similarly. □

THEOREM 3.4. *Let $n_0>n+1$ be an even integer and set $\tilde{l}=6n_0$. Let $c_0>0$ be the constant in Theorem 2.3. Let $0<\tilde{\tau}<1$ be small enough such that $\tilde{\tau}\ll c_0/2$. For $\varphi\in P_\sigma(\tilde{\tau}; \tilde{l})$ and $k\geqq 1$, set*

(3.19) $$|\varphi|_{1,k}=\max_{1\leq|\gamma+\delta|\leq k}\sup_{x,\xi}|\partial_x^\gamma\partial_\xi^\delta\nabla_x\varphi(x,\xi)|,$$

which is finite by (1.3)-i) *and* $\sigma\leqq 1$. *Then for any* $\varphi(x,\xi)\in P_\sigma(\tilde{\tau};\tilde{l})$ *and* $a(x,\xi)\in B_0^0$ *satisfying* $|a-1|_{6n_0}^{(0;0)}(|\varphi|_{1,6n_0})^{6n_0}\ll c_0/4$, *there exists* $q(\xi,y)\in B_0^0$ *such that*

(3.20) $$A_\varphi Q_{\varphi*}=Q_{\varphi*}A_\varphi=I.$$

Here $A_\varphi=a_\varphi(X,D)$ *and* $Q_{\varphi*}=q_{\varphi*}(D,Y)$.

PROOF. Similarly to the proof of Theorem 3.3, we can write using Theorem 3.2-(3.9)-i)

(3.21) $$A_\varphi I_{\varphi*}=I+p_0(X,D),$$

where

(3.22) $$p_0(x,\xi)=\text{Os-}\iint e^{-iy\eta}t_0(x,\xi+\eta,x+y)dy\,đ\eta,$$

$$t_0(x,\xi,y)=a(x,\nabla_x\varphi^{-1}(x,\xi,y))J_\xi(x,\xi,y)-1$$
$$=\{a(x,\nabla_x\varphi^{-1}(x,\xi,y))-1\}J_\xi(x,\xi,y)+\{J_\xi(x,\xi,y)-1\}.$$

By Proposition 2.1, $\varphi\in P_\sigma(\tilde{\tau};\tilde{l})$ and $a\in B_0^0$,

(3.23) $$|p_0|_k^{(0;0)}\leqq C_k|t_0|_{2k+2n_0}^{(0;0,0)}\leqq C_k'\{|a-1|_{2k+2n_0}^{(0;0)}(|\varphi|_{1,2k+2n_0})^{6n_0}(\tilde{\tau}+1)+\tilde{\tau}\},$$

if $k\leqq 2n_0$. Thus by our assumptions on a, φ and $\tilde{\tau}$, we have

(3.24) $$|p_0|_{2n_0}^{(0;0)}<c_0.$$

Hence $(I+p_0(x,\xi))^{-1}$ exists in B_0^0 by Theorem 2.3. Therefore $A_\varphi Q_{\varphi*}=I$ holds, if we set

(3.25) $$Q_{\varphi*}=I_{\varphi*}(I+p_0(X,D))^{-1}.$$

Thus $Q_{\varphi*}A_\varphi Q_{\varphi*}=Q_{\varphi*}$, and

(3.26) $$(Q_{\varphi*}A_\varphi-I)I_{\varphi*}=0.$$

Using Theorem 3.3, we obtain from this that $Q_{\varphi*}A_\varphi=I$. (3.20) has been proved. □

REMARK. We see by taking the adjoint of (3.20) that there exists the inverse $A_{\varphi*}^{-1}$ of $A_{\varphi*}=a_{\varphi*}(Y,D)$ in the class of Fourier integral operators $Q_\varphi=q_\varphi(D,X)$, although our proof above does not work directly for the existence of $A_{\varphi*}^{-1}$.

§ 4. L^2-boundedness of Fourier integral operators.

THEOREM 4.1. *Let* $p(\xi, y)$, $q(x, \xi)\in B_0^0$ *and* $\varphi(x, \xi)\in P_\sigma(\tau)$. *Let* $s\in \boldsymbol{R}^1$. *Then there exists a constant* $C=C_{s,n}>0$ *independent of* p, q *and* φ *such that*

$$\text{(4.1)}\quad \text{i)}\quad \|\langle x\rangle^s p_{\varphi*}(D, Y)\langle x\rangle^{-s}\| \leqq C\Big(1+\sum_{1\leq|\alpha+\beta|\leq 2r}\sup_{x,\xi}|\partial_x^\alpha\partial_\xi^\beta\nabla_x\varphi(x, \xi)|\Big)^{2r}|p|_{2r}^{(0;0)},$$

$$\text{ii)}\quad \|\langle x\rangle^s q_{\varphi}(X, D)\langle x\rangle^{-s}\| \leqq C\Big(1+\sum_{1\leq|\alpha+\beta|\leq 2r}\sup_{x,\xi}|\partial_x^\alpha\partial_\xi^\beta\nabla_x\varphi(x, \xi)|\Big)^{2r}|q|_{2r}^{(0;0)},$$

where $r=2([n/2]+[5n/4]+2)+n_0+4|s|+1$, n_0 *being an even integer with* $n_0>n+1$.

