The isometry groups of manifolds admitting nonconstant convex functions

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A function $\phi: M \to R$ on a Riemannian manifold M is by definition *convex* if, for every geodesic segment $\gamma: [a, b] \to M$, the function $\phi \circ \gamma: [a, b] \to R$ is convex in the usual sense, i. e. $(\phi \circ \gamma)(\lambda t + (1-\lambda)s) \le \lambda[(\phi \circ \gamma)(t)] + (1-\lambda)[(\phi \circ \gamma)(s)]$ for all $t, s \in [a, b]$ and $\lambda \in [0, 1]$. Convex functions on Riemannian manifolds arise naturally in a number of geometric contexts, and the existence of convex functions of certain types can often be used to produce information about the structure of the manifold itself. Specifically, if M is a complete Riemannian manifold and if $\phi: M \to R$ is a convex function, then there is a C^∞ manifold N such that $M - \{x \in M \mid \phi(x) = \inf_M \phi\}$ is diffeomorphic to the product manifold $N \times R$ ([4], [5]). In particular, if the minimum set $\{x \in M \mid \phi(x) = \inf_M \phi\}$ is empty, then M itself is diffeomorphic to such a product $N \times R$. It is this case of empty minimum set and with, moreover, the manifold N compact that will be considered now and throughout this paper. The C^∞ product structure $N \times R$ on such a manifold M is obtained as a smoothing of a topological product structure that corresponds to the level sets of ϕ; specifically, there is a homeomorphism $H: M \to N \times R$ such that ϕ is constant on $H^{-1}(N \times \{\alpha\})$ for each $\alpha \in R$ and $H^{-1}(N \times \{\alpha\}) = \{x \in M \mid \phi(x) = \text{the value of } \phi \text{ on } H^{-1}(N \times \{\alpha\})\}$.

It is not necessarily the case that such an M has a product metric structure; for instance, the function e^x is convex with empty minimum set on the surface of revolution $\{(x, y, z) \in R^3 \mid x^2 + y^2 = e^x\}$ which is not metrically a product. This surface of revolution has as cross sections the level sets of the convex function, and these increase in size as the function increases. The standard (mean curvature) formula for first variation of hypersurface area combined with the observation that the mean curvature of a convex hypersurface is nonnegative shows that this size increase phenomenon extends to the general situation, at least in the case of smooth ϕ; more precisely, if ϕ is C^∞, then the $(n-1)$-volume, $n = \dim M$, of the smooth submanifold $\{x \in M \mid \phi(x) = \alpha\}$ is a nondecreasing function of α, $\alpha \in \phi(M)$. This result holds even when ϕ is not smooth, if $(n-1)$-volume is interpreted as $(n-1)$-Hausdorff measure ([2]). The volumes of the level sets can be far from constant, as shown in the example given. But

intuition suggests that, if these volumes are nonconstant (as a function of α) then the isometry group of M would be compact because the part of the topological cylinder $N \times \boldsymbol{R}$ with smaller cross section could not be isometric to a part with larger cross section. The main result of this paper is that this latter intuition is true in precise form, and that in case of noncompact isometry group M is in fact a metric product.

THEOREM. *If M is a complete Riemannian manifold with noncompact isometry group, and if there is a convex function $\phi : M \to \boldsymbol{R}$ with $\{x \in M \mid \phi(x) = \inf_M \phi\} = \varnothing$ and $\{x \in M \mid \phi(x) = \alpha\}$ compact for all $\alpha \in \boldsymbol{R}$, then M is isometric to the product $N \times \boldsymbol{R}$ of a compact C^∞ Riemannian manifold N and the real line $\boldsymbol{R}$.*

If a Riemannian manifold M is isometric to a product $N \times \boldsymbol{R}$ via an isometry $H : M \to N \times \boldsymbol{R}$, then each $H^{-1}(N \times \{\alpha\})$, $\alpha \in \boldsymbol{R}$, is a totally geodesic submanifold. And if $\phi : M \to \boldsymbol{R}$ is convex, then the function $\phi_\alpha : N \to \boldsymbol{R}$ defined by $\phi_\alpha(x) = \phi(H^{-1}(x, \alpha))$ is a convex function on N. If N is compact, then each ϕ_α is constant. If $n = \dim M \geq 2$, then dimension N is at least one, and thus the constancy of a ϕ_α implies the constancy of ϕ on a (non-constant) geodesic segment in M. A convex function $\phi : M \to \boldsymbol{R}$ on (an arbitrary Riemannian manifold) M is *strictly convex* by definition if for every nonconstant geodesic segment $\gamma : [a, b] \to M$, the strict inequality $(\phi \circ \gamma)(\lambda a + (1 - \lambda)b) < \lambda[(\phi \circ \gamma)(a)] + (1 - \lambda)[(\phi \circ \gamma)(b)]$ holds for every $\lambda \in (0, 1)$. Note that a strictly convex function cannot be constant on a nonconstant geodesic segment. Combining these observations yields the following corollary of the theorem:

COROLLARY. *If M is a complete Riemannian manifold, and if there exists on M a strictly convex function $\phi : M \to \boldsymbol{R}$ such that $\{x \in M \mid \phi(x) = \inf_M \phi\} = \varnothing$ and such that $\{x \in M \mid \phi(x) = \alpha\}$ is compact for all $\alpha \in \boldsymbol{R}$, then the isometry group of M is compact.*

This special case of the theorem was previously established by Yamaguchi ([8]).

