J. Math. Soc. Japan
Vol. 39, No. 3, 1987

On étale $SL_2(F_p)$-coverings of algebraic curves of genus 2

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(Received April 25, 1984)
(Revised Nov. 20, 1985)

§ 0. Introduction.

Let C be a connected complete non-singular curve over an algebraically closed field k of positive characteristic p. In this paper, we shall give an upper bound for the number of finite étale Galois coverings of C whose Galois groups are isomorphic to $SL_2(F_p)$ (F_p: a finite field with p elements) when the genus of C is two.

To explain the situation, let us recall some known results. Let g be a positive integer, and let Δ_g be the group generated by $2g$-letters $a_1, \cdots, a_g, b_1, \cdots, b_g$ with one defining relation $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}=1$, and let $\bar{\Delta}_g$ be the pro-finite completion of Δ_g. Let C be a curve of genus g defined over k. Then it was shown by Grothendieck [**3**], and also by Popp [**14**] that there is a surjective continuous homomorphism from $\bar{\Delta}_g$ onto the algebraic fundamental group $\pi_1(C)$ of C, and that its kernel is contained in an arbitrary open normal subgroup of $\bar{\Delta}_g$ of index prime to p. Now fix a finite group G. Let $n(C, G)$ be the number of finite étale Galois coverings of C whose Galois groups are isomorphic to G, and for any compact Riemann surface R of genus g, let $N(R, G)$ be the number of finite unramified Galois coverings of R whose Galois groups are isomorphic to G. Recall that $N(R, G)$ is uniquely determined by g, and that it is equal to the number $N(g, G)$ of normal subgroups N of Δ_g satisfying $\Delta_g/N \cong G$. It follows from the above result that $n(C, G) \leqq N(g, G)$ for any curve C of genus g, and that the equality holds if the order of G is prime to p. So we naturally ask whether or not the equality holds for some curve C if the order of G is divisible by p. The answer is negative for a p-group or a meta-abelian group (for the former case, see Hasse and Witt [**5**], Šafarevič [**15**], and for the latter case, see Katsurada [7], and Nakajima [**11**]). However, when G is a non-solvable group of order divisible by p (for example $G=SL_2(F_{p^m})$ with $p^m \geqq 4$), it seems very difficult to obtain a reasonable upper bound for $n(C, G)$ in the general case. As an attempt, in [8] we treated the special case where $G=SL_2(F_4)$ and C is a certain hyperelliptic curve in characteristic 2. In this paper, developing the method of [8], we shall give an upper bound for $n(C, SL_2(F_p))$ when the genus of C is two, and $p\neq 2, 3$. Moreover, comparing this with the result of Ihara [6], we shall show that $n(C, SL_2(F_p))$ is strictly smaller than $N(2, SL_2(F_p))$ for any curve of genus two.

To be more precise, let $\mathrm{Irr}(N, SL_2(F_p))$ (resp. $\mathrm{Surj}(N, SL_2(F_p))$) be the set of $GL_2(k)$-equivalence classes of irreducible representations of a group N into (resp. onto) $SL_2(F_p)$. Note that the $GL_2(k)$-equivalence class of an irreducible representation ρ belongs to $\mathrm{Irr}(\pi_1(C), SL_2(F_p))\setminus\mathrm{Surj}(\pi_1(C), SL_2(F_p))$ if and only if the order of $\rho(\pi_1(C))$ is prime to p. So the number $\#\mathrm{Irr}(\pi_1(C), SL_2(F_p)) -\#\mathrm{Surj}(\pi_1(C), SL_2(F_p))$ is equal to $\#\mathrm{Irr}(\Delta_g, SL_2(F_p))-\#\mathrm{Surj}(\Delta_g, SL_2(F_p))$. Moreover note that $\#\mathrm{Surj}(\pi_1(C), SL_2(F_p))$ (resp. $\#\mathrm{Surj}(\Delta_g, SL_2(F_p))$) is equal to $n(C, SL_2(F_p))$ (resp. $N(g, SL_2(F_p))$). Thus to compare $n(C, SL_2(F_p))$ and $N(g, SL_2(F_p))$, it is sufficient to compare $\#\mathrm{Irr}(\pi_1(C), SL_2(F_p))$ and $\#\mathrm{Irr}(\Delta_g, SL_2(F_p))$. Now our main result is

THEOREM A. *Assume that the genus of C is two, and $p\neq 2, 3$. Then*

$$\#\mathrm{Irr}(\pi_1(C), SL_2(F_p)) \leqq \frac{1}{3}p^6+\frac{5}{3}p^4+p^2+18p+1.$$

On the other hand, Ihara showed in [6] among others that

$$\#\mathrm{Irr}(\Delta_2, SL_2(F_p)) = p^6+16p^4-5p^2.$$

Thus we have

COROLLARY TO THEOREM A. *Let the assumptions be as above. Then $n(C, SL_2(F_p))$ is strictly smaller than $N(2, SL_2(F_p))$ for any curve C of genus two.*

Now we explain the outline of the proof of Theorem A. By the result of Lange and Stuhler [10], our problem can be reduced to the estimate of the number of stable vector bundles of rank 2 with trivial determinant which are invariant under the p-th power map on C. So in §2 and §3, we consider certain sets $\mathcal{M}$, $\mathfrak{F}\mathcal{M}$ of vector bundles which contain all such vector bundles (see Theorems 2.1 and 3.2). In §4 we construct certain subsets of the projective spaces associated with some matrices which are related to $\mathfrak{F}\mathcal{M}$. In §5 we complete the proof mainly by Theorems 1.4 and 1.5.

The author would like to thank the referee for his valuable advice.

§1. Generalization of Bezout's theorem.

In this section, we shall give a certain generalization of Bezout's theorem. Let R be a commutative ring, and $M_{mn}(R)$ be the ring of all (m, n) matrices with entries in R. Hereafter we assume that $m\geqq n$. For each element A of

$M_{mn}(R)$, let $A(i_1, \cdots, i_r; j_1, \cdots, j_s)$ denote the matrix obtained by deleting the $i_1, \cdots, i_r$-th rows and the $j_1, \cdots, j_s$-th columns from A (we write $B=A(i_1, \cdots, i_r;\ )$ if B is obtained by deleting the $i_1, \cdots, i_r$-th rows). For a while, let R be a polynomial ring $k[X_0, \cdots, X_N]$ over an algebraically closed field k. For each element A of $M_{mn}(R)$, let $\mathfrak{J}(A)$ be the ideal of R generated by all determinants $\det A(i_1, \cdots, i_{m-n};\ )$ $(1\leqq i_1<\cdots<i_{m-n}\leqq m)$. Let V_A be the algebraic set of the $N+1$-dimensional affine space A^{N+1} over k defined by $\mathfrak{J}(A)$. If $\mathfrak{J}(A)$ is a homogeneous ideal, it is regarded as a subset of the N-dimensional projective space P^N over k. Now assume that the codimension of V_A is $m-n+1$. Then we can define a cycle C_A of P^N by

$$C_A = \sum_Q \mathrm{length}(R_Q/\mathfrak{J}(A)R_Q)Q$$

where Q runs over all prime components of V_A of codimension $m-n+1$, and R_Q denotes the local ring of P^N at Q. Then we can determine the degree of C_A explicitly (see Theorem 1.5). Note that when $n=1$, this is nothing but Bezout's theorem on intersections of divisors.

For the proof of Theorem 1.5, let R be an arbitrary commutative ring, and let A be an element of $M_{mn}(R)$. For each systems $(i_1, \cdots, i_{m-n})$ and $(j_1, \cdots, j_{m-n+1})$ of integers such that $1\leqq i_1<\cdots<i_{m-n}\leqq m$ and $1\leqq j_1<\cdots<j_{m-n+1}\leqq m$, put $f_{i_1\cdots i_{m-n}}=\det A(i_1, \cdots, i_{m-n};\ )$, and $h_{j_1\cdots j_{m-n+1}}=\det A(j_1, \cdots, j_{m-n+1}; n)$.

LEMMA 1.1. *With the above notations, we have*

$$(1.1)\qquad (-1)^e h_{n\cdots m} f_{i_1\cdots i_{m-n}} = \sum_{j=n}^{m} (-1)^{e_j} h_{i_1\cdots i_{m-n}j} f_{n\cdots\check{j}\cdots m},$$

where e, e_j are integers, and $(n\cdots\check{j}\cdots m)$ means $(n\cdots j-1, j+1\cdots m)$. (Here, we make the convention that $h_{i_1\cdots i_{m-n}j}=0$ if $j=i_1, \cdots, i_{m-n-1}$ or i_{m-n}.)

PROOF. Fix integers $i_1, \cdots, i_{m-n-1}$, and i_{m-n}, and define a matrix $D=(d_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq m-n}}$ by $d_{ij}=\delta_{i,i_j}$ where δ is Kronecker's delta. Write

$$A=\begin{pmatrix} A_{11} & A_{12} \\ \underbrace{A_{21}}_{n-1} & \underbrace{A_{22}}_{1}\end{pmatrix}\begin{matrix}\}\ n-1\\ \}\ m-n+1\end{matrix},\qquad D=\begin{pmatrix} D_1 \\ D_2\end{pmatrix}\begin{matrix}\}\ n-1\\ \}\ m-n+1\end{matrix},$$

and put

$$B=\begin{pmatrix} A_{11} & 0 & 0 & D_1 \\ A_{21} & A_{21} & A_{22} & D_2 \\ 0 & A_{11} & A_{12} & 0\end{pmatrix}.$$

Then by a simple calculation, we have

$$\det B = (-1)^e h_{n\cdots m} f_{i_1\cdots i_{m-n}}$$

with some integer e. On the other hand, by Laplace's expansion theorem,

$$\det B = \sum_{1 \leq k_1 < \cdots < k_n \leq m+n-1} \det B(k_1, \cdots, k_n; n, \cdots, 2n-1) C_{k_1 \cdots k_n},$$

where $C_{k_1 \cdots k_n}$ denotes the cofactor of $\det B(k_1, \cdots, k_n; n, \cdots, 2n-1)$ in B. We have

$$\det B(k_1, \cdots, k_n; n, \cdots, 2n-1) C_{k_1 \cdots k_n}$$
$$= \begin{cases} (-1)^{e_j} h_{i_1 \cdots i_{m-n} j} f_{n \cdots \check{j} \cdots m} & \text{if } (k_1, \cdots, k_n) = (j, m+1, \cdots, m+n-1), \\ 0 & \text{otherwise.} \end{cases}$$

(Here we make the convention that $f_{n \cdots \check{j} \cdots m} = 0$ if $j > m$ or $j < n$.) This proves the assertion.

Hereafter, for an ideal $\mathfrak{J}$ of a ring R, we often use the same notation $\mathfrak{J}$ to denote the ideal $\mathfrak{J} R_{\mathfrak{Q}}$ of the local ring $R_{\mathfrak{Q}}$ of $\mathfrak{Q}$. Then we have

COROLLARY TO LEMMA 1.1. *Assume that R is a Noetherian ring. For an element A of $M_{mn}(R)$, put $\mathfrak{F} = \mathfrak{J}(A)$, $\mathfrak{F}(m) = \mathfrak{J}(A(m;))$, and $\mathfrak{H} = \mathfrak{J}(A(m; n))$. Let $\mathfrak{F}(m)'$ be the minimal pure ideal containing $\mathfrak{F}(m)$. Assume that*

$$\operatorname{height}\langle f_{n \cdots m-1}, \mathfrak{F}(m)\rangle, \operatorname{height}\langle h_{n \cdots m}, \mathfrak{F}(m)\rangle \geqq \operatorname{height} \mathfrak{F}(m) + 1.$$

Then for any prime ideal $\mathfrak{Q}$ of R such that $\operatorname{height} \mathfrak{Q} = \operatorname{height} \mathfrak{F}(m) + 1$, *we have*

$$\begin{aligned} &\operatorname{length} R_{\mathfrak{Q}} / \langle f_{n \cdots m-1}, \mathfrak{F}(m)' \rangle - \operatorname{length} R_{\mathfrak{Q}} / \langle \mathfrak{F}, \mathfrak{F}(m)' \rangle \\ &= \operatorname{length} R_{\mathfrak{Q}} / \langle h_{n \cdots m}, \mathfrak{F}(m)' \rangle - \operatorname{length} R_{\mathfrak{Q}} / \langle \mathfrak{H}, \mathfrak{F}(m)' \rangle. \end{aligned} \tag{1.2}$$

PROOF. By (1.1), we have $\langle f_{n \cdots m-1} \mathfrak{H}, \mathfrak{F}(m)' \rangle = \langle h_{n \cdots m} \mathfrak{F}, \mathfrak{F}(m)' \rangle$. Therefore, (1.2) is easily proved.

LEMMA 1.2. *Let R be a Cohen Macaulay local ring of Krull dimension n_0. Let $\mathfrak{F}$ be an ideal of height $n_0 - 1$, and T be an element of R such that* $\operatorname{height}\langle \mathfrak{F}, T \rangle = n_0$. *Then $\mathfrak{F}$ is a pure ideal of R if and only if*

$$\sum_{\mathfrak{P}} \operatorname{length} R_{\mathfrak{P}} / \mathfrak{F} \cdot \operatorname{length} R / \langle \mathfrak{P}, T \rangle = \operatorname{length} R / \langle \mathfrak{F}, T \rangle \tag{1.3}$$

where $\mathfrak{P}$ runs over all prime divisors of $\mathfrak{F}$ of height $n_0 - 1$.

PROOF. Assume that $\mathfrak{F}$ is pure. Then $T \bmod \mathfrak{F}$ is not a zero divisor in $R/\mathfrak{F}$. Thus (1.3) is nothing but "the associative formula of multiplicities" (for example, see the proof of Prop. 1 of Chap. IV § 1.3 in Šafarevič [**16**]). Conversely assume that (1.3) holds. Let $\mathfrak{F}'$ be the minimal pure ideal containing $\mathfrak{F}$. Then $\mathfrak{F}$ can be expressed as $\mathfrak{F} = \mathfrak{F}' \cap \mathfrak{q}$ with an ideal $\mathfrak{q}$ whose radical $\sqrt{\mathfrak{q}}$ is the maximal ideal of R. Then by the assumption and the associative formula, we have

$$\text{length}\, R/\langle \mathfrak{F}', T\rangle = \text{length}\, R/\langle \mathfrak{F}, T\rangle$$

and so we have $\langle \mathfrak{F}, T\rangle = \langle \mathfrak{F}', T\rangle$. This implies $\mathfrak{F}' = \mathfrak{F}$. q. e. d.

Now fix positive integers l, n. A system $\{\mathfrak{J}^{(i,j)}, g^{(k,j)}\}$ $(1 \leqq i,\ k \leqq l,\ 1 \leqq j \leqq n)$ of ideals and elements of R is called general if:

(1) $\mathfrak{J}^{(i,j)} \supset \langle g^{(k,j)}_{1\leqq k\leqq i}\rangle$, $\mathfrak{J}^{(1,j)} = \langle g^{(1,j)}\rangle$, and $\mathfrak{J}^{(i,1)} = \langle g^{(k,1)}_{1\leqq k\leqq i}\rangle$,

(2) $\mathfrak{J}^{(i,j)} \supset \mathfrak{J}^{(i-1,j)}$, and $\mathfrak{J}^{(i,j-1)} \supset \mathfrak{J}^{(i-1,j)}$,

(3) for any $2 \leqq i \leqq l$, $2 \leqq j \leqq n$ such that $\mathfrak{J}^{(i-1,j)}$, $\mathfrak{J}^{(i,j)}$, $\mathfrak{J}^{(i,j-1)} \neq R$, we have

$$\begin{aligned} \text{height}\, \mathfrak{J}^{(i,j)} &= \text{height}\langle \mathfrak{J}^{(i-1,j)}, g^{(i,j)}\rangle \\ &= \text{height}\langle \mathfrak{J}^{(i-1,j)}, g^{(1,j-1)}\rangle = \text{height}\, \mathfrak{J}^{(i,j-1)} \\ &= \text{height}\, \mathfrak{J}^{(i-1,j)} + 1 = i. \end{aligned}$$

For each ring S and its ideal $\mathfrak{J}$, put $i(\mathfrak{J}, S) = \text{length}\, S/\mathfrak{J}$.

PROPOSITION 1.3. *Let R be a Cohen Macaulay domain which is a finitely generated k-algebra, and let $\{\mathfrak{J}^{(i',j')}, g^{(k',j')}\}$ $(1 \leqq i',\ k' \leqq l,\ 1 \leqq j' \leqq n)$ be a general system. Assume that for any $2 \leqq i \leqq l$, $2 \leqq j \leqq n$, we have*

$$\langle g^{(i,j)} \mathfrak{J}^{(i,j-1)}, \mathfrak{J}^{(i-1,j)}\rangle = \langle g^{(1,j-1)} \mathfrak{J}^{(i,j)}, \mathfrak{J}^{(i-1,j)}\rangle. \tag{1.4}$$

Then the ideal $\mathfrak{J}^{(i-1,j)} R_{\mathfrak{Q}}$ of $R_{\mathfrak{Q}}$ is pure for any $2 \leqq i \leqq l$, $1 \leqq j \leqq n$, and for any prime ideal $\mathfrak{Q}$ of R of height i. Moreover

$$\begin{aligned} i(\mathfrak{J}^{(i,j)}; R_{\mathfrak{Q}}) = &\sum_{\mathfrak{P}} i(\mathfrak{J}^{(i-1,j)}; R_{\mathfrak{P}})\, i(\langle g^{(i,j)}, \mathfrak{P}\rangle; R_{\mathfrak{Q}}) \\ &- \sum_{\mathfrak{P}} i(\mathfrak{J}^{(i-1,j)}; R_{\mathfrak{P}})\, i(\langle g^{(1,j-1)}, \mathfrak{P}\rangle; R_{\mathfrak{Q}}) + i(\mathfrak{J}^{(i,j-1)}; R_{\mathfrak{Q}}) \end{aligned} \tag{1.5}$$

where $\mathfrak{P}$ runs over all prime ideals of $R_{\mathfrak{Q}}$ of height $i-1$.

PROOF. The assertion (1.5) is a direct consequence of the purity of $\mathfrak{J}^{(i-1,j)}$, and (1.3), (1.4) (cf. Corollary to Lemma 1.1). So we prove the purity of $\mathfrak{J}^{(i,j)} R_{\mathfrak{Q}}$ for any prime ideal $\mathfrak{Q}$ of R of height $i+1$ by induction on (i, j). The assertion holds for $i=1$ or $j=1$. Assume that the assertion holds for any (i', j') such that $i' \leqq i$, $j' \leqq j$, $i'+j' < i+j$. For any prime ideal $\mathfrak{Q}$ of height $i+1$ which contains $\mathfrak{J}^{(i,j)}$, put $S = R_{\mathfrak{Q}}$. Then we can take a prime element T of R such that $T \in \mathfrak{Q}$ and $\{\overline{\mathfrak{J}^{(i',j')}}, \overline{g^{(k',j')}}\}$ $(1 \leqq i',\ k' \leqq i+1,\ 1 \leqq j' \leqq n)$ is a general system in $R/\langle T\rangle$. Here for a subset or an element $\mathfrak{J}$ of R, $\bar{\mathfrak{J}}$ denotes the image of $\mathfrak{J}$ under the canonical surjection $R \to R/\langle T\rangle$. Put $\bar{S} = S/TS$. Then by the inductive hypothesis, $\overline{\mathfrak{J}^{(i-1,j)}} \bar{S}$ is a pure ideal of $\bar{S}$. Thus an element T mod $\langle \mathfrak{J}^{(i-1,j)}, g^{(i,j)}\rangle$ is not a zero divisor in $S/\langle \mathfrak{J}^{(i-1,j)}, g^{(i,j)}\rangle$. Thus we have

$$i(\langle \overline{\mathfrak{J}^{(i-1,j)}}, \overline{g^{(i,j)}}\rangle; \bar{S}) = \sum_{\mathfrak{P}} i(\langle \mathfrak{J}^{(i-1,j)}, g^{(i,j)}\rangle; S_{\mathfrak{P}})\, i(\langle \mathfrak{P}, T\rangle; S).$$

Similarly we have

$$i(\langle\overline{\mathfrak{J}^{(i-1,j)}}, \overline{g^{(1,j-1)}}\rangle\,;\bar{S}) = \sum_{\mathfrak{P}} i(\langle\mathfrak{J}^{(i-1,j)}, g^{(1,j-1)}\rangle\,; S_{\mathfrak{P}})\, i(\langle\mathfrak{P}, T\rangle\,; S).$$

Moreover, by the inductive hypothesis, $\mathfrak{J}^{(i,j-1)}$ is pure, so we have

$$i(\overline{\mathfrak{J}^{(i,j-1)}}\,;\bar{S}) = \sum_{\mathfrak{P}} i(\mathfrak{J}^{(i,j-1)}\,; S_{\mathfrak{P}})\, i(\langle\mathfrak{P}, T\rangle\,; S).$$

By the assumption on the purity of $\mathfrak{J}^{(i-1,j)}$, $\overline{\mathfrak{J}^{(i-1,j)}}$, and by (1.4),

$$i(\overline{\mathfrak{J}^{(i,j)}}\,;\bar{S}) = \sum_{\mathfrak{P}} i(\mathfrak{J}^{(i,j)}\,; S_{\mathfrak{P}})\, i(\langle\mathfrak{P}, T\rangle\,; S),$$

where $\mathfrak{P}$ runs over all prime ideals of S of height i. Thus by Lemma 1.2, $\mathfrak{J}^{(i,j)}S$ is pure. This completes the proof.