PROOF. Let $f\in\mathcal{S}$. Then by Theorem 3.2 and Proposition 2.1,

$$\text{(4.2)}\quad \begin{aligned}\|\langle x\rangle^s p_{\varphi*}(D, Y)\langle x\rangle^{-s}f\|^2 &= (\langle x\rangle^{-s}p_\varphi(D, X)\langle x\rangle^{2s}p_{\varphi*}(D, Y)\langle x\rangle^{-s}f, f)\\ &= (r(X, D)\langle x\rangle^{-s}f, \langle x\rangle^{-s}f)\\ &= (q(X, D)f, f)\leqq \|q(X, D)f\|\|f\|\end{aligned}$$

for some $r(x, \xi)\in B_{2s}^0$ and $q(x, \xi)\in B_0^0$. By Calderón-Vaillancourt theorem [1],

$$\text{(4.3)}\quad \|q(X, D)f\|\leqq C|q|_{r_0}^{(0;0)}\|f\|,\qquad r_0=2([n/2]+[5n/4]+2).$$

(3.9)-i), (2.5) and (2.7) yield

$$\text{(4.4)}\quad |q|_k^{(0;0)}\leqq C_{k,s}(|p|_{2k+2n_0+8|s|})^2 \times\Big(1+\sum_{1\leq|\alpha+\beta|\leq 2k+2n_0+8|s|+1}\sup_{x,\xi}|\partial_x^\alpha\partial_\xi^\beta\nabla_x\varphi(x, \xi)|\Big)^{2k+2n_0+8|s|}.$$

(4.1)-i) then follows from (4.2)~(4.4). Taking the adjoint of (4.1)-i), we obtain (4.1)-ii). □

Using Theorem 4.1, we can prove the following theorem in quite the same way as in the proof of Lemma 3.3 of [2] (cf. Appendix of [2]).

THEOREM 4.2. *Let* $a_\pm(x, \xi)\in B_0^0$ *and* $b_\pm(\xi, y)\in B_0^0$ *satisfy for some* $L\geqq 0$, $-1<\theta_0<\theta_1<1$ *and for any* $l\geqq 1$

$$\text{(4.5)}\quad |\partial_x^\alpha\partial_\xi^\beta a_\pm(x, \xi)|\leqq\begin{cases} C_{\alpha\beta}\langle x\rangle^{-L-|\alpha|}, & \pm\cos(x, \xi)\equiv\pm x\xi/|x||\xi|\geqq\theta_0,\\ C_{\alpha\beta}\langle x\rangle^{-|\alpha|}, & \pm\cos(x, \xi)\leqq\theta_0,\end{cases}$$

$$\text{(4.6)}\quad |\partial_y^\alpha\partial_\xi^\beta b_\pm(\xi, y)|\leqq\begin{cases} C_{\alpha\beta}\langle y\rangle^{-|\alpha|}, & \pm\cos(\xi, y)\geqq\theta_1,\\ C_{\alpha\beta l}\langle y\rangle^{-l}, & \pm\cos(\xi, y)\leqq\theta_1,\end{cases}$$

and for some $\delta_0>0$, $\mu_0>0$

$$(4.7)\qquad \begin{cases} a_{\pm}(x, \xi)=0 & \text{for } |x|\leqq\delta_0 \text{ or } |\xi|\leqq\mu_0, \\ b_{\pm}(\xi, y)=0 & \text{for } |y|\leqq\delta_0 \text{ or } |\xi|\leqq\mu_0. \end{cases}$$

Set for $f\in\mathcal{S}$ *and* $t\in\boldsymbol{R}^1$

$$(4.8)\qquad T_{\pm}(t)f(x)=\text{Os-}\iint e^{i(x\xi-t|\xi|^2/2-y\xi)}a_{\pm}(x, \xi)b_{\pm}(\xi, y)f(y)dy\,d\xi.$$

Then for $\pm t\geqq 0$ *and* $s_1, s_2\geqq 0$ *with* $s_1+s_2\leqq L$,

$$(4.9)\qquad \|\langle x\rangle^{s_1}T_{\pm}(t)\langle y\rangle^{s_2}\|\leqq C\langle t\mu_0\rangle^{-L+s_1+s_2}$$

for some constant C independent of t.

REMARK. In Theorem 4.2, $T_{\pm}(t)$ is a pseudodifferential operator, which simplifies its proof in contrast to Lemma 3.3 of [2], where we have treated the Fourier integral operator version of $T_{\pm}(t)$.

§ 5. Micro-local estimates for $(-\Delta/2-z)^{-1}$.

In this section we summarize the micro-local estimates for the free resolvent $R_0(z)=(H_0-z)^{-1}$, $H_0=-\Delta/2$, $\operatorname{Im} z\neq 0$, which will be used in the next section in proving those for the perturbed resolvent $R(z)=(H-z)^{-1}$, $H=H_0+V(x)$, with $V(x)$ a smooth long-range potential.

Let $p_{\pm}(x, \xi)\in B_0^0$ satisfy for some $-1<\mu_{\pm}<1$ and $\delta_0>0$ and for any $l\geqq 1$

$$(5.1)\qquad p_{\pm}(x, \xi)=0 \qquad \text{for } |x|\leqq\delta_0 \text{ or } |\xi|\leqq\delta_0,$$

and

$$(5.2)\qquad |\partial_x^{\alpha}\partial_{\xi}^{\beta}p_{\pm}(x, \xi)|\leqq C_{\alpha\beta l}\langle x\rangle^{-l} \qquad \text{if } \cos(x, \xi)\lessgtr\mu_{\pm}$$

in respective signs. We write $P_{\pm}=p_{\pm}(X, D)$.

We denote by $L_s^2=L_s^2(\boldsymbol{R}^n)$ $(s\in\boldsymbol{R}^1)$ the weighted L^2 space with the norm

$$(5.3)\qquad \|f\|_s=\left(\int_{\boldsymbol{R}^n}\langle x\rangle^{2s}|f(x)|^2dx\right)^{1/2}.$$

$\|\ \|_{s\to s'}$ $(s, s'\in\boldsymbol{R}^1)$ will denote the operator norm from L_s^2 into $L_{s'}^2$. It is well-known (cf. e.g. Kuroda [8]) that $R_0(z)$ $(\operatorname{Im} z\neq 0)$ has boundary values as $\operatorname{Im} z\to 0$ in the space $B(L_s^2, L_{-s}^2)$ $(s>1/2)$ of bounded operators from L_s^2 into L_{-s}^2. We denote the boundary values by $R_0(\lambda\pm i0)\in B(L_s^2, L_{-s}^2)$ for $\lambda\neq 0$. Then the following estimates are known ([8]):

$$(5.4)\qquad \|R_0(\lambda\pm i0)\|_{s\to -s}\leqq C\lambda^{-1/2} \qquad \text{for } \lambda\geqq\lambda_0(>0).$$

The following theorem is a special case of Theorem 1 of [3].