The general idea of the proof of the theorem is to make use of the deformation retraction of $\{x \in M \mid \phi(x) \geq \alpha\}$ onto $\{x \in M \mid \phi(x) = \alpha\}$, $\alpha \in \phi(M)$, developed in [4]; from a refined version of the arguments there, a retraction will be obtained that is distance nonincreasing (this is closely related to constructions in [2], also). It follows then that an $(n-1)$-dimensional manifold in $\{x \in M \mid \phi(x) \geq \alpha\}$, $\alpha \in \phi(M)$, that is in the $\boldsymbol{Z}_2$ homology class of $\{x \in M \mid \phi(x) = \alpha\}$ has $(n-1)$-dimensional volume at least as great as the volume of $\{x \in M \mid \phi(x) = \alpha\}$. Here $(n-1)$-dimensional volume is in the sense of Hausdorff measure. This fact will be seen to imply that all the level sets $\{X \in M \mid \phi(x) = \alpha\}$ has equal $(n-1)$-volume if the isometry group of M is noncompact. Moreover, it will follow in

this latter case that each level set is absolutely area minimizing in its homology class. Then it will be shown that consequently each level set must be a C^{∞} totally geodesic submanifold. The establishment of this fact uses the result of [3] on the Hausdorff dimension of the singular set of mod 2 absolutely minimizing current. In the case of the convex function ϕ being smooth (even just C^2), it is only a matter of calculation to see that M must have a metric product structure if the level sets of ϕ are totally geodesic (cf., the first result and remarks following it in [1]). In the general case wherein ϕ need not be C^2, the desired result on metric product structure is established by consideration of suitable smooth approximations of ϕ.

The carrying out in detail of this program for the proof of the theorem will depend on the establishment of two lemmas, which will be established independently of the main line of the argument and which seem to be of independent interest, also:

LEMMA 1. *If $\phi : M \rightarrow \mathbf{R}$ is a convex function on a complete Riemannian manifold, and if $\{x \in M \mid \phi(x) = \alpha\}$ is compact for all $\alpha \in \phi(M)$, then, for each $\alpha \leq \phi(M)$ there exists a distance nonincreasing retraction of $\{x \in M \mid \phi(x) \geq \alpha\}$ onto $\{x \in M \mid \phi(x) = \alpha\}$.*

LEMMA 2. *If $\phi : M \rightarrow \mathbf{R}$ is a convex function of a complete Riemannian manifold, if $\{x \in M \mid \phi(x) = \inf_M \phi\}$ is empty, and if for each $\alpha \in \phi(M)$ $\{x \in M \mid \phi(x) = \alpha\}$ is a compact totally geodesic hypersurface, then M is isometric to a Riemannian product $N \times \mathbf{R}$, where N is a compact connected C^{∞} Riemannian manifold, via an isometry $I : M \rightarrow N \times \mathbf{R}$ such that for each $\beta \in \mathbf{R}$ $I^{-1}(N \times \{\beta\})$ is a level set $\{x \in M \mid \phi(x) = \alpha\}$ for some unique $\alpha \in \phi(M)$.*

Lemma 1 is a global statement by nature. Lemma 2, on the other hand, will be obtained by applying globally an argument that is essentially local in character, and consequently various local versions of it also hold. The details of this point will become clear as the proof of Lemma 2 is presented.

Section 1 presents the proof of the main theorem, Lemmas 1 and 2 being assumed. Lemma 1 is proved in Section 2, and Lemma 2 in Section 3.

Throughout, the following notations will be used for brevity: with $\phi : M \rightarrow \mathbf{R}$ fixed by the context, $M_{\beta}^{\alpha} = \{x \in M \mid \beta \leq \phi(x) \leq \alpha\}$, $M_{\alpha} = \{x \in M \mid \phi(x) \geq \alpha\}$ and $M^{\alpha} = \{x \in M \mid \phi(x) \leq \alpha\}$.

1. Proof of the main theorem.

Throughout this section, ϕ and M will be as in the hypothesis of the theorem: M is to be a complete Riemannian manifold, $\phi: M \rightarrow \mathbf{R}$ a convex function without minimum and with compact level sets.