Put $l=m-n+1$. We say that an element A of $M_{mn}(R)$ has a general system if a system $\{\mathfrak{J}(A(i+j, \cdots, m\,; j+1, \cdots, n)),\ \det A(j, \cdots, j+\check{k}-1, \cdots, m\,; j+1, \cdots, n)\}$ $(1\leqq k,\ i\leqq l,\ 1\leqq j\leqq n)$ of ideals and elements of R is general. We make the convention that $A(i+j, \cdots, m\,; j+1, \cdots, n)=A(i+n, \cdots, m\,;\ )$ if $j=n$, and $A(i+j, \cdots, m\,; j+1, \cdots, n)=A$ for $j=n$ and $i=l$. Hereafter, we use the same symbol H to denote the divisor on a variety defined by the polynomial H. Define a submatrix A_j of A by $A_j=A(m-n+j+1, \cdots, m\,; j+1, \cdots, n)$. Let $F_{(ij)}$ be the divisor of Spec$R$ defined by the element $F_{(ij)}=\det A_j(j, \cdots, j+\check{i}-1, \cdots, m-n+j\,;\ )$ of R. For any divisors $D_1, \cdots, D_r$, let $D_1\cdot D_2\cdots D_r$ denote the intersection product of them. We write $i(\mathfrak{J}(A)\,;\mathfrak{Q})=i(\mathfrak{J}(A)\,; R_{\mathfrak{Q}})$. Then by (1.5), (1.2), we easily obtain

THEOREM 1.4. *Let R be as in Proposition* 1.3, *and for each element A of $M_{mn}(R)$, define a cycle C_A of* Spec$R$ *by*

$$C_A = \sum_{Q} i(\mathfrak{J}(A)\,; Q)Q\,,$$

where Q runs over all prime components of V_A of codimension $m-n+1$. Put $l=m-n+1$, and assume that A has a general system. Then C_A is expressed as

$$C_A = \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\cdots\sum_{i_{l-1}=1}^{i_{l-2}} (F_{(l,i_1)}-F_{(1,i_1-1)})\cdot(F_{(l-1,i_2)}-F_{(1,i_2-1)}) \cdots (F_{(2,i_{l-1})}-F_{(1,i_{l-1}-1)})\cdot F_{(1,i_{l-1})}\,.$$

(Here, we put $F_{(ij)}=0$ for $j=0$.) If R is a graded ring, and if the $F_{(ij)}$'s are all homogeneous, as a cycle of Proj R, *C_A has a similar expression.*

THEOREM 1.5. *Let $R=k[X_0, \cdots, X_N]$. Assume that an element A of $M_{mn}(R)$ has a general system, and that the determinant of any square submatrix of A is homogeneous. Then*

$$\sum_{Q} i(\mathfrak{J}(A); Q) \deg Q = \sum_{i_1=1}^{n} \cdots \sum_{i_{l-1}=1}^{i_{l-2}} (\deg F_{(l, i_1)} - \deg F_{(1, i_1-1)}) \\ \cdots (\deg F_{(2, i_{l-1})} - \deg F_{(1, i_{l-1}-1)}) \deg F_{(1, i_{l-1})},$$

where Q runs over all prime components of $V_A \subset \mathrm{Proj}\, R$ of codimension l.

REMARK. A result analogous to Theorem 1.5 is proved by Chern in [**1**].

§ 2. Representation of vector bundles.

Let C be a connected complete non-singular curve of genus 2 over an algebraically closed field k. Hereafter, we assume that the characteristic of k is not 2. Then we shall represent some subset of isomorphism classes of vector bundles which contains all isomorphism classes of stable vector bundles of rank 2 with trivial determinant in an algebra-geometric manner. An answer to this problem has been obtained by Narasimhan and Ramanan in [**12**]. But for the proof of our main theorem, we need another formulation (see Theorem 2.1).

By a vector bundle on C, we mean a locally free sheaf on C, and by a line bundle on C, we mean a locally free sheaf of rank one. As is well known, a divisor D on C defines a line bundle in a standard way. This line bundle is denoted by L_D. For any two line bundles L_1, L_2, we often abbreviate $L_1 \otimes L_2$ as $L_1 L_2$. We write $L_{D^{-1}} = L_D^{-1}$. Let E, F be vector bundles on C. Let (W, i, p) denote an extension $0 \to E \xrightarrow{i} W \xrightarrow{p} F \to 0$ of F by E. We denote by $(W, i, p)_U$ the equivalence class of (W, i, p). The set of all equivalence classes of extensions of F by E is denoted by $\mathrm{Ext}(F, E)$. For each extension (W, i, p) of F by E, put

$$\delta((W, i, p)_U) = \partial(\mathrm{id}_F), \tag{2.1}$$

where $\partial : H^0(C, \mathrm{Hom}(F, F)) \to H^1(C, \mathrm{Hom}(F, E))$ is the connecting homomorphism and id_F is the identity map on F. As is well known, δ defines a bijection from $\mathrm{Ext}(F, E)$ to $H^1(C, \mathrm{Hom}(F, E))$.

For each k-module M of dimension n, let $\mathrm{P}.M$ denote the set of all one-dimensional submodules of M. Then $\mathrm{P}.M$ can be regarded as the set of k-valued points of an $(n-1)$-dimensional projective space. We often use the same notation $\mathrm{P}.M$ to denote this projective space. For each non-zero element w of M, let $\langle w \rangle$ be the k-module generated by w. Then two non-trivial extensions (W, i, p) and (W', i', p') of F by E are called quasi-equivalent if $\langle \delta((W, i, p)_U) \rangle = \langle \delta((W', i', p')_U) \rangle$. We denote by $(W, i, p)_T$ the quasi-equivalence class of (W, i, p). The set of all quasi-equivalence classes of non-trivial extensions of F by E is denoted by $\mathrm{P.Ext}(F, E)$ and the natural bijection from $\mathrm{P.Ext}(F, E)$ to $\mathrm{P}.H^1(C, \mathrm{Hom}(F, E))$ is denoted by ε. We often abbreviate $(W, i, p)_U$ (resp. $(W, i, p)_T$) as $(W)_U$ (resp. $(W)_T$).

For each vector bundle V, let $[V]$ (resp. $[[V]]$) denote the corresponding isomorphism class (resp. S-equivalence class). (For the word "S-equivalence class", see Seshadri [**17**].) We say that V (or $[V]$) is represented in a subset S of $\mathrm{Ext}(F, E)$ if there is an element $(V_1, i_1, p_1)_U$ of S such that $[V]=[V_1]$. Then V (or $[V]$) is called represented by an element w of $H^1(C, \mathrm{Hom}(F, E))$ if $\delta((V_1, i_1, p_1)_U)=w$. In particular if (V_1, i_1, p_1) is non-trivial and $(V_1, i_1, p_1)_T$ belongs to a subset S' of $\mathrm{P.Ext}(F, E)$, V (or $[V]$) is called represented in S' by $\langle w\rangle$ or by $(V_1, i_1, p_1)_T$.

Now there are six Weierstrass points on C. The set of Weierstrass points is denoted by $\mathcal{W}$. A line bundle L_Q is called a Weierstrass line bundle if $Q\in\mathcal{W}$. We use the same symbol θ to denote the divisor class of a divisor θ. Let J be the Jacobian variety of C and put $J(n)=\{\theta\in J;\ \theta^n=1\}$.

Let $\pi: C\to P^1$ be the finite Galois morphism from C to a one dimensional projective space P^1 of degree two with six ramification points $Q\in\mathcal{W}$, and let σ be the non-trivial element of the Galois group $G=\mathrm{Gal}(C/P^1)$. Fix a point P_0 of C, and put $K=L_{P_0}\otimes L_{P_0'}$ with $P_0'=\sigma(P_0)$. Then σ acts on $H^1(C, \mathrm{Hom}(K, K^{-1}))$ in a natural manner. Thus we have

$$H^1(C, \mathrm{Hom}(K, K^{-1})) = H^1(C, \mathrm{Hom}(K, K^{-1}))_+ \oplus H^1(C, \mathrm{Hom}(K, K^{-1}))_-,$$

where $H^1(C, \mathrm{Hom}(K, K^{-1}))_\pm=\{\xi\in H^1(C, \mathrm{Hom}(K, K^{-1}));\ \sigma(\xi)=\pm\xi\}$ (cf. § 2.3). Put $\mathrm{Ext}(K, K^{-1})_i=\delta^{-1}(H^1(C, \mathrm{Hom}(K, K^{-1}))_i)$ for $i=+,-$.

Now put

$$\mathcal{S} = \{[[V]];\ V \text{ is a semi-stable vector bundle of rank 2 with trivial determinant}\}.$$

Then we have the following

THEOREM 2.1. *Assume that P_0 is a non-Weierstrass point. Put $P^*=\mathrm{P.Ext}(K, K)_- \amalg \coprod_{Q\in\mathcal{W}} \mathrm{P.Ext}(L_Q, L_Q^{-1})$ (a disjoint union), and $W_{ns}=\{(W_{[Q]})_T;\ Q\in\mathcal{W}\}$ (for the definition of $(W_{[Q]})_T$, see (2) of Lemma 2.4). Then every element of $P^*\setminus W_{ns}$ represents a semi-stable vector bundle, and the (set-theoritical) map*

$$\begin{array}{cccc} \bar{\varepsilon}: & P^*\setminus W_{ns} & \longrightarrow & \mathcal{S} \\ & \Downarrow & & \Downarrow \\ & (W, i, p)_T & \longrightarrow & [[W]] \end{array}$$

is surjective. Moreover

$$\bar{\varepsilon}^{-1}([[W]]) = \begin{cases} 2 & \text{if } W \text{ is stable}, \\ \text{infinite} & \text{if } [[W]]=[[L_\theta\oplus L_\theta^{-1}]] \text{ with } \theta\in J(2), \\ 1 & \text{otherwise}. \end{cases}$$

The proof of Theorem 2.1 will be done after Lemma 2.17 (see Propositions 2.7 and 2.10, and Corollary to Proposition 2.16).

§ 2.1. **Bilinear maps and homomorphisms of vector bundles.** Let E, F, and V be vector bundles on C. Then define a bilinear map $\Theta_V: H^0(C, \mathrm{Hom}(V, F)) \times H^1(C, \mathrm{Hom}(F, E)) \to H^1(C, \mathrm{Hom}(V, E))$ by

$$\Theta_V(f, w) = f^*(w) \tag{2.2}$$

for $f \in H^0(C, \mathrm{Hom}(V, F))$ and $w \in H^1(C, \mathrm{Hom}(F, E))$, where $f^*: H^1(C, \mathrm{Hom}(F, E)) \to H^1(C, \mathrm{Hom}(V, E))$ is the homomorphism induced by f. Now fix an extension $0 \to E \to W \to F \to 0$, and put $w = \delta((W)_U)$. Then by the commutative diagram in the proof of Lemma 3.1 in Narasimhan and Ramanan [**12**], for any $f \in H^0(C, \mathrm{Hom}(V, F))$,

$$\Theta_V(f, w) = \partial(f) \tag{2.3}$$

where $\partial: H^0(C, \mathrm{Hom}(V, F)) \to H^1(C, \mathrm{Hom}(V, E))$ is the connecting homomorphism. Thus we obtain the following isomorphism

$$\Delta : H^0(C, \mathrm{Hom}(V, W))/H^0(C, \mathrm{Hom}(V, E)) \cong \ker \Theta_V(\ , w). \tag{2.4}$$

Here we regard $H^0(C, \mathrm{Hom}(V, E))$ as a submodule of $H^0(C, \mathrm{Hom}(V, W))$ in a natural manner, and $\Theta_V(\ , w): H^0(C, \mathrm{Hom}(V, F)) \to H^1(C, \mathrm{Hom}(V, E))$ is the homomorphism induced by Θ_V. Define a bilinear map, which we also denote by Θ_V, from $H^1(C, \mathrm{Hom}(F, E)) \times H^0(C, \mathrm{Hom}(E, V))$ to $H^1(C, \mathrm{Hom}(F, V))$ in the same manner as above. Then similarly to (2.4),

$$\Delta : H^0(C, \mathrm{Hom}(W, V))/H^0(C, \mathrm{Hom}(F, V)) \cong \ker \Theta_V(w, \ ). \tag{2.5}$$

Hereafter, we often abbreviate Θ_V as Θ if no confusion arises.

Now for the σ of $G = \mathrm{Gal}(C/P^1)$, and a vector bundle U, there is a natural isomorphism

$$\sigma^* : H^i(C, U) \cong H^i(C, \sigma^* U). \tag{2.6}$$

In particular, if U is σ-invariant, the isomorphism $\sigma^* U \cong U$ induces an isomorphism $H^i(C, \sigma^* U) \cong H^i(C, U)$. Then composing this with the σ^* in (2.6), we obtain an isomorphism from $H^i(C, U)$ to itself, which will be also denoted by σ. Since the characteristic of k is not 2, $H^i(C, U)$ is decomposed as

$$H^i(C, U) = H^i(C, U)_+ \oplus H^i(C, U)_-,$$

where $H^i(C, U)_\pm = \{\xi \in H^i(C, U); \sigma(\xi) = \pm\xi\}$. Now let E_1, E_2, E_3 be vector bundles. Let p, q be non-negative integers such that $p+q=1$, and let $\Theta: H^p(C, \mathrm{Hom}(E_1, E_2)) \times H^q(C, \mathrm{Hom}(E_2, E_3)) \to H^1(C, \mathrm{Hom}(E_1, E_3))$ (resp. Θ': $H^p(C, \sigma^*\mathrm{Hom}(E_1, E_2)) \times H^q(C, \sigma^*\mathrm{Hom}(E_2, E_3)) \to H^1(C, \sigma^*\mathrm{Hom}(E_1, E_3))$) be the bilinear map stated above. Then clearly

$$\Theta'(\sigma^* w_1, \sigma^* w_2) = \sigma^*(\Theta(w_1, w_2)). \tag{2.7}$$

From this we obtain the following

LEMMA 2.2. (1) *Let E_1, E_2 and E_3 be σ-invariant vector bundles, and let Θ be as above. Then Θ is decomposed as $\Theta=\Theta_{++}+\Theta_{+-}+\Theta_{-+}+\Theta_{--}$ with*

$$\Theta_{ij}: H^p(C, \mathrm{Hom}(E_1, E_2))_i \times H^q(C, \mathrm{Hom}(E_2, E_3))_j \to H^1(C, \mathrm{Hom}(E_1, E_3))_{k(i,j)}$$

where $k(i, j)=+$ or $-$ according as $i=j$ or not. In particular, for any element w of $H^1(C, \mathrm{Hom}(E_2, E_3))_-$, the homomorphism $\Theta(\ , w)$ is decomposed as $\Theta(\ , w)=\Theta(\ , w)_+ + \Theta(\ , w)_-$ with

$$\Theta(\ , w)_i : H^0(C, \mathrm{Hom}(E_1, E_2))_i \to H^1(C, \mathrm{Hom}(E_1, E_3))_{j(i)},$$

where $j(i)=+$ or $-$ according as $i=-$ or $+$. The homomorphism $\Theta(w,\ )$ is decomposed in a similar manner.
(2) *Let D be a divisor on C, and let W be a vector bundle. Then $H^0(C, \mathrm{Hom}(L_D, W))\neq 0$ if and only if $H^0(C, \mathrm{Hom}(L_{\sigma^*(D)}, \sigma^*W))\neq 0$.*

Now let E, F be σ-invariant vector bundles on C, and assume that $(W, i, p)_U$ belongs to $\mathrm{Ext}(F, E)$. Then we obtain an extension $0\to E\to \sigma^*W\to F\to 0$, and by (2.1), (2.3), (2.7) and by the definition of the action of σ on $H^1(C, \mathrm{Hom}(F, E))$, we have

$$\delta((\sigma^*W)_U)=\sigma\circ\delta((W)_U).$$

Thus we have $\delta((\sigma^*W)_U)=+\delta((W)_U)$ or $-\delta((W)_U)$ according as $(W, i, p)_U$ belongs to $\mathrm{Ext}(F, E)_+$ or $\mathrm{Ext}(F, E)_-$. Thus there is the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \sigma^*W & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pm\mathrm{id}} & & \downarrow{\scriptstyle h_\sigma} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & E & \longrightarrow & W & \longrightarrow & F & \longrightarrow & 0. \end{array} \tag{2.8}$$

Thus we can define an action σ' of σ on $H^i(C, W)$ by

$$\sigma'(u)=h_\sigma^*\circ\sigma^*(u)$$

for $u\in H^i(C, W)$. Then we have

PROPOSITION 2.3. *Let E, F, and V be σ-invariant vector bundles. Let $(W)_U \in \mathrm{Ext}(F, E)_-$. Then we can define an action of σ' on $H^0(C, \mathrm{Hom}(V, W))$ such that*

$$H^0(C, \mathrm{Hom}(V, E))_{j(i)} \subset H^0(C, \mathrm{Hom}(V, W))_i$$

for $i=+, -$, and that the isomorphism Δ in (2.4) is decomposed as $\Delta=\Delta_+ + \Delta_-$ with

$$\Delta_i : H^0(C, \mathrm{Hom}(V, W))_i / H^0(C, \mathrm{Hom}(V, E))_{j(i)} \cong \ker\Theta(\ , w)_i.$$

We can also define an action of σ on $H^0(C, \mathrm{Hom}(W, V))$ with the same properties.

PROOF. The action σ' of σ on $H^0(C, W)$ induces an action of σ on $H^0(C, \mathrm{Hom}(V, W))$, which will be also denoted by σ'. Then by (2.8) we obtain the following commutative diagram:

$$\begin{array}{ccccccc} 0\to H^0(C, \mathrm{Hom}(V, E)) & \to & H^0(C, \mathrm{Hom}(V, W)) & \to & H^0(C, \mathrm{Hom}(V, F)) & \to & H^1(C, \mathrm{Hom}(V, E)) \\ \downarrow{\scriptstyle -\sigma} & & \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle -\sigma} \\ 0\to H^0(C, \mathrm{Hom}(V, E)) & \to & H^0(C, \mathrm{Hom}(V, W)) & \to & H^0(C, \mathrm{Hom}(V, F)) & \to & H^1(C, \mathrm{Hom}(V, E)). \end{array}$$

Thus the assertion is proved.

§2.2. **Čech cohomology and cup product.** Let $H^i(\mathcal{U}, V)$ be the i-th cohomology group of a vector bundle V with respect to an open covering $\mathcal{U}$. As is well known, if $\mathcal{U}$ is an affine open covering, there is an isomorphism $\iota_{\mathcal{U}}$ from $H^i(\mathcal{U}, V)$ to $H^i(C, V)$. For each element u of the 1-cocycle group $Z^1(\mathcal{U}, V)$, let $[u]$ denote the corresponding cohomology class. Let $\pi: C\to P^1$ be the morphism stated before. For each point P of C, define an affine open subset U_P of C by $U_P=\pi^{-1}(P^1\setminus\{\pi(P)\})$. If $\pi(Q_1)\neq\pi(Q_2)$, $\mathcal{U}=\{U_{Q_1}, U_{Q_2}\}$ forms an affine open covering of C. Then for each element u of the group $\Gamma(U_{Q_1}\cap U_{Q_2}; V)$ of the sections of V over $U_{Q_1}\cap U_{Q_2}$, define an element $\{u^{\lambda\mu}\}$ of $Z^1(\mathcal{U}, V)$ by $u^{12}=-u^{21}=u$, $u^{11}=u^{22}=0$. $\{u^{\lambda\mu}\}$ is uniquely determined by u. Hence we often simply write $\{u^{\lambda\mu}\}=\{u\}$, and $[\{u^{\lambda\mu}\}]=[u]$.