THEOREM 5.1. *Let* $s>1/2$ *and* $\lambda_0>0$. *Let* $-1<\mu_{\pm}<1$ *and* $\lambda_0>\delta_0^2/2$. *Then* *there exists a constant $C>0$ such that for $\lambda\geqq\lambda_0$*

$$\|P_{\mp}R_0(\lambda\pm i0)\|_{s\to s-1}\leqq C\lambda^{-1/2}. \tag{5.5}$$

PROOF. We consider $P_-R(\lambda+i0)$ only. The other case can be treated similarly. We decompose P_- as $P_-=P_-^0+P_-^1$, where P_-^j $(j=0, 1)$ are pseudodifferential operators with symbols $p_-^j(x, \xi)\in B_0^0$ which satisfy (5.1) and for any $l\geqq 1$

$$|\partial_x^\alpha\partial_\xi^\beta p_-^0(x, \xi)|\leqq C_{\alpha\beta l}\langle x\rangle^{-l} \tag{5.6}$$

and for some μ_-' with $1>\mu_-'>\mu_-$

$$p_-^1(x, \xi)=0 \qquad \text{for } \cos(x, \xi)>\mu_-'. \tag{5.7}$$

Then from (5.4) and (5.6) follows

$$\|P_-^0R_0(\lambda+i0)\|_{s\to s-1}\leqq C\lambda^{-1/2} \tag{5.8}$$

for $\lambda\geqq\lambda_0$.

The other part $P_-^1R(\lambda+i0)$ can be treated in quite the same way as in the proof of Theorems 3.3 and 3.5 of [3]. In fact, the proof becomes much simpler in our case than there due to the non-existence of the perturbation. We omit the details. □

COROLLARY 5.2. *Let $s>1/2$ and $\lambda_0>0$. Let $-1<\mu_\pm<1$ and $\lambda_0>\delta_0^2/2$. Let $\varphi(x, \xi)\in P_\sigma(\tilde{\tau}; 6n_0)$ with an even $n_0>n+1$ and small $0<\tilde{\tau}\ll 1$, and assume that $|\nabla_\xi\varphi(x, \xi)-x|<c_1\langle x\rangle$ for a sufficiently small constant $c_1>0$. Set $P_\varphi^\pm=p_{\pm\varphi}(X, D)$. Then there exists a constant $C>0$ such that for $\lambda\geqq\lambda_0$*

$$\|P_\varphi^\mp R_0(\lambda\pm i0)\|_{s\to s-1}\leqq C\lambda^{-1/2}. \tag{5.9}$$

PROOF. By Theorem 3.2, $\tilde{P}^\mp=I_\varphi\cdot P_\varphi^\mp$ is, for a given $N\geqq 0$, expressed as a sum of two pseudodifferential operators R^N and $Q^\mp$ with $\|\langle x\rangle^N R^N\langle x\rangle^N\|<\infty$ and the symbols $q^\mp(x, \xi)\in B_0^0$ of $Q^\mp$ satisfying (5.1) and (5.2) with δ_0 and $\mu_\pm$ replaced by some other constants $0<\delta_0'<\delta_0$ and $\mu_\pm'\lesseqgtr\mu_\pm$, $-1<\mu_\pm'<1$. Here, in order to let $\delta_0'>0$ and $-1<\mu_\pm'<1$, we have used the assumption that c_1 in $|\nabla_\xi\varphi(x, \xi)-x|<c_1\langle x\rangle$ is sufficiently small and the asymptotic expansion of the symbols of $\tilde{P}^\mp$ (see Theorem 3.2-(3.9)-ii) and Proposition 2.1 and note that $|\nabla_\xi\varphi^{-1}(\xi, y, \eta)-y|<c_1'\langle y\rangle$ for some small $c_1'>0$ if $|\nabla_\xi\varphi(x, \xi)-x|<c_1\langle x\rangle$). Since by Theorem 3.3, there exists a symbol $r(x, \xi)\in B_0^0$ such that $R_\varphi I_\varphi\cdot=I$, we have $P_\varphi^\mp=R_\varphi\tilde{P}^\mp$. Therefore by Theorems 4.1, 5.1 and (5.4), we have taking $N=s$

$$\begin{aligned}\|P_\varphi^\mp R_0(\lambda\pm i0)\|_{s\to s-1} &\leqq \|R_\varphi\|_{s-1\to s-1}\|R^N\|_{-s\to s-1}\|R_0(\lambda\pm i0)\|_{s\to -s} \\ &\quad +\|R_\varphi\|_{s-1\to s-1}\|Q^\mp R_0(\lambda\pm i0)\|_{s\to s-1} \\ &\leqq C\lambda^{-1/2}. \quad \square\end{aligned} \tag{5.10}$$

THEOREM 5.3. *Let $s\geqq 0$ and $\lambda_0>0$. Let $-1<\mu_-<\mu_+<1$ and $\lambda_0>\delta_0^2/2$. Then there exists a constant $C>0$ such that for $\lambda\geqq\lambda_0$*

$$\|P_\mp R_0(\lambda\pm i0)P_\pm\|_{-s\to s}\leqq C\lambda^{-1/2}. \tag{5.11}$$

PROOF. We consider $P_-R_0(\lambda+i0)P_+$ only. The other case can be treated similarly. We decompose P_- as $P_-=P_-^0+P_-^\infty$, where P_-^j $(j=0, \infty)$ are the pseudodifferential operators with symbols $p_-^0(x, \xi)$ and $p_-^\infty(x, \xi)\in B_0^0$ which satisfy (5.1), (5.2) and

$$\begin{cases} p_-^0(x, \xi)=0 & \text{for } |\xi|>\sqrt{2\lambda}/2, \\ p_-^\infty(x, \xi)=0 & \text{for } |\xi|<\sqrt{2\lambda}/4. \end{cases} \tag{5.12}$$