For two vector bundles E, F, we often identify $\mathrm{Hom}(F, E)$ with $F^*\otimes E$, where F^* is the dual of F. If E, F are line bundles, we can naturally identify $\mathrm{Hom}(V, E)$ with $\mathrm{Hom}(V, F)\otimes\mathrm{Hom}(F, E)$. Then the bilinear map Θ_V defined by (2.2) is nothing but the cup product

$$\cup : H^0(C, \mathrm{Hom}(V, F))\times H^1(C, \mathrm{Hom}(F, E)) \to H^1(C, \mathrm{Hom}(V, F)\otimes\mathrm{Hom}(F, E)).$$

Thus if we take an affine open covering $\mathcal{U}=\{U_\lambda\}$, we have

$$\Theta_V(f, w) = \iota_{\mathcal{U}}([\{f^\lambda\otimes w^{\lambda\mu}\}]) \tag{2.9}$$

for $f=\{f^\lambda\}\in H^0(C, \mathrm{Hom}(V, F))$ $(=H^0(\mathcal{U}, \mathrm{Hom}(V, F)))$, and $w=\iota_{\mathcal{U}}([\{w^{\lambda\mu}\}])\in H^1(C, \mathrm{Hom}(F, E))$. In particular if we fix w,

$$\ker\Theta_V(\ , w) = \left\{\begin{array}{l}\{f^\lambda\}\in H^0(C, \mathrm{Hom}(V, F)) ; f^\lambda\otimes w^{\lambda\mu}=b^\lambda-b^\mu \\ \qquad\qquad \text{in } U_\lambda\cap U_\mu \text{ with } b^\lambda\in\Gamma(U_\lambda, \mathrm{Hom}(V, E))\end{array}\right\}. \tag{2.10}$$

Similarly, the bilinear map $\Theta_V: H^1(C, \mathrm{Hom}(F, E))\times H^0(C, \mathrm{Hom}(E, V))\to H^1(C, \mathrm{Hom}(F, V))$ in §2.1 is nothing but the cup product

$$\cup : H^1(C, \mathrm{Hom}(F, E))\times H^0(C, \mathrm{Hom}(E, V)) \to H^1(C, \mathrm{Hom}(F, E)\otimes\mathrm{Hom}(E, V)).$$

§2.3. **Representation of unstable vector bundles.** For vector bundles E, F, and V, put

$$S(F, E; V) = \{(W)_U \in \mathrm{Ext}(F, E) \ ; \ H^0(C, \mathrm{Hom}(V, W)) \neq 0\},$$

$$S'(F, E; V) = \{(W)_U \in \mathrm{Ext}(F, E) \ ; \ W \text{ has a sub-vector bundle } V\}.$$

Clearly we have $S'(F, E; V) \subset S(F, E; V)$.

LEMMA 2.4. *Let θ_0 be the unit element of the Jacobian J. Let $K = L_{P_0} \otimes L_{P'_0}$ be the line bundle stated before.*

(1) *Let $\theta \in J$, and $\theta \neq \theta_0$. Then* $\mathrm{P}.S(K, K^{-1}; L_\theta)$ *forms a one-dimensional linear subspace of* $\mathrm{P}.\mathrm{Ext}(K, K^{-1})$.

(2) *For any $Q \in C$,* $\mathrm{P}.S'(K, K^{-1}; K \otimes L_Q^{-1})$ *consists of one element. This element will be denoted by $(W_{[Q]})_T$. $W_{[Q]}$ is non-semistable. Conversely if an element $(W)_T$ of* $\mathrm{P}.\mathrm{Ext}(K, K^{-1})$ *represents a non-semistable vector bundle, there exists a unique $Q \in C$ such that $(W)_T = (W_{[Q]})_T$.*

PROOF. (1) By (2.4), we have

$$\delta(S(K, K^{-1}; L_\theta)) = \{w \in H^1(C, \mathrm{Hom}(K, K^{-1})) \ ; \ \ker \Theta_{L_\theta}(\ , w) \neq 0\}.$$

Since $\theta \neq \theta_0$, $H^0(C, \mathrm{Hom}(L_\theta, K))$ is generated by a single element γ. Fix this γ. Then an element w of $H^1(C, \mathrm{Hom}(K, K^{-1}))$ belongs to $\delta(S(K, K^{-1}; L_\theta))$ if and only if $w \in \ker \Theta_{L_\theta}(\gamma, \ )$. By (2.2), $\Theta_{L_\theta}(\gamma, \ )$ is surjective. Thus by the Riemann-Roch theorem, $\ker \Theta_{L_\theta}(\gamma, \ )$ is a two-dimensional k-submodule of $H^1(C, \mathrm{Hom}(K, K^{-1}))$. This completes the proof.

(2) In a way similar to (1), we can prove that $\mathrm{P}.S(K, K^{-1}; K \otimes L_Q^{-1})$ consists of one element. Let $(W)_T \in \mathrm{P}.S(K, K^{-1}; K \otimes L_Q^{-1})$, and let h be a non-zero element of $H^0(C, \mathrm{Hom}(K \otimes L_Q^{-1}, W))$. Assume that $(W)_T \notin \mathrm{P}.S'(K, K^{-1}; K \otimes L_Q^{-1})$. Then $h(P) = 0$ for some $P \in C$, where $h(P)$ denotes the image of h under the natural homomorphism from $H^0(C, \mathrm{Hom}(K \otimes L_Q^{-1}, W))$ to the fibre of $\mathrm{Hom}(K \otimes L_Q^{-1}, W)$ at P. Then by Lemma 5.3 in Narasimhan-Ramanan [**12**], $H^0(C, \mathrm{Hom}(K \otimes L_Q^{-1} \otimes L_P, W)) \neq 0$. Thus by (2.4), $H^0(C, \mathrm{Hom}(K \otimes L_Q^{-1} \otimes L_P, K)) \neq 0$. Thus we have $K \otimes L_Q^{-1} \otimes L_P \cong K$, and $W \cong K^{-1} \oplus K$, which is a contradiction. Thus we have $(W)_T \in \mathrm{P}.S'(K, K^{-1}; K \otimes L_Q^{-1})$. Conversely assume that an element $(W)_T$ of $\mathrm{P}.\mathrm{Ext}(K, K^{-1})$ represents a non-semistable vector bundle. Then $H^0(C, \mathrm{Hom}(L_1, W)) \neq 0$ for some line bundle L_1 of positive degree. Then by (2.4), $H^0(C, \mathrm{Hom}(L_1, K)) \neq 0$. Since we have $W \neq K^{-1} \oplus K$, the degree of L_1 is 1, and $L_1 = K \otimes L_Q^{-1}$ for some $Q \in C$. This proves the assertion.

Now let $L = L_P \otimes L_{P'}$ with $P \in C$. Then L^m is a σ-invariant line bundle. Hence we can define submodules $H^i(C, L^m)_+$ and $H^i(C, L^m)_-$ of $H^i(C, L^m)$ as in §2.1. From now on, let $K = L_{P_0} \otimes L_{P'_0}$ with P_0 a non-Weierstrass point. Fix $R \in C$, $R \neq P_0, P'_0$ and a non-zero element x_R of $H^0(C, \mathrm{Hom}(K \otimes L_R^{-1} \otimes L_{R'}^{-1}))$. Then for any $P \in C$, $P \neq P_0, P'_0$, there is a unique element x_P of $H^0(C, \mathrm{Hom}(K \otimes L_P^{-1} \otimes L_{P'}^{-1}))$ such that $x_R - x_P$ belongs to the field k. Moreover put $x_{P_0}=x_{P'_0}=1_k$, the unit element of k, and take a rational function y on C such that $y^2=\prod_{Q\in\mathcal{W}}x_Q$.

LEMMA 2.5. (1) *For the covering* $\mathcal{U}=\{U_Q, U_{P_0}\}$ $(\pi(Q)\neq\pi(P_0))$, *let* ι_Q *denote the isomorphism* $\iota_{\mathcal{U}}: H^1(\mathcal{U}, K^*\otimes K^{-1})\to H^1(C, K^*\otimes K^{-1})$ *in* §2.2. *Then* $\{\iota_Q([x_Q^{-i-1}y])\}_{i=0}^3$, *and* $\{\iota_Q([x_Q^{-1}])\}$ *form bases for* $H^1(C, K^*\otimes K^{-1})_-$, *and* $H^1(C, K^*\otimes K^{-1})_+$, *respectively. Moreover for any* $P\in C$ *such that* $\pi(P)\neq\pi(P_0)$, $\iota_P([x_P^{-1}y])=\sum_{i=0}^3\lambda_P^i\iota_Q([x_Q^{-i-1}y])$ *with* $\lambda_P=x_Q-x_P$, $\iota_P([x_P^{-4}y])=\iota_Q([x_Q^{-4}y])$, *and* $\iota_P([x_P^{-1}])=\iota_Q([x_Q^{-1}])$.
(2) *Let* $P\in C$, $Q\in\mathcal{W}$. *For the covering* $\mathcal{U}=\{U_P, U_{P_0}\}$ $(\pi(P)\neq\pi(P_0))$, *let* $\iota_{Q,P}$ *be the isomorphism from* $H^1(\mathcal{U}, L_Q^*\otimes L_Q^{-1})$ *to* $H^1(C, L_Q^*\otimes L_Q^{-1})$ *in* §2.2. *Then* $\{\iota_{Q,Q}([x_Q^{-i+1}y])\}_{i=1}^3$ *forms a basis for* $H^1(C, L_Q^*\otimes L_Q^{-1})$. *Moreover, we have* $\iota_{Q,P}([x_Qx_P^{-1}y])=\sum_{i=1}^3\lambda_P^{i-1}\iota_{Q,Q}([x_Q^{-i+1}y])$ *for any* $P\in C$ *such that* $\pi(P)\neq\pi(P_0)$.

PROOF. The first part of (1) can be easily proved. Put $U_1=U_{P_0}$, $U_2=U_Q$, and $U_3=U_P$, and $x=x_Q$, $\lambda=\lambda_P$. Then an element $x_P^{-1}y=x^{-1}(1-x^{-1}\lambda)^{-1}y$ of $\Gamma(U_1\cap U_3, K^*\otimes K^{-1})$ can be expressed as

$$x_P^{-1}y=x^{-1}\Big(\sum_{i=0}^{3}\lambda^ix^{-i}y+x^{-4}\lambda^4(1-x^{-1}\lambda)^{-1}y\Big).$$

Here, $x^{-5}\lambda^4(1-x^{-1}\lambda)^{-1}y\in\Gamma(U_2\cap U_3, K^*\otimes K^{-1})$. This proves the second part of (1). Similarly the rest of the assertion can be proved.

By (1) of the above lemma, $\iota_P([x_P^{-4}y])$, and $\iota_P([x_P^{-1}])$ do not depend on a point P such that $\pi(P)\neq\pi(P_0)$. Hence we write $\zeta_{P_0}=\zeta_{P'_0}=\iota_P([x_P^{-4}y])$, and $\eta=\iota_P([x_P^{-1}])$. We also write $\zeta_P=\iota_P([x_P^{-1}y])$ and $\zeta_{Q,P}=\iota_{Q,P}([x_Qx_P^{-1}y])$ for $P\neq P_0, P'_0$, and $Q\in\mathcal{W}$. Moreover define a subset C_θ of $\mathrm{P.Ext}(K, K^{-1})_-$ by $C_\theta=\varepsilon^{-1}(\{\langle\zeta_P\rangle; P\in C\})$ or $C_\theta=\varepsilon^{-1}(\{\langle a\zeta_P+b\zeta_Q\rangle; a, b\in k,\ a\neq0 \text{ or } b\neq0\})$ according as $\theta=\theta_0$ or $\theta=P_0P'_0P^{-1}Q^{-1}\neq\theta_0$.

LEMMA 2.6. (1) $\langle\eta\rangle\in\varepsilon(\mathrm{P.}S'(K, K^{-1}; L_{\theta_0})_+)$ *and* $C_{\theta_0}\subset\mathrm{P.}S(K, K^{-1}; L_{\theta_0})_-$. *In particular if* $P\in\mathcal{W}$, $\langle\zeta_P\rangle=\varepsilon((W_{[P]})_T)$.
(2) *Let* $L_\theta=K\otimes L_P^{-1}\otimes L_Q^{-1}$, *with* $\theta\in J(2)$, $\theta\neq\theta_0$. *Then,* $C_\theta\subset\mathrm{P.}S(K, K^{-1}; L_\theta)_-$.
(3) *For any* $P, Q\in\mathcal{W}$, $\langle\zeta_{Q,P}\rangle\in\varepsilon(\mathrm{P.}S(L_Q, L_Q^{-1}; K\otimes L_P^{-1}\otimes L_Q^{-1}))$.
(In the above, for a line bundle $L=K$ or L_Q, we identify $H^1(C, \mathrm{Hom}(L, L^{-1}))$ with $H^1(C, L^*\otimes L^{-1})$.) Hereafter we write $(W_{[Q,\theta]})_T=\varepsilon^{-1}(\langle\zeta_{Q,P}\rangle)$.

PROOF. Let $\mathcal{U}=\{U_Q, U_{P_0}\}$ be as above. Let $\theta=\theta_0$. Then for any $P\in C$, x_P belongs to $H^0(\mathcal{U}, L_\theta^*\otimes K)$. Moreover for any $P\neq P_0, P'_0$, we have $x_Px_Q^{-1}=1_k-\lambda_Px_Q^{-1}$ with $1_k\in\Gamma(U_{P_0}, L_\theta^*\otimes K^{-1})$, $\lambda_Px_Q^{-1}\in\Gamma(U_Q, L_\theta^*\otimes K^{-1})$, and $x_{P_0}x_Q^{-1}\in\Gamma(U_Q, L_\theta^*\otimes K^{-1})$. Thus by (2.10), $\Theta_{L_\theta}(x_P, \eta)=0$ for any $P\in C$. Thus by (2.4), η belongs to $\delta(S(K, K^{-1}; L_\theta)_+)$. A similar calculation shows that for any $P\in C$, the homomorphism $\Theta_{K\otimes L_P^{-1}}(\ ,\eta): H^0(C, (K\otimes L_P^{-1})^*\otimes K))\to H^1(C, (K\otimes L_P^{-1})^*\otimes K^{-1})$

is injective. Hence by (2) of Lemma 2.4, and (2.4), η represents a semistable vector bundle. This proves the first part of (1). Similarly, for any $P\in C$, $\Theta_{L_\theta}(x_P, \zeta_P)=0$. This proves the second part of (1). In particular, if P belongs to $\mathcal{W}$, 1_k belongs to $H^0(C, (K\otimes L_P^{-1})^*\otimes K)$, and $1_k x_P^{-1} y$ belongs to $\Gamma(U_{P_0}, (K\otimes L_P^{-1})^*\otimes K^{-1})$. By (2.10), this implies that $\Theta_{K\otimes L_P^{-1}}(l_k, \zeta_P)=0$. Thus the last part of (1) is proved by (2.4), and (2) of Lemma 2.4. Similarly, (2) and (3) can be proved.

COROLLARY. *Let $\theta=\theta_0$. Then for any $x_P\in H^0(C, L_\theta^*\otimes K)$, the kernel of the homomorphism $\Theta_{L_\theta}(x_P,\ ): H^1(C, K^*\otimes K^{-1})\to H^1(C, L_\theta^*\otimes K^{-1})$ is generated by ζ_P and η over k.*

PROOF. It follows from the proof of (1) of the above lemma that ζ_P and η belong to $\ker\Theta_{L_\theta}(x_P,\ )$. On the other hand, similarly to (1) of Lemma 2.4, $\ker\Theta_{L_\theta}(x_P,\ )$ is two dimensional. This completes the proof.

PROPOSITION 2.7. *An element $(W)_T$ of P^* represents a non-semistable vector bundle if and only if $(W)_T=(W_{[Q]})_T$ for some $Q\in\mathcal{W}$.*

PROOF. The "if" part follows from (1) of Lemma 2.6. On the other hand, as seen in §2.1, if $(W)_T\in \mathrm{P.Ext}(K, K^{-1})_-$, $\sigma^*W\cong W$. Thus the "only if" part follows from Lemma 5.1 in Narasimhan-Ramanan [12], (2) of Lemma 2.2, and (2) of Lemma 2.4.

Now recall that a line bundle L of degree 0 can be expressed as $L=K\otimes L_P^{-1}\otimes L_Q^{-1}$ with $P, Q\in C$. Then,

PROPOSITION 2.8. (1) *Let $L_\theta=K\otimes L_P^{-1}\otimes L_Q^{-1}$ with $\theta\notin J(2)$. Assume that $P, Q\notin\mathcal{W}$. Then $\mathrm{P.}S'(K, K^{-1}; L_\theta)_-$ consists exactly of one element. This element represents a vector bundle $L_\theta\oplus L_\theta^{-1}$. Assume that exactly one of P and Q belongs to $\mathcal{W}$. Then $\mathrm{P.}S'(K, K^{-1}; L_\theta)_-$ is empty.*
(2) *Let $L_\theta=K\otimes L_P^{-1}\otimes L_Q^{-1}$ with $\theta\in J(2)$, $\theta\neq\theta_0$. Then $\mathrm{P.}S(K, K^{-1}; L_\theta)=\mathrm{P.}S(K, K^{-1}; L_\theta)_-$.*

PROOF. By Lemma 5.3 in Narasimhan-Ramanan [12], and (2) of Lemma 2.4,

$$\mathrm{P.}S(K, K^{-1}; L_\theta)=\mathrm{P.}S'(K, K^{-1}; L_\theta)\cup\{(W_{[P]})_T, (W_{[Q]})_T\}.$$

By (1) of Lemma 2.5, $\mathrm{P.Ext}(K, K^{-1})_-$ forms a linear subspace of $\mathrm{P.Ext}(K, K^{-1})$ of codimension one. By (2) of Lemma 2.4, and the assumption, $\{(W_{[P]})_T, (W_{[Q]})_T\}\not\subset\mathrm{P.Ext}(K, K^{-1})_-$. Thus by (1) of Lemma 2.4, and by Bezout's theorem, $\mathrm{P.}S(K, K^{-1}; L_\theta)_-=\mathrm{P.Ext}(K, K^{-1})_-\cap\mathrm{P.}S(K, K^{-1}; L_\theta)$ consists exactly of one element. Thus the assertion (1) follows from (2) of Lemma 2.2, and Proposition 2.7. Similarly, (2) can be proved.

Similarly to Lemma 2.4, we obtain the following (see also Lemma 5.8 in Narasimhan-Ramanan [**12**]).

PROPOSITION 2.9. (1) *Fix a Weierstrass line bundle L. Let $\theta \in J$, $\theta \neq \theta_0$. Put $L_\theta = K \otimes L_P^{-1} \otimes L_Q^{-1}$. Then* $\mathrm{P}.S(L, L^{-1}; L_\theta)$ *is empty or consists exactly of one element according as $L_P \neq L$, $L_Q \neq L$ or one of L_P, L_Q coincides with L. Moreover in the latter case, the element of* $\mathrm{P}.S(L, L^{-1}; L_\theta)$ *represents a vector bundle $L_\theta \oplus L_\theta^{-1}$ if $\theta \notin J(2)$.*
(2) *For any Weierstrass line bundle L,* $\mathrm{P}.S(L, L^{-1}; L_{\theta_0})$ *consists exactly of one element.*

PROPOSITION 2.10. (1) *For any $\theta \in J$, $\theta \notin J(2)$, there exists exactly one element $(W)_T$ of P^* which has a subline bundle L_θ. Moreover $W \cong L_\theta \oplus L_\theta^{-1}$.*
(2) *Let $L_\theta = K \otimes L_P^{-1} \otimes L_Q^{-1}$, with $\theta \in J(2)$, $\theta \neq \theta_0$. Then* $\mathrm{P}.S(K, K^{-1}; L_\theta)_- = C_\theta$, *and*

$$\mathrm{P}.S(K, K^{-1}; L_\theta)_- = \mathrm{P}.S'(K, K^{-1}; L_\theta)_- \cup \{(W_{[Q]})_T, (W_{[P]})_T\}.$$

(3) $\mathrm{P}.S(K, K^{-1}; L_{\theta_0})_- = C_{\theta_0}$, *and*

$$\mathrm{P}.S(K, K^{-1}; L_{\theta_0})_- = \mathrm{P}.S'(K, K^{-1}; L_{\theta_0})_- \cup \{(W_{[Q]})_T ; Q \in \mathcal{W}\}.$$

PROOF. (1) follows from (1) of Proposition 2.8 and (1) of Proposition 2.9, and (2) follows from (2) of Lemma 2.6, and (2) of Proposition 2.8. Moreover by (2) of Lemma 2.6, to prove (3), it suffices to prove $\mathrm{P}.S(K, K^{-1}; L_{\theta_0})_- \subset C_{\theta_0}$. Let $w \in \mathrm{P}.S(K, K^{-1}; L_{\theta_0})_-$. Then there exists a non-zero element x of $H^0(C, L_{\theta_0}^* \otimes K)$ such that $\theta(x, w) = 0$. We have $x = x_Q$ for some $Q \in C$. Thus the assertion follows from Corollary to Lemma 2.6.