We use the relation:

$$R_0(\lambda+i\varepsilon)=i\int_0^\infty e^{it(\lambda+i\varepsilon-H_0)}dt, \qquad \varepsilon>0. \tag{5.13}$$

$P_-^0R_0(\lambda+i0)P_+$ can be written as a pseudodifferential operator with symbol

$$q(x, \xi, y)=p_-^0(x, \xi)(|\xi|^2/2-\lambda)^{-1}p_+^R(\xi, y), \tag{5.14}$$

where $p_+^R(\xi, y)$ is defined through (2.5) with $p(z, \eta, y)=p_+(z, \eta)$. By using (2.8)-ii), it is easy to see that, for an arbitrarily given $L\geqq 0$, $p_+^R(\xi, y)$ is split into the sum of two terms $\tilde{p}_+^R(\xi, y)$ and $\tilde{\tilde{p}}_+^R(\xi, y)\in B_0^0$ such that for any $l\geqq 0$

$$\begin{cases} |\partial_y^\alpha\partial_\xi^\beta\tilde{p}_+^R(\xi, y)|\leqq C_{\alpha\beta l}\langle y\rangle^{-l} & \text{if } \cos(x, \xi)<\mu_+, \\ \tilde{p}_+^R(\xi, y)=0 \quad \text{for } |y|\leqq\delta_0 \text{ or } |\xi|\leqq\delta_0, \\ \|\langle x\rangle^L\tilde{\tilde{p}}_+^R(D, Y)\langle x\rangle^L\|<\infty. \end{cases} \tag{5.15}$$

By (5.12) for $p_-^0(x, \xi)$ and Theorem 4.1, one has for $s\geqq 0$

$$\|P_-^0R_0(\lambda+i0)\|_{s\to s}\leqq C\lambda^{-1}. \tag{5.16}$$

Thus taking $L=s$, we have

$$\|P_-^0R_0(\lambda+i0)\tilde{\tilde{P}}_+^R\|_{-s\to s}\leqq\|P_-^0R_0(\lambda+i0)\|_{s\to s}\|\tilde{\tilde{P}}_+^R\|_{-s\to s}\leqq C\lambda^{-1}. \tag{5.17}$$

Since $P_-^0e^{-itH_0}\tilde{P}_+^R$ has the form of $T_+(t)$ in Theorem 4.2, we have for $t\geqq 0$ and any $l\geqq 1$

$$\|P_-^0e^{-itH_0}\tilde{P}_+^R\|_{-2s\to 2s}\leqq C_l\langle t\rangle^{-l}. \tag{5.18}$$

Hence, using (5.13), one has

$$\|P_-^0R_0(\lambda+i0)\tilde{P}_+^R\|_{-2s\to 2s}\leqq C. \tag{5.19}$$

(5.12) for $p_-^0(x, \xi)$, (5.14) and Theorem 4.1 yield

$$\|P_-^0R_0(\lambda+i0)\tilde{P}_+^R\|_{0\to 0}\leqq C\lambda^{-1}. \tag{5.20}$$

Interpolating (5.19) and (5.20), we get

$$\|P_-^0 R_0(\lambda+i0)\tilde{P}_+^R\|_{-s\to s} \leqq C\lambda^{-1/2}. \tag{5.21}$$

Combining (5.17) and (5.21), we obtain

$$\|P_-^0 R_0(\lambda+i0)P_+\|_{-s\to s} \leqq C\lambda^{-1/2}. \tag{5.22}$$

On the other hand, Theorem 4.2-(4.9), (5.12) and (5.15) imply that for $t\geqq 0$, $s\geqq 0$ and $l\geqq 1$

$$\|P_-^\infty e^{-itH_0}P_+\|_{-s\to s} \leqq C_l\langle t\sqrt{\lambda}\rangle^{-l}. \tag{5.23}$$

By (5.13) we then have

$$\|P_-^\infty R_0(\lambda+i0)P_+\|_{-s\to s} \leqq C\lambda^{-1/2}. \tag{5.24}$$

This and (5.22) yield (5.11). □

COROLLARY 5.4. *Let* $s\geqq 0$ *and* $\lambda_0>0$. *Let* $-1<\mu_-<\mu_+<1$ *and* $\lambda_0>\delta_0^2/2$. *Let* $\varphi_j(x, \xi)\in P_\sigma(\tilde{\tau}; 6n_0)$ $(j=1, 2)$ *with an even* $n_0>n+1$ *and small* $0<\tilde{\tau}\ll 1$, *and assume that* $|\nabla_\xi\varphi_j(x, \xi)-x|<c_1\langle x\rangle$ $(j=1, 2)$ *for a sufficiently small constant* $c_1>0$. *Set* $P_{\varphi_1}^\mp=p_{\mp\varphi_1}(X, D)$ *and* $P_{\varphi_2^*}^\pm=p_{\pm\varphi_2^*}(Y, D)$. *Then there exists a constant* $C>0$ *such that for* $\lambda\geqq\lambda_0$

$$\|P_{\varphi_1}^\mp R_0(\lambda\pm i0)P_{\varphi_2^*}^\pm\|_{-s\to s} \leqq C\lambda^{-1/2}. \tag{5.25}$$