§ 2.4. **Representation of stable vector bundles.** Finally we consider the representation of stable vector bundles in P^* (cf. Proposition 2.16 and its corollary). Let V be a stable vector bundle of rank two with trivial determinant. Put

$$S_V^* = \{(W, i, p)_U \in \mathrm{Ext}(K, K^{-1}) \ ; \ H^0(C, \mathrm{Hom}(W, V)) \neq 0\},$$

$$G_V = \{((W, i, p)_T, \langle f \rangle) \in \mathrm{P}.S_V^* \times \mathrm{P}.H^0(C, \mathrm{Hom}(K^{-1}, V)) \ ; \text{ there is a non-zero homomorphism } g: W \to V \text{ such that } gi = f\}$$

and

$$G_V' = \{(\langle w \rangle, \langle f \rangle) \in \mathrm{P}.H^1(C, \mathrm{Hom}(K, K^{-1})) \times \mathrm{P}.H^0(C, \mathrm{Hom}(K^{-1}, V)) ; \Theta(w, f) = 0\}.$$

Let

$$\nabla : \mathrm{P}.\mathrm{Ext}(K, K^{-1}) \times \mathrm{P}.H^0(C, \mathrm{Hom}(K^{-1}, V)) \to \mathrm{P}.H^1(C, \mathrm{Hom}(K, K^{-1})) \times \mathrm{P}.H^0(C, \mathrm{Hom}(K^{-1}, V))$$

be the natural bijection. Then by Lemma 3.2 in Narasimhan-Ramanan [**12**], we have

$$(2.11) \qquad \nabla(G_V)=G'_V.$$

Let $\{\omega_i\}_{i=1}^5$, $\{u_j\}_{j=1}^m$, $\{\xi_k\}_{k=1}^n$ be bases for $H^1(C, \mathrm{Hom}(K, K^{-1}))$, $H^0(C, \mathrm{Hom}(K^{-1}, V))$, $H^1(C, \mathrm{Hom}(K, V))$, respectively. Then for any $1\leqq i\leqq 5$, and $1\leqq j\leqq m$, we have

$$\Theta(\omega_i, u_j)=\sum_{k=1}^n a_{ijk}\xi_k$$

with $a_{ijk}\in k$. Then by (2.11), G_V forms an algebraic subset of $\mathrm{P.Ext}(K, K^{-1})\times \mathrm{P.}H^0(C, \mathrm{Hom}(K^{-1}, V))$, and is isomorphic to

$$\left\{((X_1, \cdots, X_5), (Y_1, \cdots, Y_m))\in P^4\times P^{m-1}\ ;\ \sum_{i=1}^5\sum_{j=1}^m a_{ijk}X_iY_j=0\ (1\leqq k\leqq n)\right\}.$$

PROPOSITION 2.11. *The projections*

$$\mathrm{pr}_1 : G_V\to \mathrm{P.}S_V^* \quad \text{and} \quad \mathrm{pr}_2 : G_V\to \mathrm{P.}H^0(C, \mathrm{Hom}(K^{-1}, V))$$

are biregular morphisms.

To prove this we need several lemmas.

LEMMA 2.12. (1) *Let $0\to E\xrightarrow{i} W\to F\to 0$ be an extension of a vector bundle F by E. Let V be a vector bundle. Then $gi\neq 0$ for any non-zero element g of $H^0(C, \mathrm{Hom}(W, V))$ if and only if $H^0(C, \mathrm{Hom}(F, V))=0$.*
(2) *Let L be a line bundle. Let V be a vector bundle such that $H^0(C, \mathrm{Hom}(L, V))=0$, and let $0\to L\to W\xrightarrow{g_0} L^{-1}\to 0$ be an extension. Then the homomorphism $\alpha : H^0(C, \mathrm{Hom}(L^{-1}, V))\to H^0(C, \mathrm{Hom}(W, V))$ defined by $f\mapsto fg_0$ is an isomorphism.*

PROOF. The two assertions follow directly from the long exact sequences of vector bundles.

LEMMA 2.13. *Let V be a stable vector bundle of rank two with trivial determinant. Then we have $\dim H^0(C, \mathrm{Hom}(W, V))\leqq 1$ for any element $(W)_T$ of $\mathrm{P.Ext}(K, K^{-1})$.*

PROOF. Assume that $\dim H^0(C, \mathrm{Hom}(W, V))>1$. Then by Proposition 4.3 and its corollary in Narasimhan-Seshadri [**13**], W is non-semistable. Thus by (2) of Lemma 2.4, and (2) of Lemma 2.12, $\dim H^0(C, \mathrm{Hom}(W, V))=\dim H^0(C, \mathrm{Hom}(K^{-1}\otimes L_Q, V))$ for some $Q\in C$. Thus by Lemma 5.4 in Narasimhan-Ramanan [**12**], V is not stable, which is a contradiction.

By this lemma, we easily obtain the following.

COROLLARY. *Let V be as above. Assume that $(W_1, i_1, p_1)_T=(W_2, i_2, p_2)_T$ in*

P. Ext(K, K^{-1}). *Then for any element h_i of $H^0(C, \mathrm{Hom}(W_i, V))$ ($i=1, 2$), we have*

$$h_1 i_1 = a h_2 i_2$$

with some $a \in k$.

Now for a while, we assume that V is a stable vector bundle of rank two with trivial determinant.

LEMMA 2.14. *For any non-zero element f of $H^0(C, \mathrm{Hom}(K^{-1}, V))$, there is a non-trivial extension $0 \to K^{-1} \xrightarrow{i} W \to K \to 0$, and a non-zero element g of $H^0(C, \mathrm{Hom}(W, V))$ such that $gi=f$.*

PROOF. We divide the proof into two cases.
Case 1. Assume that $f(Q) \neq 0$ for any $Q \in C$, where $f(Q)$ denotes the image of f under the natural homomorphism from $H^0(C, \mathrm{Hom}(K^{-1}, V))$ to the fibre of $\mathrm{Hom}(K^{-1}, V)$ at Q. Then the homomorphism $f: K^{-1} \to V$ is injective at any fibre. Thus there is a non-trivial extension $0 \to K^{-1} \xrightarrow{f} V \to K \to 0$. Hence this extension and the identity homomorphism $\mathrm{id}_V; V \to V$ satisfy the required condition.
Case 2. Assume that $f(Q)=0$ for some $Q \in C$. Let $(W_{[Q]}, i, p)_T$ be the element of P. Ext(K, K^{-1}) in (2) of Lemma 2.4, and $g_0: W_{[Q]} \to K^{-1} \otimes L_Q$ be the homomorphism in (2) of Lemma 2.12. Then by (1) of Lemma 2.12, the map $c = g_0 i$ is non-zero, and it is induced by the canonical section of L_Q. Hence, by Lemma 5.3 in Narasimhan-Ramanan [**12**], f admits a factorization into c, followed by a homomorphism $h: K^{-1} \otimes L_Q \to V$. By (1) of Lemma 2.12, the map $h \circ g_0: W_{[Q]} \to V$ is non-zero, and satisfies the required condition.

LEMMA 2.15. *Let $(W_j, i_j, p_j)_T \in$ P. Ext(K, K^{-1}) for $j=1, 2$, and let h_j be a non-zero element of $H^0(C, \mathrm{Hom}(W_j, V))$. Then, if $\langle h_1 i_1 \rangle = \langle h_2 i_2 \rangle$ in* P. $H^0(C, \mathrm{Hom}(K^{-1}, V))$, $(W_1, i_1, p_1)_T = (W_2, i_2, p_2)_T$.

PROOF. We also divide the proof into two cases.
Case 1. Assume that $h_1 \circ i_1(Q) \neq 0$ for any $Q \in C$. Then W_1 is stable. In fact, if not, by Proposition 4.3 in Narasimhan-Seshadri [**13**], and by (2) of Lemma 2.4, we have $(W_1)_T = (W_{[Q]})_T$ for some Q. Then by (2) of Lemma 2.12, for the homomorphism $g_0: W_{[Q]} \to K^{-1} \otimes L_Q$, there is a homomorphism $h: K^{-1} \otimes L_Q \to V$ such that $h_1 = h \circ g_0$. Hence the map $h_1 \circ i_1$ admits a factorization into $g_0 \circ i_1$, followed by h. This contradicts Lemma 5.3 in Narasimhan-Ramanan [**12**]. In the same manner, W_2 is stable. Hence h_1 and h_2 are isomorphisms. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{-1} & \xrightarrow{i_1} & W_1 & \xrightarrow{p_1} & K & \longrightarrow & 0 \\ & & {\scriptstyle u}\downarrow & & {\scriptstyle h}\downarrow & & & & \\ 0 & \longrightarrow & K^{-1} & \xrightarrow{i_2} & W_2 & \xrightarrow{p_2} & K & \longrightarrow & 0, \end{array}$$

where $u \neq 0$, and h is an isomorphism. From this, the assertion can be easily proved in this case.

Case 2. Assume that $h_1 \circ i_1(Q)=0$ for some $Q \in C$. Then by Lemma 5.3 in Narasimhan-Ramanan [12], we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{-1} & \xrightarrow{i_1} & W_1 & \xrightarrow{p_1} & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow {\scriptstyle h_1} & & & & \\ 0 & \longrightarrow & K^{-1}\otimes L_Q & \xrightarrow{i'} & V & \xrightarrow{p'} & K\otimes L_Q^{-1} & \longrightarrow & 0 . \end{array}$$

Thus by (1) of Lemma 2.12, we have $p' \circ h_1=0$. Thus it follows from the exact sequence

$$0 \to H^0(C, \mathrm{Hom}(W_1, K^{-1}\otimes L_Q)) \to H^0(C, \mathrm{Hom}(W_1, V)) \to H^0(C, \mathrm{Hom}(W_1, K\otimes L_Q^{-1}))$$

that we have $H^0(C, \mathrm{Hom}(W_1, K^{-1}\otimes L_Q)) \neq 0$. Since W_1 has a trivial determinant, the dual W_1^* of W_1 is isomorphic to W_1. Hence we have $H^0(C, \mathrm{Hom}(K\otimes L_Q^{-1}, W_1)) \neq 0$. In the same manner, we have $H^0(C, \mathrm{Hom}(K\otimes L_Q^{-1}, W_2)) \neq 0$. Thus the assertion follows from (2) of Lemma 2.4.

PROOF OF PROPOSITION 2.11. The assertion follows from (2.11), Corollary to Lemma 2.13, and Lemmas 2.14 and 2.15.

Now the main result in this subsection is the following.

PROPOSITION 2.16. (1) *For any stable vector bundle V of rank two with trivial determinant, there are at most two Weierstrass line bundles L_1, L_2 such that V can be represented in* $\mathrm{P.Ext}(L_i, L_i^{-1})$ $(i=1, 2)$. *Moreover, if we fix L_i, the representation of V in* $\mathrm{P.Ext}(L_i, L_i^{-1})$ *is unique.*
(2) *For any stable vector bundle V of rank two with trivial determinant, there are exactly two elements $(V_1)_T$, $(V_2)_T$ of* $\mathrm{P.Ext}(K, K^{-1})_-$ *such that* $H^0(C, \mathrm{Hom}(V_i, V)) \neq 0$ $(i=1, 2)$.

By Propositions 2.7 and 2.16, and (1) of Lemma 2.12, we have

COROLLARY. *Any stable vector bundle of rank two with trivial determinant can be represented in P^* exactly in two ways.*

To prove the proposition, we need the following lemma.

LEMMA 2.17. (1) *Assume that an element $(W_0)_T$ of P^* represents a stable vector bundle. Then there are exactly two elements $(V_1)_T$, $(V_2)_T$ of* $\mathrm{P.Ext}(K, K^{-1})_-$ *such that* $H^0(C, \mathrm{Hom}(V_i, W_0)) \neq 0$ *for* $i=1, 2$.
(2) *Let V be as above. For each line bundle L of degree one, put* $T_{V,L}^* = \{(W)_T \in \mathrm{P.Ext}(L, L^{-1});\ H^0(C, \mathrm{Hom}(W, V)) \neq 0\}$. *Then we have* $\# T_{V,L}^* \leq 1$.

PROOF. Similarly to Proposition 2.11, we have a biregular morphism from $P.H^0(C, \mathrm{Hom}(L^{-1}, V))$ to $T^*_{V,L}$. Thus (2) is proved. Now we prove (1). As seen in § 2.2, for any element $(W_0)_T$ of $P.\mathrm{Ext}(K, K^{-1})_-$, we have

$$P.S^*_{W_0} \cap P.\mathrm{Ext}(K, K^{-1})_- = P.S^{*+}_{W_0} \cup P.S^{*-}_{W_0},$$

where

$$P.S^{*i}_{W_0} = \{(W)_T \in P.\mathrm{Ext}(K, K^{-1})_- ; H^0(C, \mathrm{Hom}(W, W_0))_i \neq 0\}$$

for $i=+, -$. Moreover, by Lemma 2.13, $P.S^{*+}_{W_0} \cap P.S^{*-}_{W_0} = \emptyset$. Now fix $i=+$ or $-$. Then by Propositions 2.3 and 2.11, we have

$$P.S^{*i}_{W_0} \subset \mathrm{pr}_1 \circ \mathrm{pr}_2^{-1}(P.H^0(C, \mathrm{Hom}(K^{-1}, W_0))_i).$$

Conversely, if an element $(W)_T$ of $P.\mathrm{Ext}(K, K^{-1})$ belongs to

$$\mathrm{pr}_1 \circ \mathrm{pr}_2^{-}(P.H^0(C, \mathrm{Hom}(K^{-1}, W_0)_i),$$

by (2.11), we have $\Theta(\delta((W)_U), f) = 0$ for some non-zero element f of $H^0(C, \mathrm{Hom}(K^{-1}, W_0))_i$. Thus by (1) of Lemma 2.2, and (1) of Lemma 2.6, $(W)_T \in P.\mathrm{Ext}(K, K^{-1})_-$. Hence, by Proposition 2.3, $(W)_T \in P.S^{*i}_{W_0}$.

Now by Proposition 2.3, we have

$$H^0(C, \mathrm{Hom}(K^{-1}, W_0))_i / H^0(C, \mathrm{Hom}(K^{-1}, K^{-1}))_{j(i)} \cong \ker \Theta_{K^{-1}}(\ \ , \delta((W_0)_U))_i.$$

Clearly,

$$\dim H^0(C, \mathrm{Hom}(K^{-1}, K^{-1})) = \dim H^0(C, \mathrm{Hom}(K^{-1}, K^{-1}))_+ = 1,$$

and

$$\dim \ker \Theta_{K^{-1}}(\ \ , \delta((W_0)_U)) = \dim \ker \Theta_{K^{-1}}(\ \ , \delta((W_0)_U))_+ \geqq 1$$

by the Riemann-Roch theorem. Hence, we have $\dim H^0(C, \mathrm{Hom}(K^{-1}, W_0))_- = 1$, and $\dim H^0(C, \mathrm{Hom}(K^{-1}, W_0))_+ \geqq 1$. On the other hand, since W_0 is stable,

$$\dim H^0(C, \mathrm{Hom}(K^{-1}, W_0)) = 2 \tag{2.12}$$

by the Riemann-Roch theorem and Serre's duality. This completes the proof.

PROOF OF PROPOSITION 2.16. (1) Let $\{f_1, f_2\}$ be a basis for $H^0(C, \mathrm{Hom}(K^{-1}, V))$ (cf. (2.12)). Then the exterior product $f_1 \wedge f_2$ belongs to $H^0(C, K^2)$. Hence it vanishes at most at two Weierstrass points. Hence at most for two Weierstrass points P, two elements $f_1(P), f_2(P)$ of the fibre of $\mathrm{Hom}(K^{-1}, V)$ at P are linearly dependent over k. Thus (1) follows from Lemma 5.3 in Narasimhan-Ramanan [12], and Lemma 2.17.

(2) Assume that V has a dual Weierstrass subline bundle L_Q^{-1}. Then the assertion follows from Lemma 2.17. Assume that V has no dual Weierstrass subline bundle. Then by Proposition 2.11, and (2.12), $P.S^*_V$ is an algebraic curve in $P.\mathrm{Ext}(K, K^{-1})$. Hence, $P.S^*_V \cap P.\mathrm{Ext}(K, K^{-1})_-$ is non-empty. Take an

element $(W_0)_T$ of $\mathrm{P}.S_V^* \cap \mathrm{P}.\mathrm{Ext}(K, K^{-1})_-$. Then by Proposition 2.7, and (2) of Lemma 2.12, W_0 is isomorphic to V. Thus the assertion (2) follows from Lemma 2.17.

PROOF OF THEOREM 2.1. The assertion follows from Propositions 2.7 and 2.10, and Corollary to Proposition 2.16.

§ 3. Classification of FM bundles.

Throughout § 3 and § 4, let p be a prime different from 2.

Now put $\mathcal{M}=\{[V] ; V \text{ is represented in } P^*\}$. In this section we investigate some subset of $\mathcal{M}$ which is related to the representation of the fundamental group $\pi_1(C)$ in $GL_2(k)$. A vector bundle V is called an FM bundle if there is a non-zero homomorphism from F^*V to V, where F^*V is the pull back of V by the p-th power absolute Frobenius map on C. The set of isomorphism classes of FM bundles which belong to $\mathcal{M}$ is denoted by $\mathcal{FM}$. For each representation ρ of $\pi_1(C)$ in $GL_2(k)$, let $[\rho]$ denote the $GL_2(k)$ equivalence class of ρ. The next proposition is our main tool

PROPOSITION 3.1 (Lange and Stuhler [**10**], see also [**8**]). (1) *Put*

$$S_2^F = \{[V] \; ; \; V \textit{ is a vector bundle of rank two with trivial determinant, and } F^*V \cong V\}$$

and

$$H(\pi_1(C), SL_2(F_p)) = \{[\rho] ; \rho(\pi_1(C)) \subset SL_2(F_p)\} .$$

Then all elements of S_2^F are semistable, and there is a bijection

$$\Phi : S_2^F \to H(\pi_1(C), SL_2(F_p)).$$

(2) *Put*

$$\mathrm{Irr}(\pi_1(C), SL_2(F_p)) = \{[\rho] \in H(\pi_1(C), SL_2(F_p)) ; \rho \textit{ irreducible}\} ,$$

and

$$\mathcal{FM}_{s1} = \{[V] \in S_2^F ; V \textit{ is stable}\} .$$

Then we have $\Phi(\mathcal{FM}_{s1}) = \mathrm{Irr}(\pi_1(C), SL_2(F_p))$.

Now we shall classify FM bundles. Let κ be the canonical bundle. Put

$$\mathcal{FM}_{s2} = \{[V] \in \mathcal{M} \; ; \; V \text{ is a stable vector bundle which is represented in } \mathrm{P}.\mathrm{Ext}(L, L^{-1}) \text{ for some line bundle } L \text{ such that } L \otimes L \cong \kappa, \text{ and } F^*V \text{ is represented in } \mathrm{Ext}(L^{-1}, L)\},$$

$$\mathcal{FM}_{ss1} = \{[L_\theta \oplus L_\theta^{-1}] \; ; \; \theta \in J(p+1) \cup J(p-1), \text{ and } \theta \notin J(2)\} ,$$

$$\mathcal{FM}_{ss2} = \{[V] \in \mathcal{M} \; ; \; V \text{ is represented in } \mathrm{Ext}(L_\theta, L_\theta^{-1}) \text{ for some } \theta \in J(2)\} ,$$

and

$$\mathcal{FM}_{ns}=\{[W_{[Q]}]\ ;\ Q\in\mathcal{W}\}.$$

Then,

THEOREM 3.2. *The set $\mathcal{FM}$ is decomposed as*

$$\mathcal{FM}=\mathcal{FM}_{s1}\cup\mathcal{FM}_{s2}\cup\mathcal{FM}_{ss1}\cup\mathcal{FM}_{ss2}\cup\mathcal{FM}_{ns}.$$

PROOF. Assume that V is stable, and belongs to $\mathcal{FM}$. If F^*V is semistable, we have $F^*V\cong V$. If F^*V is not semistable, by Korollar 2.6 in [**10**], F^*V has a subline bundle L such that $L\otimes L\cong\kappa$. They by (2) of Lemma 2.12, V has a subline bundle L^{-1}. Conversely if $[V]$ belongs to $\mathcal{FM}_{s2}$, again by (2) of Lemma 2.12, $[V]\in\mathcal{FM}$. If we have $V\cong L_\theta\oplus L_\theta^{-1}$ with $\theta\in J$, we have $F^*V\cong L_\theta^p\oplus L_\theta^{-p}$. Therefore we have $F^*V\cong V$ if and only if $L_\theta^p\cong L_\theta$ or $L_\theta^p\cong L_\theta^{-1}$. Let $0\to L_\theta^{-1}\to V\to L_\theta\to 0$ be an extension with $\theta\in J(2)$. Then clearly we have $H^0(C,\mathrm{Hom}(F^*V,L_\theta^p))\neq 0$. Since we have $L_\theta^p\cong L_\theta^{-1}$, we have $H^0(C,\mathrm{Hom}(F^*V,L_\theta^{-1}))\neq 0$. Thus by (2.5), $[V]$ belongs to $\mathcal{FM}$. Similarly, we have $\mathcal{FM}_{ns}\subset\mathcal{FM}$. This proves the assertion.