PROOF. As in the proof of Corollary 5.2, using Theorem 3.2, we can decompose $\tilde{P}_1^\mp=I_{\varphi_1^*}P_{\varphi_1}^\mp$ and $\tilde{P}_2^\pm=P_{\varphi_2^*}^\pm I_{\varphi_2}$ as $\tilde{P}_j^\mp=R_j^N+Q_j^\mp$ $(j=1, 2)$ for a given $N\geqq 0$ such that $\|\langle x\rangle^N R_j^N\langle x\rangle^N\|<\infty$ and the symbols $q_j^\mp(x, \xi)\in B_0^0$ of $Q_j^\mp$ satisfy (5.1) and (5.2) with δ_0 and $\mu_\pm$ replaced by some other constants $0<\delta_0'<\delta_0$ and $\mu_\pm'\lessgtr\mu_\pm$, $-1<\mu_-'<\mu_+'<1$. Here, in order to let $\delta_0'>0$ and $-1<\mu_-'<\mu_+'<1$, we have used the assumption that c_1 in $|\nabla_\xi\varphi_j(x, \xi)-x|<c_1\langle x\rangle$ is sufficiently small and the asymptotic expansion of the symbols of $\tilde{P}_j^\mp$ (cf. Theorem 3.2-(3.9)-ii) and Proposition 2.1). Since Theorem 3.3 implies the existence of the symbols $r^1(x, \xi)$ and $r^2(\xi, y)\in B_0^0$ such that $R_{\varphi_1}^1 I_{\varphi_1^*}=I$ and $I_{\varphi_2}R_{\varphi_2^*}^2=I$, we have $P_{\varphi_1}^\mp=R_{\varphi_1}^1\tilde{P}_1^\mp$ and $P_{\varphi_2^*}^\pm=\tilde{P}_2^\pm R_{\varphi_2^*}^2$. Therefore we can estimate as follows:

$$\begin{aligned}
&\|P_{\varphi_1}^\mp R_0(\lambda\pm i0)P_{\varphi_2^*}^\pm\|_{-s\to s} \\
&\leqq \|R_{\varphi_1}^1\|_{s\to s}\{\|R_1^N\|_{-s-1\to s}\|R_0(\lambda\pm i0)\|_{s+1\to -s-1}\|R_2^N\|_{-s\to s+1} \\
&\quad+\|R_1^N\|_{-1-s\to s}\|R_0(\lambda\pm i0)Q_2^\pm\|_{-s\to -1-s}+\|Q_1^\mp R_0(\lambda\pm i0)\|_{s+1\to s}\|R_2^N\|_{-s\to s+1} \\
&\quad+\|Q_1^\mp R_0(\lambda\pm i0)Q_2^\pm\|_{-s\to s}\}\,\|R_{\varphi_2^*}^2\|_{-s\to -s}.
\end{aligned} \tag{5.26}$$

Taking $N=s+1$, (5.4), Theorems 4.1, 5.1 and 5.3 conclude the proof. □

§ 6. Micro-local estimates for $(-\Delta/2+V-z)^{-1}$.

We assume the following

ASSUMPTION (L). $V(x)$ is a real-valued C^∞ function such that for all multi-index α

$$|\partial_x^\alpha V(x)| \leqq C_\alpha \langle x\rangle^{-|\alpha|-\varepsilon} \tag{6.1}$$

for some constants $C_\alpha>0$ and ε with $0<\varepsilon<1$.

Let $H=H_0+V(x)$ and $R(z)=(H-z)^{-1}$, $\mathrm{Im}\, z\neq 0$. In this section, we extend Theorems 5.1 and 5.3 to $R(z)$.

In [2] and [4], we have constructed a phase function $\varphi(x, \xi)$ and an amplitude function $a(x, \xi)$ which satisfy the following

PROPOSITION 6.1 (cf. [5, Theorem 2.1]). *Let* $-1<\theta_0<\theta_1<1$, $d>0$, $0<\delta\ll 1$ *and* $R\gg 1$. *Then there exist real-valued* C^∞ *functions* $\varphi(x, \xi)$ *and* $a(x, \xi)$ *which satisfy the following properties:*

i) *For* $|\xi|\geqq d/2$, $\cos(x, \xi)\in[-1, \theta_0]\cup[\theta_1, 1]$ *and* $|x|\geqq R/2$

$$|\nabla_x\varphi(x, \xi)|^2/2+V(x) = |\xi|^2/2. \tag{6.2}$$

φ *satisfies for all* $(x, \xi)\in \boldsymbol{R}^{2n}$ *and* α, β

$$|\partial_x^\alpha\partial_\xi^\beta(\varphi(x, \xi)-x\xi)| \leqq C_{\alpha\beta}\langle x\rangle^{1-\varepsilon-|\alpha|}\langle\xi\rangle^{-1} \tag{6.3}$$

and

$$\varphi(x, \xi) = x\xi \qquad \text{for } |x|\leqq R/4 \text{ or } |\xi|\leqq d/4, \tag{6.4}$$

where constants $C_{\alpha\beta}$ *are independent of* $(x, \xi)\in\boldsymbol{R}^{2n}$ *and* R.

ii) $a(x, \xi)$ *satisfies for all* $(x, \xi)\in\boldsymbol{R}^{2n}$ *and* α, β

$$\begin{cases} |\partial_x^\alpha\partial_\xi^\beta a(x, \xi)| \leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}, \\ |\partial_x^\alpha\partial_\xi^\beta(a(x, \xi)-1)| \leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|-\varepsilon}\langle\xi\rangle^{-1} \\ \qquad \text{for } \cos(x, \xi)\in[-1, \theta_0-\delta]\cup[\theta_1+\delta, 1],\ |\xi|\geqq d,\ |x|\geqq R, \\ a(x, \xi) = 0 \quad \text{for } \cos(x, \xi)\in[\theta_0, \theta_1] \text{ or } |x|\leqq R/2 \text{ or } |\xi|\leqq d/2 \end{cases} \tag{6.5}$$

where $C_{\alpha\beta}$ *are independent of* (x, ξ) *and* R. *Further set*

$$t(x, \xi) = e^{-i\varphi(x,\xi)}(-\Delta/2+V(x)-|\xi|^2/2)(e^{i\varphi(x,\xi)}a(x, \xi)). \tag{6.6}$$

Then for any $L\geqq 1$

$$|\partial_x^\alpha\partial_\xi^\beta t(x, \xi)| \leqq \begin{cases} C_{\alpha\beta L}\langle x\rangle^{-L}\langle\xi\rangle & \text{for } \cos(x, \xi)\in[-1, \theta_0-\delta]\cup[\theta_1+\delta, 1], \\ C_{\alpha\beta}\langle x\rangle^{-1-|\alpha|}\langle\xi\rangle & \text{otherwise.} \end{cases} \tag{6.7}$$

In particular, $t(x, \xi)\in B^1_{-1}$.