Now for each σ-invariant line bundle L, put

$$\mathcal{FM}(L)=\{(V)_T\in\mathrm{P.Ext}(L,L^{-1})_-\ ;\ [V]\in\mathcal{FM}\}.$$

Then $\mathcal{FM}(L)$ is decomposed as $\mathcal{FM}(L)=\mathcal{FM}(L,+)\cup\mathcal{FM}(L,-)$ where

$$\mathcal{FM}(L,i)=\{(V)_T\in\mathcal{FM}(L);H^0(C,\mathrm{Hom}(F^*V,V))_i\neq 0\}.$$

In the rest of this section, we shall consider the finiteness property of $\mathcal{FM}(L)$ when $L=K$ or L_Q with $Q\in\mathcal{W}$. For that purpose, let $L=L_Q$ and, $g:H^1(C,L^{-3})_-\to H^1(C,L^{-3p})_-$ be the p-linear homomorphism induced by the p-th power absolute Frobenius map on C. Let $\Theta':H^1(C,L^{-3p})_-\times H^0(C,L^{p-1})_i\to H^1(C,L^{-2p-1})_{j(i)}$ be the cup product, and $\rho:H^1(C,L^{-2p-1})_{j(i)}\to H^1(C,L^{-p-3})_{j(i)}$ be the surjective homomorphism induced by the exact sequence $0\to L^{-2p-1}\to L^{-p-3}\to(\mathcal{O}_C/\mathcal{J}_Q^{p-2})\otimes L^{-p-3}\to 0$, where $\mathcal{J}_Q$ is the ideal sheaf of Q. We note that there is an isomorphism τ from $H^1(C,L^{-2})$ to $H^1(C,L^{-3})_-$. Then we define a bilinear mapping $\Psi:H^1(C,L^{-2})\times H^0(C,L^{p-1})_i\to H^1(C,L^{-p-3})_{j(i)}$ by $\Psi=\rho\circ\Theta'\circ(g\circ\tau\times\mathrm{id})$. Then we have

LEMMA 3.3. *Let L be as above. Let $\Theta:H^1(C,L^{-2p})_-\times H^0(C,L^{p-1})_i\to H^1(C,L^{-p-1})_{j(i)}$ be the cup product, $f:H^1(C,L^{-2})\to H^1(C,L^{-2p})_-$ be the p-linear map induced by the p-th power absolute Frobenius map on C, and $h:H^1(C,L^{-p-3})_{j(i)}\to H^1(C,L^{-p-1})_{j(i)}$ be the homomorphism induced by the exact sequence $0\to L^{-p-3}\to L^{-p-1}\to(\mathcal{O}_C/\mathcal{J}_Q^2)\otimes L^{-p-1}\to 0$. Then*

(3.1) $\Theta\circ(f\times\mathrm{id})=h\circ\Psi$.

(3.2) $\Psi(w,\ ):H^0(C,L^{p-1})_i\to H^1(C,L^{-p-3})_{j(i)}$ *is injective for any non-zero $w\in H^1(C,L^{-2})$.*

PROOF. Clearly Ψ satisfies (3.1). To prove (3.2), let $\{w^{\alpha\beta}\}$ be an element of $Z^1(\mathcal{V}, L^{-2})$, and assume that for some non-zero u of $H^0(C, L^{p-1})_l$, $\{(w^{\alpha\beta})^p u\}$ belongs to the 1-coboundary group $B^1(\mathcal{V}, L^{-p-3})$. Then there is an element $\{a^\alpha\}$ of the 0-cochain group $C^0(\mathcal{V}, L^{-p-3})$ such that $(w^{\alpha\beta})^p u = a^\alpha - a^\beta$ in $U_\alpha \cap U_\beta$. Put $v^\alpha = u^2 d(u^{-1}a^\alpha)$, where d denotes the derivation. Then $\{v^\alpha\}$ defines an element of $H^0(C, \kappa \otimes L^{-3})$, where κ denotes the canonical line bundle on C. Since $H^0(C, \kappa\otimes L^{-3})=0$, there is a function c^α on U_α such that $u^{-1}a^\alpha = (c^\alpha)^p$. Estimating the orders of u^{-1} and a^α at any point of U_α, we see that c^α belongs to $\Gamma(U_\alpha, L^{-2})$. This implies that $\{w^{\alpha\beta}\}$ belongs to $B^1(\mathcal{V}, L^{-2})$. This proves the assertion.

The following lemma can be easily proved by (1.2), and Bezout's theorem. Hence we omit the proof.

LEMMA 3.4. *Let $m \geqq n$. Let F be an (m, n) matrix. Assume that any non-zero component of F is a homogeneous polynomial in $X_1, \cdots, X_r$ of degree p, and that the algebraic subset $V_{F(m;\ )}$ of* Proj $k[X_1, \cdots, X_r]$ *associate with the matrix $F(m;\ )$ is at least of dimension one. Then V_F is non-empty.*

Now we have

PROPOSITION 3.5. *$\mathcal{FM}_{s2}$ is a finite set.*

PROOF. For any line bundle L such that $L\otimes L \cong \kappa$, and for $l=+, -$, put

$$\mathcal{N}(L, l) = \{\langle w\rangle \in \mathrm{P}.H^1(C, L^{-2})\ ;\ \Theta(f(w), u)=0 \text{ for some non-zero element } u\in H^0(C, L^{p-1})_l\}.$$

Now for any two line bundles L_1, L_2 such that $L_1\otimes L_1\cong\kappa$, $L_2\otimes L_2\cong\kappa$, there is an element θ of $J(2)$ such that $L_2 \cong L_\theta \otimes L_1$. Clearly, this isomorphism induces a bijection from $\mathcal{N}(L_1, l)$ to $\mathcal{N}(L_2, l)$. Thus by Theorems 2.1 and 3.2, and Proposition 2.3, it suffices to show that for a Weierstrass line bundle L, the set $\mathcal{N}(L, l)$ is finite. To show this, let $\{w_i\}_{i=1}^3$, $\{u_i\}_{i=1}^n$ be bases for $H^1(C, L^{-2})$, $H^0(C, L^{p-1})_l$, respectively. By a direct calculation we have $\dim H^1(C, L^{-p-3})_{j(l)} - \dim H^1(C, L^{-p-1})_{j(l)} = 1$. Hence we can take a basis $\{\xi_i\}_{i=1}^m$ for $H^1(C, L^{-p-3})_{j(l)}$ such that $\{h(\xi_i)\}_{i=1}^{m-1}$ forms a basis for $H^1(C, L^{-p-1})_{j(l)}$, and $h(\xi_m)=0$. Then we have

$$\Psi(w_k, u_j) = \sum_{i=1}^m a_{ijk}\xi_i$$

with $a_{ijk}\in k$. We identify the projective space $\mathrm{P.Ext}(L, L^{-1})$ with $\mathrm{Proj}\, k[X_1, X_2, X_3]$. We also identify $H^1(C, L^{-2})$ with $H^1(C, \mathrm{Hom}(L, L^{-1}))$. Now define a matrix $A=(A_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ with entries in $k[X_1, X_2, X_3]$ by $A_{ij} = \sum_{k=1}^3 a_{ijk} X_k^p$.

Then by construction, and by (3.1), we have

$$\varepsilon(V_A)=\{\langle w\rangle\in \mathrm{P}.H^1(C,L^{-2})\ ;\ \Psi(w,u)=0$$
$$\text{for some non-zero element } u \text{ of } H^0(C,L^{p-1})_l\}$$

and, $\varepsilon(V_{A(m;\,)})=\mathfrak{N}(L,l)$. Assume that $\mathfrak{N}(L,l)$ is not a finite set. Then, by Lemma 3.4, V_A is non-empty. This contradicts Lemma 3.3. q. e. d.

Now by Theorems 2.1 and 3.2, and Proposition 3.5, we have

THEOREM 3.6. (1) *For any $Q\in\mathcal{W}$, $\mathfrak{FM}(L_Q)$ is a finite set.*
(2) *$\mathfrak{FM}(K)\supset\bigcup_{\theta\in J(2)}C_\theta$, and $\mathfrak{FM}(K)\setminus\bigcup_{\theta\in J(2)}C_\theta$ is a finite set.*

§ 4. Construction of FM matrices.

In this section, we shall construct certain matrices called FM matrices, which are related to the representation of FM bundles. Let L_1, L_2 be σ-invariant line bundles on C, and let $(W)_U\in\mathrm{Ext}(L_1,L_1^{-1})_-$ and $(V)_U\in\mathrm{Ext}(L_2,L_2^{-1})_-$. Put $\omega=\delta((W)_U)$ and $\upsilon=\delta((V)_U)$. Now assume that $\deg L_1-3\geqq\deg L_2>0$. Then we have a commutative diagram of exact sequences

$$\text{(4.1)}\qquad\begin{array}{ccccccc}
 & & 0 & & & & 0\\
 & & \downarrow & & & & \downarrow\\
0\to & & H^0(C,\mathrm{Hom}(W,L_2^{-1}))_{j(i)} & \longrightarrow & H^0(C,\mathrm{Hom}(W,V))_i & \longrightarrow & H^0(C,\mathrm{Hom}(W,L_2))_i\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q_{13\omega}}\\
0\to & & H^0(C,\mathrm{Hom}(L_1^{-1},L_2^{-1}))_{j(i)} & \xrightarrow{p_{21\upsilon}} & H^0(C,\mathrm{Hom}(L_1^{-1},V))_i & \xrightarrow{p_{22\upsilon}} & H^0(C,\mathrm{Hom}(L_1^{-1},L_2))_i\to 0\\
 & {\scriptstyle q_{21\omega}} & \downarrow & & \downarrow{\scriptstyle q_{22\upsilon\omega}} & & \downarrow{\scriptstyle q_{23\omega}}\\
0\to & & H^1(C,\mathrm{Hom}(L_1,L_2^{-1}))_i & \xrightarrow{p_{31\upsilon}} & H^1(C,\mathrm{Hom}(L_1,V))_{j(i)} & \xrightarrow{p_{32\upsilon}} & H^1(C,\mathrm{Hom}(L_1,L_2))_{j(i)}\to 0\\
 & {\scriptstyle q_{31\omega}} & \downarrow & & & & \\
 & & H^1(C,\mathrm{Hom}(W,L_2^{-1}))_{j(i)} & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

where $q_{21\omega}$, $q_{22\upsilon\omega}$ and $q_{23\omega}$ are the connecting homomorphisms. For each $(V)_U\in\mathrm{Ext}(L_2,L_2^{-1})_-$, let $\Theta_V:H^1(C,\mathrm{Hom}(L_1,L_1^{-1}))\times H^0(C,\mathrm{Hom}(L_1^{-1},V))\to H^1(C,\mathrm{Hom}(L_1,V))$ be the bilinear map in § 2.1. Moreover take bilinear maps $\Theta':H^1(C,\mathrm{Hom}(L_1,L_1^{-1}))_-\times H^0(C,\mathrm{Hom}(L_1^{-1},L_2))_i\to H^1(C,\mathrm{Hom}(L_1,L_2))_{j(i)}$, and $\Theta'':H^1(C,\mathrm{Hom}(L_1,L_1^{-1}))_-\times H^0(C,\mathrm{Hom}(L_1^{-1},L_2^{-1}))_{j(i)}\to H^1(C,\mathrm{Hom}(L_1,L_2^{-1}))_i$ in § 2.1. We often write $\Theta_\upsilon=\Theta_V$ if $\upsilon=\delta((V)_U)$. Then by (2.3) and the diagram (4.1), for any $\upsilon\in H^1(C,\mathrm{Hom}(L_2,L_2^{-1}))_-$, we can choose a homomorphism $r_{23\upsilon}:H^0(C,\mathrm{Hom}(L_1^{-1},L_2))_i\to H^0(C,\mathrm{Hom}(L_1^{-1},V))_i$ (resp. $r_{32\upsilon}:H^1(C,\mathrm{Hom}(L_1,V))_{j(i)}\to H^1(C,\mathrm{Hom}(L_1,L_2^{-1}))_i$) such that $p_{22\upsilon}\circ r_{23\upsilon}=\mathrm{id}$ (resp. $r_{32\upsilon}\circ p_{31\upsilon}=\mathrm{id}$) and $p_{32\upsilon}\circ\Theta_\upsilon\circ(\mathrm{id}\times r_{23\upsilon})=\Theta'$ (resp. $r_{32\upsilon}\circ\Theta_\upsilon\circ(\mathrm{id}\times p_{21\upsilon})=\Theta''$). Moreover fixing $\omega=\delta((W)_U)\in H^1(C,\mathrm{Hom}(L_1,L_1^{-1}))_-$ and $\upsilon=\delta((V)_U)\in H^1(C,\mathrm{Hom}(L_2,L_2^{-1}))_-$ and define a homomorphism $\Phi_{\omega\upsilon}$ from $H^0(C,\mathrm{Hom}(W,L_2))_i$ to $H^1(C,\mathrm{Hom}(W,L_2^{-1}))_{j(i)}$ by the snake lemma. Then we have

(4.2) $$H^0(C, \mathrm{Hom}(W, V))_i / H^0(C, \mathrm{Hom}(W, L_2^{-1}))_{j(i)} \cong \ker \Phi_{\omega\upsilon}.$$

Moreover by the definition of the addition in $\mathrm{Ext}(L_2, L_2^{-1})$ and by a diagram chase, for any $\upsilon_1, \upsilon_2 \in H^1(C, \mathrm{Hom}(L_2, L_2^{-1}))_-$ and $a_1, a_2 \in k$,

(4.3) $$\Phi_{\omega, a_1\upsilon_1 + a_2\upsilon_2} = a_1\Phi_{\omega, \upsilon_1} + a_2\Phi_{\omega, \upsilon_2}$$

(for the definition of the addition in $\mathrm{Ext}(L_2, L_2^{-1})$, see §1, Chapter VII in Mitchel [18]). Now fix a basis $\{\upsilon_l\}$ for $H^1(C, \mathrm{Hom}(L_2, L_2^{-1}))_-$. For each υ_l, define a bilinear map Θ'_{υ_l} from $H^1(C, \mathrm{Hom}(L_1, L_1^{-1}))_- \times H^0(C, \mathrm{Hom}(L_1^{-1}, L_2))_i$ to $H^1(C, \mathrm{Hom}(L_1, L_2^{-1}))_i$ by $\Theta'_{\upsilon_l} = r_{32\upsilon_l} \circ \Theta_{\upsilon_l} \circ (\mathrm{id} \times r_{23\upsilon_l})$. Moreover for each $\upsilon = \sum_l a_l \upsilon_l \in H^1(C, \mathrm{Hom}(L_2, L_2^{-1}))_-$ with $a_l \in k$, put $\Theta'_\upsilon = \sum_l a_l \Theta'_{\upsilon_l}$. The map Θ'_{υ_l} depends on choices of $r_{32\upsilon_l}$ and $r_{23\upsilon_l}$. However, the map $q_{31\omega} \circ \Theta'_\upsilon(\omega, \ ) \circ q_{13\omega}$ does not depend on choices of $r_{32\upsilon_l}$ and $r_{23\upsilon_l}$, and it coincides with $\Phi_{\omega\upsilon_l}$. Thus by (4.3), and by the definition, we have

(4.4) $$q_{31\omega} \circ \Theta'_\upsilon(\omega, q_{13\omega}(\gamma)) = \Phi_{\omega\upsilon}(\gamma)$$

for any $\upsilon \in H^1(C, \mathrm{Hom}(L_2, L_2^{-1}))_-$, $\omega \in H^1(C, \mathrm{Hom}(L_1, L_1^{-1}))_-$, and $\gamma \in H^0(C, \mathrm{Hom}(W, L_2))_i$.

Now let L be a σ-invariant line bundle such that $\deg L^p - 3 \geqq \deg L > 0$. Put $L_1 = L^p$, and $L_2 = L$, and let $\{\omega_k\}_{k=1}^{m_0}$, $\{\xi_i\}_{i=1}^{m_1}$, and $\{\eta_i\}_{i=1}^{m_2}$ be bases for $H^1(C, \mathrm{Hom}(L, L^{-1}))_-$, $H^1(C, \mathrm{Hom}(L^p, L))_{j(i)}$ and $H^1(C, \mathrm{Hom}(L^p, L^{-1}))_i$, respectively. Moreover let $\{u_j\}_{j=1}^{n_1}$, and $\{v_j\}_{j=1}^{n_2}$ be bases for $H^0(C, \mathrm{Hom}(L^{-p}, L))_i$, and $H^0(C, \mathrm{Hom}(L^{-p}, L^{-1}))_{j(i)}$ respectively. Let $f: H^1(C, \mathrm{Hom}(L, L^{-1})) \to H^1(C, \mathrm{Hom}(L^p, L^{-p}))$ be the p-linear map in §3. With respect to the basis $\{\omega_l\}$, fix a set $\{\Theta'_{\omega_l}\}$ of bilinear maps satisfying (4.4). Then for any $1 \leqq k$, $l \leqq m_0$, and $1 \leqq j \leqq n_1$, we have

$$\Theta'(f(\omega_k), u_j) = \sum_{i=1}^{m_1} a_{ijk}\xi_i, \quad \Theta'_{\omega_l}(f(\omega_k), u_j) = \sum_{i=1}^{m_2} b_{ijkl}\eta_i,$$

and for any $1 \leqq k \leqq m_0$, and $1 \leqq j \leqq n_2$, we have

$$\Theta''(f(\omega_k), v_j) = \sum_{i=1}^{m_2} d_{ijk}\eta_i,$$

where a_{ijk}, b_{ijkl}, and d_{ijk} are elements of the field k.

Now let $X_1, \cdots, X_{m_0}$ be variables over k, and define an (m_1+m_2, n_1+n_2)-matrix $F = \begin{pmatrix} (A_{ij}) & 0 \\ (B_{ij}) & (D_{ij}) \end{pmatrix}$ with entries in $k[X_1, \cdots, X_{m_0}]$ by

$$A_{ij} = \sum_{k=1}^{m_0} a_{ijk}X_k^p, \qquad B_{ij} = \sum_{1 \leq k, l \leq m_0} b_{ijkl}X_k^p X_l, \qquad \text{and} \qquad D_{ij} = \sum_{k=1}^{m_0} d_{ijk}X_k^p.$$

This matrix F will be called an FM matrix of the type (L, i).

Via the isomorphism s from the symmetric algebra $\mathcal{S}(H^1(C, \mathrm{Hom}(L, L^{-1}))_-)$ of $H^1(C, \mathrm{Hom}(L, L^{-1}))_-$ to $k[X_1, \cdots, X_{m_0}]$ such that $s(\omega_i)=X_i$, we identify $\mathrm{P.Ext}(L, L^{-1})_-$ with $\mathrm{Proj}\, k[X_1, \cdots, X_{m_0}]$. Then we have

THEOREM 4.1. $V_F=\mathfrak{F}\mathcal{M}(L, i)$.

PROOF. Recall that

$$H^0(C, \mathrm{Hom}(F^*W, L))_i \cong \ker \Theta'(f(\omega), \ )$$

and

$$H^0(C, \mathrm{Hom}(F^*W, L^{-1}))_{j(i)} \cong \ker \Theta''(f(\omega), \ )$$

for $\omega=\delta((W)_U)\in H^1(C, \mathrm{Hom}(L, L^{-1}))_-$ by (4.1). Thus the assertion follows immediately from (4.2) and (4.4).

REMARK 4.2. For the proof of our main theorem, we express Θ_υ in terms of Čech cocycles. Let $\{f^\lambda\}$ (resp. $\{e^\lambda\}$) be the set of local equations defining a line bundle L_1 (resp. L_2). Then $\{f^{\lambda\mu}\}=\{f^\lambda(f^\mu)^{-1}\}$ (resp. $\{e^{\lambda\mu}\}=\{e^\lambda(e^\mu)^{-1}\}$) is an element of $Z^1(\mathcal{U}, \mathcal{O}^*)$ defining L_1 (resp. L_2). Let $0\to L_2^{-1}\to V\to L_2\to 0$ be an extension of line bundles. Then V can be defined by a cocycle $\{\Phi^{\lambda\mu}\}\in Z^1(\mathcal{U}, GL_2(\mathcal{O}))$ of the form $\Phi^{\lambda\mu}=\begin{pmatrix}(e^{\lambda\mu})^{-1} & s^{\lambda\mu}\\ 0 & e^{\lambda\mu}\end{pmatrix}$ with $s^{\lambda\mu}\in\Gamma(U_\lambda\cap U_\mu, \mathcal{O})$. Moreover if we put $\upsilon^{\lambda\mu}=e^{\lambda\mu}s^{\lambda\mu}(e^\mu)^2$, $\{\upsilon^{\lambda\mu}\}$ belongs to $Z^1(\mathcal{U}, \mathrm{Hom}(L_2, L_2^{-1}))$ and it represents $\upsilon=\delta((V)_U)$ (cf. Corollary to Theorem 10 and the proof of Theorem 13 in Gunning [4]).