Let $c_0>0$ be the constant in Theorem 2.3. Let $\tilde{\tau}>0$ be small enough and $n_0>n+1$ be an even integer such that $\tilde{\tau}\ll c_0/2$, so that Theorem 3.3 holds for the phase functions in $P_\sigma(\tilde{\tau}; 6n_0)$. Let $-1<\mu_-<\mu_+<1$ and $\delta_0>0$ be fixed and let $c_1>0$ be the constant in Corollaries 5.2 and 5.4. Then for given $d>0$ and $-1<\theta_0<\theta_1<1$, we can choose $R>1$ in Proposition 6.1 so large that $\varphi(x, \xi)\in P_{1-\varepsilon}(\tilde{\tau}; 6n_0)$; $|\nabla_\xi\varphi(x, \xi)-x|<c_1\langle x\rangle$; and $a(x, \xi)$ restricted to the region $G=\{(x, \xi)\in \boldsymbol{R}^{2n}|\cos(x, \xi)\in[-1, \theta_0-\delta]\cup[\theta_1+\delta, 1], |\xi|\geqq d, |x|\geqq R\}$ can be extended to an $\tilde{a}(x, \xi)\in B_0^0$ with $|\tilde{a}-1|_{6n_0}^{(0;0)}(|\varphi|_{1,6n_0})^{6n_0}\ll c_0/4$. Hence Theorem 3.4-(3.20) holds for $\varphi(x, \xi)$ and $\tilde{a}(x, \xi)$. These are possible due to (6.3), (6.4) and (6.5). We remark that we can switch to another larger $R>1$ and smaller $c_1>0$ if necessary in the following, since the estimates in Proposition 6.1 are uniform in large R as far as $-1<\theta_0<\theta_1<1$ and $d>0$ are fixed.

We then define the Fourier integral operators:

$$Jf(x) = \text{Os-}\iint e^{i(\varphi(x,\xi)-y\xi)}a(x, \xi)f(y)dyd\xi\,, \tag{6.8}$$

$$\tilde{J}f(x) = \text{Os-}\iint e^{i(\varphi(x,\xi)-y\xi)}\tilde{a}(x, \xi)f(y)dyd\xi\,, \tag{6.9}$$

$$\begin{aligned} Tf(x) &\equiv (HJ-JH_0)f(x) \\ &= \text{Os-}\iint e^{i(\varphi(x,\xi)-y\xi)}t(x, \xi)f(y)dyd\xi \end{aligned} \tag{6.10}$$

for $f\in\mathcal{S}$. The second equality in (6.10) is easily seen by (6.6).

PROPOSITION 6.2. *For* $\operatorname{Im} z\neq 0$,

$$R(z)J = JR_0(z)-R(z)TR_0(z)\,. \tag{6.11}$$

Proof is easy by using $T=HJ-JH_0$.

PROPOSITION 6.3. *The inverse* $\tilde{J}^{-1}$ *exists in the class of conjugate Fourier integral operators with phase function* $\varphi(x, \xi)$ *and* B_0^0*-symbol.*

Proof is clear by Theorem 3.4-(3.20), the definition of $\tilde{a}(x, \xi)$, and our choice of $R\gg 1$.

Let $p_\pm(x, \xi)\in B_0^0$ satisfy (5.1) and (5.2) for $-1<\mu_-<\mu_+<1$ and $\delta_0>0$ fixed above.

PROPOSITION 6.4. *If* $\theta_1+\delta<\mu_+$ [*resp.* $\mu_-<\theta_0-\delta$] *and* $d\ll\delta_0$, *then for any* $s_1, s_2\in\boldsymbol{R}^1$

$$\|(\tilde{J}-J)\tilde{J}^{-1}P_+\|_{s_1\to s_2} < \infty\,. \qquad [\textit{resp.}\ \|(\tilde{J}-J)\tilde{J}^{-1}P_-\|_{s_1\to s_2} < \infty.] \tag{6.12}$$

Proof is easy by using the calculus (Theorems 2.2, 3.2 and 3.4), the asymptotic expansion of the symbols (Proposition 2.1), and the L^2-boundedness theorem (Theorem 4.1). Here we switch to another larger $R>1$ if necessary, and use Theorems 3.2-(3.9)-i), $\theta_1+\delta<\mu_+$, $d\ll\delta_0$, (6.3) and (6.4) in order to separate the supports of the symbols $\tilde{p}_-(x, \xi)$ and $\tilde{p}_+(x, \xi)\in B_0^0$ of $(\tilde{J}-J)\tilde{J}^{-1}$ and P_+ in the sense that $\tilde{\mu}_\pm$ in (5.2) for $\tilde{p}_\pm(x, \xi)$ satisfy $-1<\tilde{\mu}_-<\tilde{\mu}_+<1$ and that $\tilde{p}_+(x, \xi)=0$ for ξ with $|\xi|<\delta_0$ and $\tilde{p}_-(x, \xi)\neq 0$ for some $x\in R^n$, so that

$$\partial_\xi^\alpha D_{x'}^\alpha\{\tilde{p}_-(x, \xi)\tilde{p}_+(x', \xi')\}|_{x'=x,\xi'=\xi}=0 \qquad \text{for all } \alpha.$$

The following theorem is due to Saitō [**9**].

THEOREM 6.5. *Let* $\alpha>1/2$ *and* $\lambda_0>0$. *Then for* $\lambda\geqq\lambda_0$ *the boundary values* $R(\lambda\pm i0)=\lim_{\varepsilon\to 0}(H-(\lambda\pm i\varepsilon))^{-1}$ *exist in* $B(L_\alpha^2, L_{-\alpha}^2)$ *and satisfy for* $\lambda\geqq\lambda_0$

$$\|R(\lambda\pm i0)\|_{\alpha\to-\alpha}\leqq C\lambda^{-1/2}, \tag{6.13}$$

where $C>0$ *is independent of* $\lambda\geqq\lambda_0$.

THEOREM 6.6. *Let* $s>1/2$ *and* $\lambda_0>0$. *Let* $-1<\mu_\pm<1$ *and* $\lambda_0>\delta_0^2/2$. *Then*

$$\|R(\lambda\pm i0)P_\pm\|_{1-s\to-s}\leqq C\lambda^{-1/2}, \tag{6.14}$$

$$\|P_\mp R(\lambda\pm i0)\|_{s\to s-1}\leqq C\lambda^{-1/2}, \tag{6.15}$$

where $C>0$ *is independent of* $\lambda\geqq\lambda_0$.

PROOF. Since (6.15) is conjugate to (6.14), we have only to prove (6.14). We consider $R(\lambda+i0)P_+$ only. The other case can be treated similarly.

We first choose $\theta_0<\theta_1$ and $d>0$ of Proposition 6.1 such that $\theta_1+\delta<\mu_+$ and $d\ll\delta_0$. By Proposition 6.3, we have

$$P_+=\tilde{J}\tilde{J}^{-1}P_+=J\tilde{J}^{-1}P_++(\tilde{J}-J)\tilde{J}^{-1}P_+. \tag{6.16}$$

It is easy to see by the calculus (Theorems 3.1-ii) and 3.4) that, for a given $N\geqq 0$, $\tilde{J}^{-1}P_+$ is decomposed as a sum of two conjugate Fourier integral operators $Q_{\varphi^*}^+$ and $R_{\varphi^*}^N$ with $\|\langle x\rangle^N R_{\varphi^*}^N\langle x\rangle^N\|<\infty$ and the symbol $q^+(\xi, y)\in B_0^0$ of $Q_{\varphi^*}^+$ satisfying (5.1) and (5.2) with δ_0 and μ_+ replaced by some other $0<\delta_0'<\delta_0$ and $-1<\mu_+'<\mu_+$. Here, we switch to another larger $R>1$, if necessary in order to let $\delta_0'>0$ and $\mu_+'>-1$ by virtue of Theorem 3.1-(3.4), (6.3) and (6.4). Note that $\mu_+-\mu_+'$ can be taken arbitrarily small if we let R large enough, hence we can assume $\theta_1+\delta<\mu_+'$ $(<\mu_+)$.

Write $z=\lambda+i\varepsilon$, $\lambda\geqq\lambda_0$, $\varepsilon>0$. By (6.11) and (6.16),

$$\begin{aligned} R(z)P_+ &= R(z)JQ_{\varphi^*}^++R(z)(JR_{\varphi^*}^N+(\tilde{J}-J)\tilde{J}^{-1}P_+) \\ &= JR_0(z)Q_{\varphi^*}^+-R(z)TR_0(z)Q_{\varphi^*}^++R(z)(JR_{\varphi^*}^N+(\tilde{J}-J)\tilde{J}^{-1}P_+). \end{aligned} \tag{6.17}$$

By virtue of the condition $|\nabla_\xi\varphi(x, \xi)-x|<c_1\langle x\rangle$, the estimate conjugate to

(5.9) of Corollary 5.2 yields that the first term on the RHS of (6.17) satisfies the required estimate. The third term on the RHS of (6.17) clearly satisfies the required estimate by Theorem 6.5 and Proposition 6.4, if we take $N=s$.

To treat the second term on the RHS of (6.17), we first note by Proposition 6.1-(6.7), Corollary 5.4, (6.10), $\theta_1+\delta<\mu'_+$, $d^2/2\ll\delta_0'^2/2<\lambda_0$ and $|\nabla_\xi\varphi(x,\xi)-x|<c_1\langle x\rangle$ that for $\lambda\geqq\lambda_0$ and $\varepsilon>0$

$$\|\langle D\rangle^{-1}TR_0(z)Q^+_{\varphi^*}\|_{1-s\to s}\leqq C\lambda^{-1/2}. \tag{6.18}$$

where $\langle D\rangle^{\pm1}$ are the pseudodifferential operators with symbols $\langle\xi\rangle^{\pm1}$. This and Theorem 6.5 imply that for $\lambda\geqq\lambda_0$ and $\varepsilon>0$

$$\begin{aligned}\|R(z)TR_0(z)Q^+_{\varphi^*}\|_{1-s\to-s}&\leqq\|R(z)\langle D\rangle\|_{s\to-s}\|\langle D\rangle^{-1}TR_0(z)Q^+_{\varphi^*}\|_{1-s\to s}\\&\leqq C\lambda^{-1/2},\end{aligned}\tag{6.19}$$

since $\|R(z)\langle D\rangle\|_{s\to-s}\leqq C$ for some constant C independent of $\lambda\geqq\lambda_0$ and $\varepsilon>0$, which follows from Theorem 6.5 and [**3**], Theorem 1.4. This completes the proof of (6.14). □

THEOREM 6.7. *Let $s\geqq0$ and $\lambda_0>0$. Let $-1<\mu_-<\mu_+<1$ and $\lambda_0>\delta_0^2/2$. Then for $\lambda\geqq\lambda_0$*

$$\|P_\mp R(\lambda\pm i0)P_\pm\|_{-s\to s}\leqq C\lambda^{-1/2}, \tag{6.20}$$

where $C>0$ is independent of λ.