Then an element γ of $H^0(C, \mathrm{Hom}(L_1^{-1}, V))$ can be regarded as an element $\left\{\begin{pmatrix}d^\lambda\\ c^\lambda\end{pmatrix}\right\}$ of $C^0(\mathcal{U}, \mathcal{O})\oplus C^0(\mathcal{U}, \mathcal{O})$ satisfying

$$(4.5) \qquad f^{\lambda\mu}e^{\lambda\mu}c^\mu = c^\lambda, \qquad \text{and} \quad d^\mu f^{\lambda\mu}(e^{\lambda\mu})^{-1}+c^\mu f^{\lambda\mu}s^{\lambda\mu} = d^\lambda$$

in $U_\lambda\cap U_\mu$ (cf. the proof of Lemma 16 in [4]). Put $u^\lambda=c^\lambda(e^\lambda f^\lambda)^{-1}$ and $\delta^\lambda=d^\lambda e^\lambda(f^\lambda)^{-1}$. Then by (4.5), $\{u^\lambda\}$ belongs to $H^0(C, \mathrm{Hom}(L_1^{-1}, L_2))$, and

$$(4.6) \qquad u^\mu\upsilon^{\lambda\mu}+\delta^\mu = \delta^\lambda \qquad \text{in } U_\lambda\cap U_\mu .$$

Thus there is an isomorphism $h_{\mathcal{U}}$ from $\tilde{\Gamma}(\mathcal{U}, \mathrm{Hom}(L_1^{-1}, V))$ to $H^0(C, \mathrm{Hom}(L_1^{-1}, V))$ where

$$\tilde{\Gamma}(\mathcal{U}, \mathrm{Hom}(L_1^{-1}, V)) = \left\{\left\{\begin{pmatrix}\delta^\lambda\\ u^\lambda\end{pmatrix}\right\}\in C^0(\mathcal{U}, \mathrm{Hom}(L_1^{-1}, L_2^{-1}))\oplus H^0(C, \mathrm{Hom}(L_1^{-1}, L_2)) ;\ \left\{\begin{pmatrix}\delta^\lambda\\ u^\lambda\end{pmatrix}\right\} \text{ satisfies (4.6)}\right\}$$

and in particular, $h_{\mathcal{U}}\left(\left\{\left\{\begin{pmatrix}\delta^\lambda\\ u^\lambda\end{pmatrix}\right\}\in\tilde{\Gamma}(\mathcal{U}, \mathrm{Hom}(L_1^{-1}, V))\,;\, u^\lambda=0\right\}\right)$ is isomorphic to $H^0(C, \mathrm{Hom}(L_1^{-1}, L_2^{-1}))$. Similarly a cocycle Ξ of $Z^1(\mathcal{U}, \mathrm{Hom}(L_1, V))$ can be regarded as an element $\left\{\begin{pmatrix} b^{\lambda\mu} \\ a^{\lambda\mu} \end{pmatrix}\right\}$ of $C^1(\mathcal{U}, \mathcal{O})\oplus C^1(\mathcal{U}, \mathcal{O})$ satisfying

$$(4.7)\qquad \begin{pmatrix} b^{\mu\nu} \\ a^{\mu\nu} \end{pmatrix}-\begin{pmatrix} b^{\lambda\nu} \\ a^{\lambda\nu} \end{pmatrix}+\begin{pmatrix} (f^{\nu\mu}e^{\nu\mu})^{-1} & (f^{\nu\mu})^{-1}s^{\nu\mu} \\ 0 & (f^{\nu\mu})^{-1}e^{\nu\mu} \end{pmatrix}\begin{pmatrix} b^{\lambda\mu} \\ a^{\lambda\mu} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in $U_\lambda \cap U_\mu \cap U_\nu$ (cf. (9) of Appendix 1 in [4]). Put $\xi^{\lambda\mu}=a^{\lambda\mu}(e^{\mu})^{-1}f^{\mu}$ and $\eta^{\lambda\mu}=b^{\lambda\mu}e^{\mu}f^{\mu}$. Then $\left\{\begin{pmatrix} \eta^{\lambda\mu} \\ \xi^{\lambda\mu} \end{pmatrix}\right\}$ belongs to $C^1(\mathcal{U}, \mathrm{Hom}(L_1, L_2^{-1}))\oplus Z^1(\mathcal{U}, \mathrm{Hom}(L_1, L_2))$ and satisfies the following condition:

$$(4.8)\qquad \begin{pmatrix} \eta^{\mu\nu} \\ \xi^{\mu\nu} \end{pmatrix}-\begin{pmatrix} \eta^{\lambda\nu} \\ \xi^{\lambda\nu} \end{pmatrix}+\begin{pmatrix} 1 & \upsilon^{\nu\mu} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \eta^{\lambda\mu} \\ \xi^{\lambda\mu} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } U_\lambda \cap U_\mu \cap U_\nu.$$

Thus there is a surjective homomorphism $i_{\mathcal{U}}$ from $\tilde{Z}^1(\mathcal{U}, \mathrm{Hom}(L_1, V))$ to $H^1(C, \mathrm{Hom}(L_1, V))$, and the kernel of $i_{\mathcal{U}}$ is $\tilde{B}^1(\mathcal{U}, \mathrm{Hom}(L_1, V))$, where $\tilde{Z}^1(\mathcal{U}, \mathrm{Hom}(L_1, V))$ is a k-submodule of $C^1(\mathcal{U}, \mathrm{Hom}(L_1, L_2^{-1}))\oplus Z^1(\mathcal{U}, \mathrm{Hom}(L_1, L_2))$ consisting of all elements satisfying (4.8), and

$$\tilde{B}^1(\mathcal{U}, \mathrm{Hom}(L_1, V))=\left\{\left\{\begin{pmatrix} \eta^{\lambda\mu} \\ \xi^{\lambda\mu} \end{pmatrix}\right\} ; \begin{pmatrix} \eta^{\lambda\mu} \\ \xi^{\lambda\mu} \end{pmatrix}=\begin{pmatrix} \beta^{\mu} \\ \alpha^{\mu} \end{pmatrix}-\begin{pmatrix} 1 & \upsilon^{\mu\lambda} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta^{\lambda} \\ \alpha^{\lambda} \end{pmatrix}\right.$$
$$\text{with some } \{\beta^\lambda\}\in C^0(\mathcal{U}, \mathrm{Hom}(L_1, L_2^{-1}))$$
$$\left.\text{and } \{\alpha^\lambda\}\in C^0(\mathcal{U}, \mathrm{Hom}(L_1, L_2))\right\}.$$

Now let $\mathcal{U}=\{U_\lambda\}$ be a σ-invariant affine open covering of C. If $\upsilon=\iota_{\mathcal{U}}(\{\upsilon^{\lambda\mu}\})$ belongs to $H^1(C, \mathrm{Hom}(L_2, L_2^{-1}))_-$, we can take $\upsilon^{\lambda\mu}$ such that $\sigma(\upsilon^{\lambda\mu})=-\upsilon^{\lambda\mu}$. Then

$$H^0(C, \mathrm{Hom}(L_1^{-1}, V))_i=h_{\mathcal{U}}\left(\left\{\left\{\begin{pmatrix} \delta^{\lambda} \\ u^{\lambda} \end{pmatrix}\right\}\in \tilde{\Gamma}(\mathcal{U}, \mathrm{Hom}(L_1^{-1}, V)) ;\ \sigma(u^\lambda)=iu^\lambda,\ \sigma(\delta^\lambda)=-i\delta^\lambda\right\}\right),$$

and

$$H^1(C, \mathrm{Hom}(L_1, V))_i=i_{\mathcal{U}}\left(\left\{\left\{\begin{pmatrix} \zeta^{\lambda\mu} \\ \xi^{\lambda\mu} \end{pmatrix}\right\}\in \tilde{Z}^1(\mathcal{U}, \mathrm{Hom}(L_1, V)) ;\ \sigma(\xi^{\lambda\mu})=i\xi^{\lambda\mu},\ \sigma(\zeta^{\lambda\mu})=-i\zeta^{\lambda\mu}\right\}\right)$$

for $i=+, -$.

Now analogously to (2.9), for any $\gamma=h_{\mathcal{U}}\left(\left\{\begin{pmatrix} \delta^{\lambda} \\ u^{\lambda} \end{pmatrix}\right\}\right)\in H^0(C, \mathrm{Hom}(L_1^{-1}, V))$ and $\omega=\iota_{\mathcal{U}}(\{\omega^{\lambda\mu}\})\in H^1(C, \mathrm{Hom}(L_1, L_1^{-1}))$, we have

$$(4.9)\qquad \Theta_{\upsilon}(\omega, \gamma)=i_{\mathcal{U}}\left(\left\{\begin{pmatrix} \omega^{\lambda\mu}\delta^{\mu} \\ \omega^{\lambda\mu}u^{\mu} \end{pmatrix}\right\}\right).$$

Here, we regard $\omega^{\lambda\mu}$ and others as rational functions on C. From this, for a basis $\{\upsilon_l\}$ for $H^1(C, \mathrm{Hom}(L_2, L_2^{-1}))_-$, we can construct a set of bilinear maps from $H^1(C, \mathrm{Hom}(L_1, L_1^{-1}))_- \times H^0(C, \mathrm{Hom}(L_1^{-1}, L_2))_i$ to $H^1(C, \mathrm{Hom}(L_1, L_2^{-1}))_i$ satisfying (4.4) as follows. Let $\upsilon_l = \iota_{\mathcal{V}}([\{\upsilon_l^{\lambda\mu}\}])$. Let $\{u_k\}$, $\{\iota_{\mathcal{V}}[\{\xi_k^{\lambda\mu}\}]\}$, and $\{\iota_{\mathcal{V}}[\{\eta_k^{\lambda\mu}\}]\}$ be bases for $H^0(C, \mathrm{Hom}(L_1^{-1}, L_2))_i$, $H^1(C, \mathrm{Hom}(L_1, L_2))_{j(i)}$, and $H^1(C, \mathrm{Hom}(L_1, L_2^{-1}))_i$, respectively. For each υ_l, and u_k, we can take an element $\{\delta_{lk}^{\lambda}\}$ of $C^0(\mathcal{V}, \mathrm{Hom}(L_1^{-1}, L_2^{-1}))$ satisfying (4.6). Similarly, for each υ_l, and $i_{\mathcal{V}}(\{\xi_k^{\lambda\mu}\})$, we can take an element $\{\zeta_{lk}^{\lambda\mu}\}$ of $C^1(\mathcal{V}, \mathrm{Hom}(L_1, L_2^{-1}))$ satisfying (4.8). Moreover we can take these elements such that $\sigma(\upsilon_l^{\lambda\mu}) = -\upsilon_l^{\lambda\mu}$, $\sigma(u_k) = iu_k$, $\sigma(\delta_{lk}^{\lambda}) = -i\delta_{lk}^{\lambda}$, $\sigma(\xi_k^{\lambda\mu}) = -i\xi_k^{\lambda\mu}$, and $\sigma(\zeta_{lk}^{\lambda\mu}) = i\zeta_{lk}^{\lambda\mu}$. Then if we put $(V_l)_U = \delta(\upsilon_l)$, $\xi_{lk} = i_{\mathcal{V}}\left(\left\{\begin{pmatrix}\zeta_{lk}^{\lambda\mu}\\ \xi_k^{\lambda\mu}\end{pmatrix}\right\}\right)$, $\eta_{lk} = i_{\mathcal{V}}\left(\left\{\begin{pmatrix}\eta_k^{\lambda\mu}\\ 0\end{pmatrix}\right\}\right)$, and $\gamma_{lk} = h_{\mathcal{V}}\left(\left\{\begin{pmatrix}\delta_{lk}^{\lambda}\\ u_k\end{pmatrix}\right\}\right)$, we can take $\{\xi_{lk}, \eta_{lk}\}$ as a basis for $H^1(C, \mathrm{Hom}(L_1, V_l))_{j(i)}$. Then for each $\omega = \iota_{\mathcal{V}}(\{\omega^{\lambda\mu}\})$ of $H^1(C, \mathrm{Hom}(L_1, L_1^{-1}))_-$, $\Theta_{\upsilon_l}(\omega, \gamma_{lj})$ is expressed as

$$\Theta_{\upsilon_l}(\omega, \gamma_{lj}) = \sum_k a_{kj}\xi_{lk} + \sum_k b_{kjl}\eta_{lk} \tag{4.10}$$

where a_{kj}, and b_{kjl} are elements of the field k, and a_{kj} does not depend on υ_l Then define a bilinear map Θ'_{υ_l} from $H^1(C, \mathrm{Hom}(L_1, L_1^{-1}))_- \times H^1(C, \mathrm{Hom}(L_1^{-1}, L_2))_i$ to $H^1(C, \mathrm{Hom}(L_1, L_2^{-1}))_i$ by

$$\Theta'_{\upsilon_l}(\omega, u_j) = \iota_{\mathcal{V}}([\{\omega^{\lambda\mu}\delta_{lj}^{\mu} - \upsilon_l^{\lambda\mu}\alpha_j^{\lambda} - \sum_k a_{kj}\zeta_{lk}^{\lambda\mu}\}]) \tag{4.11}$$

where $\{\alpha_j^{\lambda}\}$ is an element of $C^0(\mathcal{V}, \mathrm{Hom}(L_1, L_2))$ such that

$$\omega^{\lambda\mu}u_j^{\mu} = \sum_k a_{kj}\xi_k^{\lambda\mu} + \alpha_j^{\mu} - \alpha_j^{\lambda}. \tag{4.12}$$

Then, by (4.9), and (4.10), the bilinear map Θ'_{υ_l} satisfies (4.4).

REMARK 4.3. For a matrix $g = (g_{ij})_{1\le i,j\le n}$ in k, and a polynomial $P(X_1, \cdots, X_n)$ over k, define a polynomial $g\circ P$ by $(g\circ P)(X_1, \cdots, X_n) = P(\sum_j g_{1j}X_j, \cdots, \sum_j g_{nj}X_j)$. For a matrix $A = (P_{ij})$ in $k[X_1, \cdots, X_n]$, put $g\circ P = (g\circ P_{ij})$. Two matrix A and A' in $k[X_1, \cdots, X_n]$ are called quasi-equivalent if $g\circ A$ and A' are equivalent with some non-singular matrix g in k. Then by construction, two FM matrices of the same type are quasi-equivalent to each other. More precisely, let F be the FM matrix in Theorem 4.1, and F' be another FM matrix of the same type as F. Then we have

$$g\circ F = \begin{pmatrix}U_{11} & 0\\ U_{21} & U_{22}\end{pmatrix} F' \begin{pmatrix}V_{11} & 0\\ V_{21} & V_{22}\end{pmatrix} \tag{4.13}$$

where g, U_{11}, U_{22}, V_{11}, and V_{22} are non-singular matrices in k of degrees m_0, m_1, m_2, n_1, and n_2, respectively, and U_{21} and V_{21} are matrices whose entries are k-linear combinations of the monomials $X_1, \cdots, X_{m_0}$.

REMARK 4.4. Let $K = L_{P_0} \otimes L_{P_0'}$ be the line bundle stated in § 2. Let L_Q be a Weierstrass line bundle, and $\mathcal{V} = \{U_Q, U_{P_0}\}$ be the affine open covering.

Fix a non-zero element z of $H^0(C, K\otimes L_Q^{-2})$. Then the homomorphism Π from $Z^1(\mathcal{U}, K^{-2})$ to $Z^1(\mathcal{U}, L_Q^{-4})$ defined by $\Pi(w)=z^2w$ induces a surjective homomorphism from $H^1(C, \mathrm{Hom}(K, K^{-1}))$ to $H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1}))$, which will be denoted also by Π. For each $(V)_U\in \mathrm{Ext}(K, K^{-1})$, let $(\bar{V})_U$ be the element of $\mathrm{Ext}(L_Q, L_Q^{-1})$ such that $\delta((\bar{V})_U)=\Pi(\delta(V)_U)$. Then the homomorphism Υ from $C^1(\mathcal{U}, K^{-p-1})\oplus Z^1(\mathcal{U}, K^{-p+1})$ to $C^1(\mathcal{U}, L_Q^{-2p-2})\oplus C^1(\mathcal{U}, L_Q^{2p+2})$ defined by $\Upsilon\binom{\eta}{\xi}=\binom{\eta z^{p+1}}{\xi z^{p-1}}$ induces a surjective homomorphism from $H^1(C, \mathrm{Hom}(K^p, V))$ to $H^1(C, \mathrm{Hom}(L_Q^p, \bar{V}))$, which will be denoted also by Υ. This induces a surjective homomorphism Υ' (resp. Υ'') from $H^1(C, \mathrm{Hom}(K^p, K))$ (resp. $H^1(C, \mathrm{Hom}(K^p, K^{-1}))$) to $H^1(C, \mathrm{Hom}(L_Q^p, L_Q))$ (resp. $H^1(C, \mathrm{Hom}(L_Q^p, L_Q^{-1}))$). Moreover the homomorphism Λ from $C^0(\mathcal{U}, L_Q^{p-1})\oplus H^0(C, L_Q^{p+1})$ to $C^0(\mathcal{U}, K^{p-1}\otimes L_Q^{-p+1})\oplus H^0(C, K^{p+1}\otimes L_Q^{-p-1})$ defined by $\Lambda\left(\binom{v}{u}\right)=\binom{vz^{p-1}}{uz^{p+1}}$ induces an injective homomorphism from $H^0(C, \mathrm{Hom}(L_Q^{-p}, \bar{V}))$ to $H^0(C, \mathrm{Hom}(K^{-p}, V))$, which will be denoted also by Λ. This induces an injective homomorphism Λ' (resp. Λ'') from $H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q))$ (resp. $H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q^{-1}))$) to $H^0(C, \mathrm{Hom}(K^{-p}, K))$ (resp. $H^0(C, \mathrm{Hom}(K^{-p}, K^{-1}))$). Moreover, we have

$$\bar{\Theta}'(\bar{f}(\Pi(\omega)), u) = \Upsilon'\Theta'(f(\omega), \Lambda'(u)), \tag{4.14}$$

$$\bar{\Theta}''(\bar{f}(\Pi(\omega)), v) = \Upsilon''\Theta''(f(\omega), \Lambda''(v)) \tag{4.15}$$

for any $\omega\in H^1(C, \mathrm{Hom}(K, K^{-1}))_-$, $u\in H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q))_i$ and $v\in H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q^{-1}))_{j(i)}$, where $\bar{f}: H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1}))\to H^1(C, \mathrm{Hom}(L_Q^p, L_Q^{-p}))$ is the p-linear map in §3, and $\bar{\Theta}'$ and $\bar{\Theta}''$ are the bilinear maps in §2.1. Let $n_1=\dim H^0(C, \mathrm{Hom}(K^{-p}, K))_i$, and $n_1'=\dim H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q))_i$. Let $\{\upsilon_l\}_{l=0}^3$, and $\{u_j\}_{j=1}^{n_1}$ be bases for $H^1(C, \mathrm{Hom}(K, K^{-1}))_-$, and $H^0(C, \mathrm{Hom}(K^{-p}, K))_i$, respectively. Let $\{\Theta_{\upsilon_l}'\}_{l=0}^3$ be the set of bilinear maps from $H^1(C, \mathrm{Hom}(K^p, K^{-p}))_- \times H^0(C, \mathrm{Hom}(K^{-p}, K))_i$ to $H^1(C, \mathrm{Hom}(K^p, K^{-1}))_i$ defined by (4.11), (4.12). Assume that $\{\Pi(\upsilon_l)\}_{l=1}^3$ forms a basis for $H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1}))$, and $\left\{h_{\mathcal{U}}\left(\left\{\binom{\delta_{lj}^{\lambda}}{u_j}\right\}\right)\right\}_{j=1}^{n_1'}$ is contained in $\Lambda(H^0(C, \mathrm{Hom}(L_Q^{-p}, \bar{V}_l))_i)$. Then $\{u_j\}_{j=1}^{n_1'}$ is contained in $\Lambda'(H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q))_i)$, and by (4.11) and (4.12), there is a set $\{\bar{\Theta}_{\Pi(\upsilon_l)}'\}_{l=1}^3$ of bilinear maps from $H^1(C, \mathrm{Hom}(L_Q^p, L_Q^{-p}))_- \times H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q))_i$ to $H^1(C, \mathrm{Hom}(L_Q^p, L_Q^{-1}))_i$ satisfying (4.4), and

$$\bar{\Theta}_{\Pi(\upsilon_l)}'(\bar{f}(\Pi(\omega)), \Lambda'^{-1}(u_j)) = \Upsilon''\Theta_{\upsilon_l}'(f(\omega), u_j) \tag{4.16}$$

for any $1\leqq j\leqq n_1'$, and $1\leqq l\leqq 3$, and $\omega\in H^1(C, \mathrm{Hom}(K, K^{-1}))_-$.