PROOF. We only consider $P_-R(\lambda+i0)P_+$. The other case can be treated similarly.

We first choose $\theta_0<\theta_1$ and $d>0$ of Proposition 6.1 such that $\theta_1+\delta<\mu_+$ and $d\ll\delta_0$. As in the proof of Theorem 6.6, for a given $N\geqq0$, $\tilde{J}^{-1}P_+$ can be decomposed as a sum of two conjugate Fourier integral operators $Q^+_{\varphi^*}$ and $R^N_{\varphi^*}$ with $\|\langle x\rangle^N R^N_{\varphi^*}\langle x\rangle^N\|<\infty$ and the symbol $q_+(\xi,y)\in B_0^0$ of $Q^+_{\varphi^*}$ satisfying (5.1) and (5.2) with δ_0 and μ_+ replaced by some other $0<\delta_0'<\delta_0$ and $-1<\mu'_+<\mu_+$. Further, switching to another larger $R>1$ if necessary and using $-1<\mu_-<\mu_+<1$ and $-1<\theta_1+\delta<\mu_+$, μ'_+ can be taken to satisfy $\mu_-<\mu'_+$ and $\theta_1+\delta<\mu'_+$.

Writing $z=\lambda+i\varepsilon$, $\lambda\geqq\lambda_0$, $\varepsilon>0$, we have from (6.17)

$$\begin{aligned}P_-R(z)P_+=&\,P_-JR_0(z)Q^+_{\varphi^*}-P_-R(z)TR_0(z)Q^+_{\varphi^*}\\&+P_-R(z)(JR^N_{\varphi^*}+(\tilde{J}-J)\tilde{J}^{-1}P_+).\end{aligned}\tag{6.21}$$

Using $-1<\mu_-<\mu'_+<1$, Theorem 3.1-i), (6.3) and (6.4) and switching to another larger $R>1$ if necessary, we separate the supports of the symbols $\tilde{p}_-(x,\xi)$ and $q_+(\xi,y)\in B_0^0$ of P_-J and $Q^+_{\varphi^*}$ in the sense that the constants μ'_- and μ'_+ corresponding to $\mu_\pm$ in (5.2) for $\tilde{p}_-$ and q_+ satisfy $-1<\mu'_-<\mu'_+<1$. Then by Corollary 5.4 and the condition $|\nabla_\xi\varphi(x, \xi)-x|<c_1\langle x\rangle$, the first term on the RHS of (6.21) satisfies the required estimate.

The third term on the RHS of (6.21) is estimated as

$$\begin{aligned}(6.22)\qquad &\|P_-R(z)(JR^N_{\varphi*}+(\tilde{J}-J)\tilde{J}^{-1}P_+)\|_{-s\to s}\\ &\leqq \|P_-R(z)\|_{s+1\to s}\|JR^N_{\varphi*}+(\tilde{J}-J)\tilde{J}^{-1}P_+)\|_{-s\to s+1},\end{aligned}$$

which is bounded by a constant times $\lambda^{-1/2}$ by Proposition 6.4 and Theorem 6.6-(6.15), if we take $N=s+1$.

The second term on the RHS of (6.21) is majorized as

$$(6.23)\qquad \|P_-R(z)TR_0(z)Q^+_{\varphi*}\|_{-s\to s}\leqq \|P_-R(z)\langle D\rangle\|_{s+1\to s}\|\langle D\rangle^{-1}TR_0(z)Q^+_{\varphi*}\|_{-s\to s+1}.$$

It follows from Theorem 6.6-(6.15) and Theorem 1.4 of [**3**] that

$$(6.24)\qquad \|P_-R(z)\langle D\rangle\|_{s+1\to s}\leqq C$$

for some constant C independent of $\lambda\geqq\lambda_0$ and $\varepsilon>0$. Therefore, by (6.10), (6.7), Corollary 5.4, $\theta_1+\delta<\mu'_+$, $d^2/2\ll\delta_0'^2/2<\lambda_0$, and $|\nabla_\xi\varphi(x, \xi)-x|<c_1\langle x\rangle$, the RHS of (6.23) is bounded by $C\lambda^{-1/2}$, which concludes the proof. □

§ 7. Micro-local approximation of $(-\Delta/2+V-z)^{-1}$.

Let $\chi_0(x)\in C_0^\infty(\boldsymbol{R}^n)$ satisfy

$$(7.1)\qquad \chi_0(x)=\begin{cases}1 & |x|\leqq 1,\\ 0 & |x|\geqq 2,\end{cases}$$

and set

$$(7.2)\qquad V_j(x)=V(x)\chi_0(x/j)$$

for $j=1, 2, 3, \cdots$. Then $V_j(x)$ satisfy (6.1) in Assumption (L) uniformly in $j\geqq 1$, Further it is easy to check the proof of Theorem 6.5 ([**9**]) to see that (6.13) holds for $R_j(\lambda\pm i0)=(H_j-(\lambda\pm i0))^{-1}$, $H_j=-\Delta/2+V_j$, uniformly in $j\geqq 1$.

Then the following theorem can be proved in quite the same way as in § 5 of [**4**] by using Theorems 6.6 and 6.7 above, and the proof is omitted here.

THEOREM 7.1. *Let* $s\geqq 0$, $\lambda_0>0$, *and* $-1<\mu_-<\mu_+<1$. *Then for* $\lambda\geqq\lambda_0$ *and* $j=1, 2, \cdots$,

$$(7.3)\qquad \|P_-(R(\lambda+i0)-R_j(\lambda+i0))P_+\|_{-s\to s}\leqq Cj^{-\varepsilon}\lambda^{-1},$$

where $C>0$ *is independent of* $\lambda\geqq\lambda_0$ *and* $j=1, 2, \cdots$.

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