REMARK 4.5. Let F be an FM matrix of the type (K, i) or (L_Q, i) with $Q\in\mathcal{W}$, and let m_1, m_2, n_1, n_2 be the integers stated before. Then there is an

$(m_1+m_2,\ n_1+n_2)$ matrix $F^{**}=\begin{pmatrix}\underbrace{(A_{ij}^{**})}_{n_1} & \underbrace{(C_{ij}^{**})}_{n_2} \\ (B_{ij}^{**}) & (D_{ij}^{**})\end{pmatrix}\begin{matrix}\}m_1\\ \}m_2\end{matrix}$ in a polynomial ring over k satisfying the following conditions:

(4.17) The components A_{ij}^{**}, B_{ij}^{**}, C_{ij}^{**}, and D_{ij}^{**} of F^{**} are homogeneous polynomials of degrees p, $p+1$, $p-1$, and p, respectively.

(4.18) F^{**} has a general system.

(For the existence of such a matrix, see the proof of Proposition 5.2.) Then, the degree of the cycle $C_{F^{**}}$ does not depend on a choice of F^{**}, and is uniquely determined by F. Therefore, we put $n(F)=\deg C_{F^{**}}$.

Hereafter, for two matrices A, B in a polynomial ring over k, we write $A \underset{\text{Dia.}}{\sim} B$ if there are non-singular diagonal matrices D_1, D_2 in k such that $B=D_1AD_2$.

§ 5. Proof of Theorem A.

In this section, let p be a prime different from 2, 3. As above, fix a non-Weierstrass point P_0, and put $K=L_{P_0}\otimes L_{P_0'}$. To prove Theorem A, first we construct FM matrices as follows.

For each Weierstrass point Q, let x_Q be the non-zero element of $H^0(C, K\otimes L_Q^{-2})$, and let $\mathcal{U}=\{U_Q, U_{P_0}\}$ be the affine open covering of C. For a while, put $U_1=U_Q$, $U_2=U_{P_0}$, and $x=x_Q$. Let $\{\omega_k\}_{k=0}^{3}$ be the basis for $H^1(C, \mathrm{Hom}(K, K^{-1}))_-$ given by $\omega_k=\iota_{\mathcal{U}}([x^{-k-1}y])$. First, let $\{u_j\}_{j=1}^{p+2}$, $\{v_j\}_{j=1}^{p-3}$ be the bases for $H^0(C, \mathrm{Hom}(K^{-p}, K))_+$, and $H^0(C, \mathrm{Hom}(K^{-p}, K^{-1}))_-$ given by $u_j=x^{j-1}$, and $v_j=x^{j-1}y$, respectively. For each $0\leqq l\leqq 3$, and $1\leqq j\leqq p+2$, define an element $\{\delta_{lj}^{\lambda}\}$ of $C^0(\mathcal{U}, \mathrm{Hom}(K^{-p}, K^{-1}))$ by

$$(5.1)\qquad (\delta_{lj}^1, \delta_{lj}^2)=\begin{cases}(0, -x^{j-l-2}y) & \text{if } 1\leqq l\leqq 3,\ p+l-1\leqq j\leqq p+2,\\ & \quad\text{or } l=0,\ 2\leqq j\leqq p+2,\\ (x^{j-l-2}y, 0) & \text{otherwise.}\end{cases}$$

Then $\{\delta_{lj}^{\lambda}\}$ satisfies (4.6). Next let $\{\xi_i\}_{i=1}^{p+1}$, $\{\eta_i\}_{i=1}^{p}$ be the bases for $H^1(C, \mathrm{Hom}(K^p, K))_-$, and $H^1(C, \mathrm{Hom}(K^p, K^{-1}))_+$ given by $\xi_i=\iota_{\mathcal{U}}([x^{-i}y])$, and $\eta_i=\iota_{\mathcal{U}}([x^{-i}])$, respectively. For each $0\leqq l\leqq 3$, and $1\leqq i\leqq p+1$, define an element ζ_{li}^{12} of $\Gamma(U_1\cap U_2, \mathrm{Hom}(K^p, K^{-1}))$ by

$$(5.2)\qquad \zeta_{li}^{12}=-x^{-i-1}y^2 \text{ or } 0 \quad \text{according as } l=0 \text{ or not.}$$

Then an element $\begin{pmatrix}\zeta_{li}^{12}\\ x^{-i}y\end{pmatrix}$ of $\Gamma(U_1\cap U_2, \mathrm{Hom}(K^p, K^{-1}))\oplus\Gamma(U_1\cap U_2, \mathrm{Hom}(K^p, K))$ defines a unique element Ξ_{li} of $\tilde{Z}^1(\mathcal{U}, \mathrm{Hom}(K^p, V_l))$ satisfying (4.8). With respect to these quantities, let us define a bilinear map Θ'_{v_l} in Remark 4.2, and construct an FM matrix $F_Q(+)$ of the type $(K, +)$. Similarly, take the bases $\{\omega_k\}_{k=0}^{3}$, $\{x^{j-1}y\}_{j=1}^{p-1}$, $\{x^{j-1}\}_{j=1}^{p}$, $\{\iota_{\mathcal{U}}([x^{-i}])\}_{i=1}^{p-2}$, and $\{\iota_{\mathcal{U}}([x^{-i}y])\}_{i=1}^{p+3}$ for $H^1(C, \mathrm{Hom}(K, K^{-1}))_-$, $H^0(C, \mathrm{Hom}(K^{-p}, K))_-$, $H^0(C, \mathrm{Hom}(K^{-p}, K^{-1}))_+$, $H^1(C, \mathrm{Hom}(K^p, K))_+$, and $H^1(C, \mathrm{Hom}(K^p, K^{-1}))_-$, respectively. Moreover, define an element $\{\delta'^{\lambda}_{lj}\}$ of $C^0(\mathcal{U}, \mathrm{Hom}(K^{-p}, K^{-1}))$ by $(\delta'^1_{lj}, \delta'^2_{lj})=(x^{j-l-2}y^2, 0)$ or $(0, -x^{j-l-2}y^2)$ according as $1\leqq l\leqq 3$, $1\leqq j\leqq p+l-5$, or not. Moreover define an element ζ'^{12}_{li} of $\Gamma(U_1\cap U_2, \mathrm{Hom}(K^p, K^{-1}))$ by $\zeta'^{12}_{li}=-x^{-i-1}y$ or 0 according as $l=0$ or not, and construct FM matrix $F_Q(-)$ of the type $(K, -)$. Now fix one of these matrices, and denote it by F. The matrix F is an element of $M_{2p+1,2p-1}(k[X_0, \cdots, X_3])$. Unfortunately, we cannot apply Theorem 1.5 to calculate the number of FM vector bundles because V_F may be one dimensional by (2) of Theorem 3.6. So we use some trick. To do this, let $c: \mathrm{P.Ext}(K, K^{-1})_- \to \mathrm{Proj}\, k[X_0, \cdots, X_3, X_4]$ be the closed immersion defined by the natural surjection $c^{\#}: k[X_0, \cdots, X_3, X_4] \to k[X_0, \cdots, X_3, X_4]/\langle X_4\rangle$. Put $P^4=\mathrm{Proj}\, k[X_0, \cdots, X_4]$. Let $\mathfrak{P}_\theta$ (resp. $\mathfrak{P}^*_\theta$) be the ideal defining the curve C_θ (resp. $C^*_\theta=c(C_\theta)$) for each $\theta\in J(2)$. Then we have

PROPOSITION 5.1. *Divide the matrix F as $F=\begin{pmatrix} \overbrace{(A_{ij})}^{n_1} & \overbrace{0}^{n_2} \\ (B_{ij}) & (D_{ij}) \end{pmatrix}\begin{matrix} \}m_1 \\ \}m_2 \end{matrix}$ as in §4. Then we can choose elements a_{ij}, b_{ijk}, d_{ij} of the field k such that the matrix*

$$F^*=F+\begin{pmatrix} \overbrace{(a_{ij}X_4^p)}^{n_1} & \overbrace{0}^{n_2} \\ (\sum_{k=0}^{3} b_{ijk}X_kX_4^p) & (d_{ij}X_4^p) \end{pmatrix}\begin{matrix} \}m_1 \\ \}m_2 \end{matrix}$$

satisfies the following conditions:

(5.3) *V_{F^*} is of codimension three in P^4, and $V_+(\langle \mathfrak{q}(F^*), X_4\rangle)=\mathrm{Proj}\, k[X_0, \cdots, X_4]/\langle \mathfrak{q}(F^*), X_4\rangle$ is of dimension zero, where $\mathfrak{q}(F^*)$ denotes the intersection of all primary components of the ideal $\mathfrak{I}(F^*)$ whose radicals are different from $\mathfrak{P}^*_\theta$ for any $\theta\in J(2)$.*

(5.4) *$i_\theta=i(\mathfrak{I}(F^*); C^*_\theta)\leqq p$ for any $\theta\in J(2)$.*

The proof of this proposition will be done after Lemma 5.5. From Proposition 5.1, we have

PROPOSITION 5.2. $\sum_{P\in P^4} i(\langle \mathfrak{q}(F^*), X_4\rangle; P) = n(F)-\sum_{\theta\in J(2)} i_\theta \deg C^*_\theta$.

PROOF. Define a matrix F^{**} in the polynomial ring $k[X_0, \cdots, X_4, T_{11}, \cdots, T_{2p+1,2p-1}]$ by $F^{**}=F^*+\begin{pmatrix} (T^p_{ij}) & (T^{p-1}_{ij}) \\ \underbrace{(T^{p+1}_{ij})}_{n_1} & \underbrace{(T^p_{ij})}_{n_2} \end{pmatrix}\begin{matrix} \}m_1 \\ \}m_2 \end{matrix}$. Clearly F^{**} has a general system. Thus by Theorem 1.5, and by the definition of $n(F)$, we have $\deg C_{F^{**}}=n(F)$ (for the definition of $C_{F^{**}}$, see Theorem 1.4). On the other hand, by the construction of F^{**}, we have $\deg C_{F^*}=\deg C_{F^{**}}$. Thus we have

$$\deg C_{F^*} = n(F).$$

Now define a cycle C'_{F^*} on P^4 by $C'_{F^*}=C_{F^*}-\sum_{\theta\in J(2)} i_\theta C^*_\theta$. Then clearly the support $\mathrm{supp}(C'_{F^*})$ of C'_{F^*} coincides with $V_+(\mathfrak{q}(F^*))$, and the support $\mathrm{supp}(C'_{F^*}\cdot X_4)$ of the intersection product of C'_{F^*} and X_4 coincides with $V_+(\langle\mathfrak{q}(F^*), X_4\rangle)$. Thus by (1) of Proposition 5.1, we have

$$\sum_{P\in P^4} i(\langle\mathfrak{q}(F^*), X_4\rangle ; P) = \deg C'_{F^*}\cdot X_4 .$$

This proves the assertion.

REMARK. We do not know whether or not F^* itself has a general system.

Now to complete the proof, we consider $i(\langle\mathfrak{q}(F^*), X_4\rangle ; P)$ for each $P\in P^4$. To do this, first we give the following lemma.

LEMMA 5.3. *Let $F=F_Q(i)$ be the* FM *matrix of the type (K, i) stated above, and let $\mathfrak{J}_Q$ be the ideal of $k[X_0, \cdots, X_3]$ defining the point $(W_{[Q]})_T$ of* $\mathrm{P.Ext}(K, K^{-1})_-=\mathrm{Proj}\, k[X_0, \cdots, X_3]$. *Let us divide F as*

$$F=\begin{pmatrix} \overbrace{F_{11}}^{(p+1)/2} & \overbrace{F_{12}}^{n_1-(p+1)/2} & \overbrace{0}^{(p-1)/2} & \overbrace{0}^{n_2-(p-1)/2} \\ F_{21} & F_{22} & 0 & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix}\begin{matrix} \}\ (p-1)/2 \\ \}\, m_1-(p-1)/2 \\ \}\ (p+1)/2 \\ \}\, m_2-(p+1)/2 \end{matrix}$$

and put

$$G=\begin{pmatrix} F_{11} & 0 & F_{12} & 0 \\ F_{31} & F_{33} & F_{32} & F_{34} \\ F_{21} & 0 & F_{22} & 0 \\ F_{41} & F_{43} & F_{42} & F_{44} \end{pmatrix}.$$

Then $\bar{F}=\begin{pmatrix} F_{22} & 0 \\ F_{42} & F_{44}\end{pmatrix}$ is an FM *matrix of the type (L_Q, i), and*

(5.5) $$\det G_{p+i}(p+j,\ p+k\ ;\ ) \equiv a_i \det \bar{F}_i(j,\ k\ ;\ )X_0^{p^2+1}+M_{ijk}X_0^{p^2} \mod \mathfrak{J}_Q^{m_{ijk}-p^2+1}$$

for any $1\leqq i\leqq p-1$, and $1\leqq j,\ k\leqq i+2$, where m_{ijk} denotes the degree of the polynomial $\det G_{p+i}(p+j,\ p+k\ ;\ )$ *in $X_0, \cdots, X_3$, M_{ijk} is an element of $\mathfrak{J}_Q^{m_{ijk}-p^2}$ $\cap k[X_1, X_2, X_3]$, and a_i is a constant depending only on i. (For the definition of $G_{p+i}(p+j,\ p+k\ ;\ )$ and others, see* § 1.)

PROOF. Let $F=F_Q(+)$. Put $x=x_Q$ and $\lambda_P=x-x_P$. Then for any $1\leqq j\leqq p+2$, $0\leqq k\leqq 3$,

(5.6) $$(x^{-k-1}y)^p x^{j-1} = \sum_{i=1}^{p+1} b_{k\,p-i-j+(p+3)/2}x^{-i}y+Q_{kj}(x)y+x^{-p-2}P_{kj}(x^{-1})y$$

where b_l is the coefficient of the l-th term of the polynomial $\prod_{\substack{P\in\mathcal{W}\\P\neq Q}}(x-\lambda_P)^{(p-1)/2}$ in x, and $Q_{kj}(x)$ is a polynomial in x, and $P_{kj}(x^{-1})$ is a polynomial in x^{-1}. Put $\alpha_{kj}^1=-x^{-p-2}P_{kj}(x^{-1})y$, and $\alpha_{kj}^2=Q_{kj}(x)y$. Then $\{\alpha_{kj}^\lambda\}$ belongs to $C^0(\mathcal{U}, \mathrm{Hom}(K^p, K))$. Thus we have

$$\Theta'(f(\omega_k), u_j)=\sum_{i=1}^{p+1}b_{kp-i-j+(p+3)/2}\xi_i \tag{5.7}$$

for any $0\leqq k\leqq 3$, and $1\leqq j\leqq p+2$. Similarly for any $j\geqq 0$, and $0\leqq k\leqq 3$, we have

$$\{(x^{-k-1}y)^p x^{j-1}y\}=\sum_{i=1}^{p}c_{kp-i-j+(p+1)/2}\{x^{-i}\} \quad \mathrm{mod}\, B^1(\mathcal{U}, \mathrm{Hom}(K^p, K^{-1})), \tag{5.8}$$

where c_l is the coefficient of the l-th term of the polynomial $\prod_{\substack{P\in\mathcal{W}\\P\neq Q}}(x-\lambda_P)^{(p+1)/2}$ in x. Thus, in particular, we have

$$\Theta''(f(\omega_k), v_j)=\sum_{i=1}^{p}c_{kp-i-j+(p+1)/2}\eta_i \tag{5.9}$$

for any $1\leqq j\leqq p-3$, and $0\leqq k\leqq 3$. Now recall that ω_k is defined by the cocycle $\{\omega_k^{\lambda\mu}\}$ such that $\omega_k^{12}=x^{-k-1}y$, and $\omega_k^{21}=-x^{-k-1}y$, and note that $\alpha_{0j}^1=0$ for any $1\leqq j\leqq p+2$. Thus by (5.1), and (5.8), we have

$$\{(\omega_0^{12})^p\delta_{lj}^2-\omega_l^{12}\alpha_{0j}^1\}=0 \quad \mathrm{mod}\, B^1(\mathcal{U}, \mathrm{Hom}(K^p, K^{-1})).$$

Thus, by (4.11), and (5.2), for any $1\leqq l\leqq 3$, and $1\leqq j\leqq p+2$, we have

$$\Theta'_{\omega_l}(f(\omega_0), u_j)=0. \tag{5.10}$$

Similarly, for any $0\leqq k\leqq 3$, and $2\leqq j\leqq p+2$, we have

$$\{(\omega_k^{21})^p\delta_{0j}^1-\omega_0^{21}\alpha_{kj}^2\}=0 \quad \mathrm{mod}\, B^1(\mathcal{U}, \mathrm{Hom}(K^p, K^{-1})).$$

We have $\zeta_{0i}^{21}=0$ by (5.2), and (4.8). Thus, we have

$$\Theta'_{\omega_0}(f(\omega_k), u_j)=0. \tag{5.11}$$

for any $0\leqq k\leqq 3$, and $2\leqq j\leqq p+2$. Similarly, by (5.1), and (5.8), we have

$$\Theta'_{\omega_0}(f(\omega_k), u_1)=\sum_{i=1}^{p}c_{kp-i+(p+1)/2}\eta_i \tag{5.12}$$

for any $0\leqq k\leqq 3$. By (1) of Lemma 2.5, $\mathfrak{J}_Q$ is generated by X_1, X_2, X_3. Thus by (5.7), (5.9), and (5.10)~(5.12), we have

(5.13) $F_{ij}=0 \ \mathrm{mod}\, \mathfrak{J}_Q^p$ except for $(i, j)=(1, 1), (2, 1), (3, 1), (3, 3)$,

$$(5.14)\qquad \begin{pmatrix} F_{11} & 0 \\ F_{31} & F_{33} \end{pmatrix} \underset{\text{Dia.}}{\widetilde{\ \ }} \left(\begin{array}{cccc:cccc} * & & * & X_0^p & & & & \\ * & & \cdot^{\cdot^{\cdot}} & & & 0 & & \\ * & \cdot^{\cdot^{\cdot}} & 0 & & & & & \\ * & X_0^p & & & & & & \\ \hdashline * & & & & & & & X_0^p \\ * & & & & & * & \cdot^{\cdot^{\cdot}} & \\ * & & 0 & & X_0^p & \cdot^{\cdot^{\cdot}} & 0 & \\ * & & & & & & & \\ X_0^{p+1} & & & & 0 & & & \end{array}\right) \bmod \mathfrak{J}_Q^p,$$

and,

$$(5.15)\qquad F_{21} \underset{\text{Dia.}}{\widetilde{\ \ }} \begin{pmatrix} X_0^p & 0 \cdots 0 \\ 0 \cdots\cdots & 0 \end{pmatrix} \bmod \mathfrak{J}_Q^p .$$

On the other hand, let Π and others be the maps defined in Remark 4.4. Then $\{\Pi(\omega_k)\}_{k=1}^3$, $\{\Upsilon'(\xi_i)\}_{i=(p+1)/2}^{p+1}$, $\{\Upsilon''(\eta_i)\}_{i=(p+1)/2}^{p}$, and $\{v_j\}_{j=(p+1)/2}^{p-3}$ form bases for $H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1}))_-$, $H^1(C, \mathrm{Hom}(L_Q^p, L_Q))_-$, $H^1(C, \mathrm{Hom}(L_Q^p, L_Q^{-1}))_+$, and $\Lambda''(H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q^{-1}))_-)$, respectively. (Recall $p \geqq 5$.) Moreover, for any $1 \leqq l \leqq 3$, $\left\{h_{\mathcal{V}}\left(\left\{\begin{pmatrix}\delta_{ij}^l \\ u_j\end{pmatrix}\right\}\right)\right\}_{j=(p+3)/2}^{p+2}$ forms a basis for $\Lambda(H^0(C, \mathrm{Hom}(L_Q^{-p}, \bar{V}_l)_+)$. Thus by Remark 4.4, $\bar{F}$ is an FM matrix of the type $(L_Q, +)$. Thus, by (5.13)~(5.15), the assertion (5.5) holds for $F = F_Q(+)$. Similarly, for $F = F_Q(-)$, the assertion holds.

COROLLARY TO LEMMA 5.3. *Let F be as above. Then for any* FM *matrix $\bar{F}'$ of the type (L_Q, i), there is a matrix F' such that* (4.13) *holds and the same relation as* (5.5) *holds for G' and $\bar{F}'$, where G' is a matrix obtained from F' in the same manner as above.*

PROOF. The assertion can be proved by (4.13), and the above lemma.

REMARK 5.4. Let $\{\omega_i\}$ be the basis for $H^1(C, \mathrm{Hom}(K, K^{-1}))_-$ given by $\omega_i = \iota_{\mathcal{V}}([x_Q^{-i-1}y])$. We identify $\mathrm{P.Ext}(K, K^{-1})_-$ with $\mathrm{Proj}\, k[X_0, \cdots, X_3]$ via the isomorphism s from $\mathcal{S}(H^1(C, \mathrm{Hom}(K, K^{-1})_-))$ to $k[X_0, \cdots, X_3]$ such that $s(\omega_i) = X_i$. Then by (1) of Lemma 2.5, we have $(W)_T \in C_{\theta_0}$ if and only if $\varepsilon((W)_T) = \langle \sum_{i=0}^3 \lambda^i \mu^{3-i} \omega_i \rangle$ with $\lambda, \mu \in k$. Thus the curve C_{θ_0} on $\mathrm{P.Ext}(K, K^{-1})_-$ can be expressed as

$$(5.16)\qquad C_{\theta_0} = \mathrm{Proj}\, k[X_0, \cdots, X_3]/\langle X_0X_2 - X_1^2,\ X_0X_3 - X_1X_2,\ X_1X_3 - X_2^2 \rangle .$$

Similarly for $L_\theta = K \otimes L_P^{-1} \otimes L_Q^{-1}$ with $\theta \in J(2)$, $\theta \neq \theta_0$, C_θ can be expressed as

$$(5.17)\qquad C_\theta = \mathrm{Proj}\, k[X_0, \cdots, X_3]/\langle X_2 - \lambda_P X_1,\ X_3 - \lambda_P^2 X_1 \rangle$$

with $\lambda_P = x_Q - x_P$. On the other hand, let $\{\zeta_i\}_{i=1}^3$ be the basis for $H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1}))$ given by $\zeta_i = \iota_{\mathcal{V}}([x_Q^{-i+1}y])$. We identify $\mathrm{P.Ext}(L_Q, L_Q^{-1})$ with $\mathrm{Proj}\, k[Y_1, Y_2, Y_3]$ via the isomorphism t from $\mathcal{S}(H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1})))$ to $k[Y_1, Y_2, Y_3]$ such that $t(\zeta_i) = Y_i$. Then by (2) of Lemma 2.5, the unique element $(W_{[Q,\theta]})_T=\varepsilon^{-1}(\langle\zeta_{Q,P}\rangle)$ of $\mathrm{P}.S(L_Q, L_Q^{-1}; K\otimes L_P^{-1}\otimes L_Q^{-1})$ can be expressed as

$$(W_{[Q,\theta]})_T = \mathrm{Proj}\, k[Y_1, Y_2, Y_3]/\langle Y_2-\lambda_P Y_1, Y_3-\lambda_P^2 Y_1\rangle. \tag{5.18}$$

Now let $P^r=\mathrm{Proj}\, k[X_0, \cdots, X_r]$ $(r\geqq 3)$. Let $c_r: \mathrm{P.Ext}(K, K^{-1})_-\to P^r$ be the closed immersion defined by the natural surjection $c_r^{\#}: k[X_0, \cdots, X_r]\to k[X_0, \cdots, X_r]/\langle X_4, \cdots, X_r\rangle$. For $1\leqq s\leqq r$, put $R_s=k[X_1, \cdots, X_r, Y_1, \cdots, \check{Y}_s, \cdots, Y_r]/\langle X_j-Y_jX_{s\, j\neq s}\rangle$, and let $\rho_r: \tilde{P}^r\to P^r$ be the blowing up with centre $c_r((W_{[Q]})_T)$ defined by the natural homomorphism $\rho_r^{\#}: k[X_1, \cdots, X_r]\to R_s$. Moreover, let $d_r: \mathrm{P.Ext}(L_Q, L_Q^{-1})\to\tilde{P}^r$ be the closed immersion defined by the natural surjection $d_r^{\#}: R_s\to R_s/\langle\bar{X}_s, \bar{Y}_4, \cdots, \bar{Y}_r\rangle$ $(1\leqq s\leqq 3)$, where $\bar{X}_i$ (resp. $\bar{Y}_i$) is the image of X_i (resp. Y_i) under the natural map from $k[X_1, \cdots, X_r, Y_1, \cdots, \check{Y}_s, \cdots, Y_r]$ to R_s. Then by (5.16), (5.17), and (5.18), for any C_θ containing $(W_{[Q]})_T$, we have

$$d_r(\{(W_{[Q,\theta]})_T\}) = \widetilde{c_r(C_\theta)}\cap\rho_r^{-1}(\{c_r((W_{[Q]})_T)\}) \tag{5.19}$$

where $\widetilde{c_r(C_\theta)}$ denotes the proper transform of $c_r(C_\theta)$.

LEMMA 5.5. *Let* $m_1=\dim H^1(C, \mathrm{Hom}(K^p, K))_{j(i)}$, *and* $n_1=\dim H^0(C, \mathrm{Hom}(K^{-p}, K))_i$. *Let F be as above. Then for any $\theta\in J(2)$, there is a matrix F_θ satisfying* (4.13), *and the following condition.*

(5.20) *There are integers* $1\leqq k\leqq m_1$, $m_1+1\leqq l$, $m\leqq 2p+1$, *and* $1\leqq n\leqq n_1$ *snch that* $\det F_\theta(k, l, m; n)\notin\mathfrak{P}_\theta$, *and* $\det F_\theta(k, l;)$, $\det F_\theta(k, m;)$ *generate the ideal* $\mathfrak{P}_\theta\mathcal{O}_{C_\theta}$.

PROOF. Let $L_\theta=K\otimes L_P^{-1}\otimes L_Q^{-1}$, and let x_P, x_Q be as above. For the affine open covering $\mathcal{U}=\{U_1, U_2\}$ such that $U_1=U_P$, $U_2=U_{P_0}$, take the basis $\{\omega_k\}_{k=1}^3$ for $H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1}))$ given by $\omega_k=\iota_{\mathcal{U}}([x_Qx_P^{-k}y])$. Moreover take the bases $\{v_j\}_{j=1}^{(p-5)/2}$ and $\{\eta_i\}_{i=1}^{(p-1)/2}$ for $H^0(C, \mathrm{Hom}(L_Q^{-p}, L_Q^{-1}))_-$ and $H^1(C, \mathrm{Hom}(L_Q^p, L_Q^{-1}))_+$ given by $v_j=x_Q^{(-p+1)/2}x_P^{j-1}y$ and $\eta_i=\iota_{\mathcal{U}}([x_Q^{(p+1)/2}x_P^{-i}])$, respectively. Moreover, for each $1\leqq l\leqq 3$, take the elements $\{u_{lj}\}_{j=1}^{(p+3)/2}$ of $\tilde{\Gamma}(\mathcal{U}, \mathrm{Hom}(L_Q^{-p}, V_l))$ given by

$$u_{lj}=\begin{cases}\left\{\begin{pmatrix}0\\ x_Q^{-(p+1)/2}x_P^{j-1}\end{pmatrix}, \begin{pmatrix}-x_Q^{(-p+1)/2}x_P^{j-l-1}y\\ x_Q^{-(p+1)/2}x_P^{j-1}\end{pmatrix}\right\} & \text{if } j-l-1\geqq 0\\ \left\{\begin{pmatrix}x_Q^{(-p+1)/2}x_P^{j-l-1}y\\ x_Q^{-(p+1)/2}x_P^{j-1}\end{pmatrix}, \begin{pmatrix}0\\ x_Q^{-(p+1)/2}x_P^{j-1}\end{pmatrix}\right\} & \text{otherwise.}\end{cases}$$

Moreover take the elements $\{\{\Xi_{li}^{\mu\nu}\}\}_{i=1}^{(p+3)/2}$ of $\tilde{Z}^1(\mathcal{U}, \mathrm{Hom}(L_Q^p, V_l))$ such that $\Xi_{li}^{12}=\begin{pmatrix}0\\ x_Q^{(p-1)/2}x_P^{-i}y\end{pmatrix}$. We identify $\mathrm{P.Ext}(L_Q, L_Q^{-1})$ with $\mathrm{Proj}\, k[X_1, X_2, X_3]$ via the isomorphism t from $\mathcal{S}(H^1(C, \mathrm{Hom}(L_Q, L_Q^{-1})))$ to $k[X_1, X_2, X_3]$ such that $t(\omega_i)=X_i$, and with respect to the above bases, let us construct an FM matrix $\bar{F}'$ of the type $(L_Q, +)$. Then by a calculation similar to that in the proof of Lemma 5.3,

$$\bar{F}' \underset{\text{Dia.}}{\widetilde{\quad}} \left(\begin{array}{cccc|ccc} * & & X_1^p & 0 & & & \\ & \cdot^{\cdot^{\cdot}} & & 0 & & 0 & \\ X_1^p & & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & & & \\ \hline & & & X_1^p(X_2+bX_3) & & * & \\ & * & & X_1^pX_3 & & * & \\ & & & 0 & & & X_1^p \\ & & & 0 & & X_1^p \cdot^{\cdot^{\cdot}} & 0 \\ & & & 0 & X_1^p & & \end{array}\right) \bmod \langle X_2^p, X_3^p\rangle \text{ with } b\in k\,.$$

By (2) of Lemma 2.5, and Lemma 2.8, the maximal ideal $\mathfrak{m}_{Q,\theta}$ of the local ring of $\mathrm{P.Ext}(L_Q, L_{\bar{Q}}^{-1})$ at $(W_{[Q,\theta]})_T$ is generated by X_2, X_3. Thus $\det \bar{F}'((p+3)/2,(p+5)/2,(p+7)/2\,;\,(p+3)/2)\notin \mathfrak{m}_{Q,\theta}$, and $\det \bar{F}'((p+3)/2,(p+5)/2\,;\,)$ and $\det \bar{F}'((p+3)/2,(p+7)/2\,;\,)$ generate $\mathfrak{m}_{Q,\theta}$. Thus the assertion follows from Corollary to Lemma 5.3, Remark 5.4, and Nakayama's lemma for the case where $(K, i)=(K, +)$. Similarly, the assertion holds for the case where $(K, i)=(K, -)$.

PROOF OF PROPOSITION 5.1. Let $s=m_1n_1+4m_2n_1+m_2n_2$. Then for each $P=(a_{11}, \cdots, a_{m_1n_1}, b_{110}, \cdots, b_{m_2n_13}, d_{11}, \cdots, d_{m_2n_2})$ of k-valued point of the affine space A^s, define a matrix $F^*(P)$ by

$$F^*(P) = F+\begin{pmatrix} (a_{ij}X_4^p) & 0 \\ \left(\sum_{k=0}^{3} b_{ijk}X_kX_4^p\right) & (d_{ij}X_4^p)\end{pmatrix}.$$

Then the function $g_\theta(P)=i(\mathfrak{J}(F^*(P))\,;\,C_\theta^*)$ defines an upper semi-continuous function on A^s, which will be also denoted by g_θ. Moreover by (4.13) and Lemma 5.5, the open set $V_\theta=\{P\in A^s\,;\,g_\theta(P)\leqq p\}$ of A^s is non-empty for any $\theta\in J(2)$. Thus there is a matrix F^* satisfying (5.4). Clearly this F^* satisfies (5.3). This proves the assertion.

PROPOSITION 5.6. *Let F^* be the matrix in Proposition* 5.1. *Then*

$$i(\langle \mathfrak{q}(F^*), X_4\rangle\,;\,c(P)) = i(\mathfrak{J}(F)\,;\,P) \tag{5.21}$$

if $P\notin C_\theta$ for any $\theta\in J(2)$.

$$i(\langle \mathfrak{q}(F^*), X_4\rangle\,;\,c((W_{[Q]})_T)) \geqq \sum_P i(\mathfrak{J}(\bar{F}_Q)\,;\,P)+n(\bar{F}_Q)-\sum_\theta i_\theta \tag{5.22}$$

where $\bar{F}_Q$ is an FM *matrix of the type (L_Q, i), P runs over all points of* $\mathrm{P.Ext}(L_Q, L_{\bar{Q}}^{-1})$ *other than $(W_{[Q,\theta]})_T$, and θ runs over all elements of $J(2)$ such that C_θ contains $(W_{[Q]})_T$.*

PROOF. (5.21) can be easily proved. To prove (5.22), let G, $\bar{F}$ be the matrices in Lemma 5.3. For the matrix G, we define a matrix G^{**} in $k[X_0, \cdots, X_4, T_{11}, \cdots, T_{2p+1,2p-1}]$ in the same manner as F^{**} for F in the proof of Proposition 5.2. Let $r=(2p+1)(2p-1)+4$. Let $\mathfrak{q}(G^{**})$ be the intersection of all the primary components of the ideal $\langle \mathfrak{J}(G^{**}), T_{11}, \cdots, T_{2p+1,2p-1}\rangle$ whose radicals are different from $(c_r^{\#})^{-1}(\mathfrak{P}_\theta)$ (for the definition of $c_r^{\#}$, see Remark 5.4). Then, by Theorem 1.5, the zero-dimensional cycle C' on P^r defined by $C'=\sum_P i(\langle \mathfrak{q}(G^{**}), X_4\rangle ; P)P$ can be expressed as

$$(5.23)\qquad C' = \sum_{i=1}^{2p-1}\sum_{j=1}^{i}(G^{**}_{(3,i)}-G^{**}_{(1,i-1)})(G^{**}_{(2,j)}-G^{**}_{(1,j-1)})G^{**}_{(1,j)}T_{11}\cdots T_{2p+1,2p-1}X_4 - \sum_{\theta\in J(2)} i_\theta c_r(C_\theta)X_4 \qquad \text{(cf. Proposition 5.2).}$$

On the other hand, let $\rho_r: \tilde{P}^r\to P^r$ be the blowing up with centre $c_r((W_{[Q]})_T)$. For each closed subscheme H of P^r, we denote by $\mathcal{J}_{\tilde{H}}$ the ideal sheaf of $\mathcal{O}_{\tilde{P}^r}$ defining the proper transform $\tilde{H}$ of H under ρ_r. Let

$$\tilde{\mathfrak{J}}(G^{**}) = \sum_{1\le i,j\le 2p+1}\mathcal{J}_{\widetilde{\det G^{**}(i,j;\ )}} + \sum_{1\le i\le 2p+1}\sum_{1\le j\le 2p-1}\mathcal{J}_{\tilde{T}_{ij}},$$

and for each P of $\tilde{P}^r$, let $\tilde{\mathfrak{q}}(G^{**})_P$ be the intersection of all the primary components of the stalk $\tilde{\mathfrak{J}}(G^{**})_P$ whose radicals are different from $(\mathcal{J}_{\widetilde{c_r(C_\theta)}})_P$ for any $\theta\in J(2)$. Then, by Lemma 1.1, and (5.16), (5.17), analogously to (5.23), the zero-dimensional cycle $\tilde{C}'$ on $\tilde{P}^r$ defined by

$$\tilde{C}' = \sum_P i(\langle \tilde{\mathfrak{q}}(G^{**})_P, (\mathcal{J}_{\tilde{X}_4})_P\rangle ; P)\,P$$

can be expressed as

$$(5.24)\qquad \tilde{C}' = \sum_{i=1}^{2p-1}\sum_{j=1}^{i}(\tilde{G}^{**}_{(3,i)}-\tilde{G}^{**}_{(1,i-1)})(\tilde{G}^{**}_{(2,j)}-\tilde{G}^{**}_{(1,j-1)})\tilde{G}^{**}_{(1,j)}\tilde{T}_{11}\cdots \tilde{T}_{2p+1,2p-1}\tilde{X}_4 - \sum_{\theta\in J(2)} i_\theta \widetilde{c_r(C_\theta)}\,\tilde{X}_4.$$

We note that for any divisor H on P^r, we have $\rho_r^* H=\tilde{H}+mE$, where m denotes the multiplicity of H at $c_r((W_{[Q]})_T)$, and E denotes the exceptional divisor. Thus, analogously to Example 7.1.11 in Fulton [2],

$$(5.25)\quad i(\langle \mathfrak{q}(G^{**}), X_4\rangle ; c_r((W_{[Q]})_T)) = \sum_{i=1}^{2p-1}\sum_{j=1}^{i}(m_{3,i}-m_{1,i-1})(m_{2,j}-m_{1,j-1})m_{1,j} - \sum_{\theta\in J(2)} i_\theta m_\theta + \sum_{P\in \operatorname{supp} E} i(\langle \tilde{\mathfrak{q}}(G^{**})_P, (\mathcal{J}_{\tilde{X}_4})_P\rangle ; P),$$

where m_{ij}, and m_θ denote the multiplicities of the divisor $G^{**}_{(i,j)}$ and the curve $c_r(C_\theta)$ at $c_r((W_{[Q]})_T)$, respectively. By (5.5), the first term on the right hand side of (5.25) is equal to $n(\bar{F})$. By (5.16), and (5.17), $m_\theta=1$ or 0 according as C_θ contains $(W_{[Q]})_T$ or not. Moreover by (5.19), if P belongs to $V_{\bar{F}}$, and $P\neq (W_{[Q,\theta]})_T$ for any $\theta\in J(2)$, $d_r(P)$ belongs to $\operatorname{supp}\tilde{C}'\cap\operatorname{supp}E$. On the other hand, if $\tilde{P}\notin\bigcup_{\theta\in J(2)}\widetilde{c_r(C_\theta)}$, then we have $i(\langle\tilde{\mathfrak{q}}(G^{**})_{\tilde{P}}, (\mathcal{J}_{\tilde{X}_4})_{\tilde{P}}\rangle ; \tilde{P})=i(\langle\tilde{\mathfrak{J}}(G^{**})_{\tilde{P}}, (\mathcal{J}_{\tilde{X}_4})_{\tilde{P}}\rangle ; \tilde{P})$. Thus by (5.5), we have $i(\langle\tilde{\mathfrak{q}}(G^{**})_{d_r(P)}, (\mathcal{J}_{\tilde{X}_4})_{d_r(P)}\rangle ; d_r(P))\geqq i(\mathfrak{J}(\bar{F}) ; P)$

for any $P\in V_{\bar{F}}$ such that $P\neq(W_{[Q,\theta]})_T$ for any $\theta\in J(2)$. This proves the assertion.

PROOF OF THEOREM A. By Theorems 2.1, 3.2, and 4.1, and Propositions 5.1, 5.2, and 5.6, we have

$$\begin{aligned}(5.26)\qquad &\mathrm{Irr}(\pi_1(C),\ SL_2(F_p))\\ &\leqq 1/2(n(F_K(+))+n(F_K(-))-\sum_{Q\in\mathcal{W}}(n(F_{L_Q}(+))+n(F_{L_Q}(-)))\\ &\quad+\sum_{\theta\in J(2)}p(n_\theta-\deg C_\theta))\end{aligned}$$

where $F_L(l)$ is an FM matrix of the type (L, l), and $n_\theta=\#\{Q\in\mathcal{W};\ (W_{[Q]})_T\in C_\theta\}$. By Proposition 2.9, $n_\theta=6$ or 2 according as $\theta=\theta_0$ or not, and by (5.16), (5.17), $\deg C_\theta=3$ or 1 according as $\theta=\theta_0$ or not. To calculate $n(F_L(l))$, let $m_1=\dim H^1(C, \mathrm{Hom}(L^p, L))_{j(l)}$ and $n_1=\dim H^0(C, \mathrm{Hom}(L^{-p}, L))_l$. Then by Definition 4.5, we have

$$(5.27)\qquad n(F_K(l))=\sum_{i=1}^{m_1+n_1}\sum_{j=1}^{i}(p+e_{3,i})(p+e_{2,j})(pj+e_{1,j})$$

where $e_{3,i}=1$ or 0 according as $n_1-2\leqq i\leqq n_1$, $e_{2,j}=1$ or 0 according as $n_1-1\leqq j\leqq n_1$, and $e_{1,j}=1$ or 0 according as $j\geqq n_1$ or not. Thus we have

$$(5.28)\qquad n(F_K(l))=4/3\ p^6+5/3\ p^4+2p^3n_1-p^3+2pn_1-p+1.$$

Similarly, we have

$$(5.29)\qquad n(F_{L_Q}(l))=\sum_{i=1}^{m_1+n_1}\sum_{j=1}^{i}(p+f_{3,i})(p+f_{2,j})pj$$

where $f_{3,i}=1$ or 0 according as $n_1-1\leqq i\leqq n_1$ or not, and $f_{2,j}=1$ or 0 according as $j=n_1$ or not. Thus we have

$$(5.30)\qquad n(F_{L_Q}(l))=p^6/6-p^4/6+p^3n_1+pn_1.$$

We have $n_1=p+2$, $p-1$, $(p+3)/2$ or $(p-3)/2$ according as $(L, l)=(K, +)$, $(K, -)$, $(L_Q, +)$ or $(L_Q, -)$. Thus Theorem A is proved.

REMARK 5.7. The above estimate is weaker than the one which was announced in [9]. In a sequel paper, we shall give a complete proof for this stronger estimate.

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