In [4], it was shown that such an M is the product $N \times \mathbf{R}$ of a compact Riemannian manifold N and the real line $\mathbf{R}$. It is actually enough to prove the theorem for the case that M, and hence N, are orientable. To see that it is enough to treat this case, suppose M is not orientable. The orientable two-fold cover $C: \tilde{M} \rightarrow M$ trivially satisfies the hypothesis of the theorem (with $\phi \circ C$ as the convex function). If the isometry group of $\tilde{M}$ is compact then the isometry group of M must also be compact because every isometry of M "lifts" to be one of $\tilde{M}$. On the other hand, if $\tilde{M}$ is isometric to a product $N_1 \times \mathbf{R}$, then M must also be isometric to a product. Indeed, in this case, the nontrivial covering transformation of $\tilde{M}$ leaves the convex function invariant and can be considered as an isometry of the Riemannian product $N_1 \times \mathbf{R}$. Recall that the convex function on $\tilde{M} = N_1 \times \mathbf{R}$ is constant on $N_1 \times \{a\}$ for every $a \in \mathbf{R}$. The nontrivial covering transformation is identity on $\mathbf{R}$ because otherwise the function has a non-connected level set and from Theorem A of [4], this implies the existence of minimum set of the convex function, a contradiction. It then follows that M is isometric to $N \times \mathbf{R}$, where N is N_1 modulo the action of the covering transformation. Thus from now on, only the case of M and N orientable need be and will be considered.

A diffeomorphism of a product $N \times \mathbf{R}$, N compact, to itself necessarily takes $N \times \{a\}$, $a \in \mathbf{R}$, to a compact submanifold of $N \times \mathbf{R}$ that separates $N \times \mathbf{R}$ into two components that both have noncompact closures in $N \times \mathbf{R}$. Each of these components determines an end of $N \times \mathbf{R}$, and thus there is an obvious sense in which the diffeomorphism either preserves the two ends of $N \times \mathbf{R}$ or interchanges the two ends. The product of two end-preserving diffeomorphisms is end-preserving, of an end-preserving and an end-interchanging is end-interchanging, and of two end-interchanging is end-preserving. In particular, the isometry group $I(M)$ of the manifold M, diffeomorphic but not necessarily isometric to $N \times \mathbf{R}$, can be decomposed into two disjoint subsets $I_0(M)$ and $I_1(M)$, consisting respectively of the end-preserving and end-interchanging elements of $I(M)$. The set $I_0(M)$ is in fact a subgroup of $I(M)$. Moreover, if $I_0(M)$ is compact, then so is $I(M)$ because $I_1(M)$ is then either empty or equals $\{\alpha\beta \mid \alpha \in I_0(M)\}$, where β is a fixed element of $I_1(M)$. To prove the theorem it is thus necessary only to prove that if $I_0(M)$ is noncompact, then M is a metric product (with $\mathbf{R}$). From now on, $I_0(M)$ will be supposed noncompact. The proof that the metric on M is a product in this case will be given as a sequence of numbered steps.

(1) *If* $\alpha \in \phi(M)$, $p \in M$ *and* $\phi(p) = \alpha$, *then*

$$\limsup \{ d(\eta(p), M_\alpha^\alpha(\phi)) \mid \eta \in I_0(M), \ \phi(\eta(p)) > \alpha \} = \infty,$$

where d is the Riemannian distance on M.

PROOF OF (1). Let $D_\alpha =$ the Riemannian diameter of $M_\alpha^\alpha(\phi)$; then $D_\alpha < \infty$ because $M_\alpha^\alpha(\phi)$ is compact. If $\eta \in I_0(M)$ satisfies $\phi(\eta(p)) < \alpha$, and if $d(\eta(p), M_\alpha^\alpha(\phi)) > D_\alpha$, then $\phi(\eta^{-1}(p)) > \alpha$. To see this, note first that p is in the component of $M - \eta(M_\alpha^\alpha(\phi))$ on which ϕ is unbounded above, because $M_\alpha(\phi) \cap \eta(M_\alpha^\alpha(\phi)) = \emptyset$ by the condition $d(\eta(p), M_\alpha^\alpha(\phi)) > D_\alpha$. Thus $\eta^{-1}(p)$ is in the component of $\eta^{-1}(M - \eta(M_\alpha^\alpha(\phi))) = M - M_\alpha^\alpha(\phi)$ on which ϕ is unbounded above, because η^{-1} is end-preserving. Hence $\phi(\eta^{-1}(p)) > \alpha$. Also $d(\eta^{-1}(p), M_\alpha^\alpha(\phi)) = d(p, \eta(M_\alpha^\alpha(\phi))) \geq d(p, \eta(p)) - D_\alpha \geq d(\eta(p), M_\alpha^\alpha(\phi)) - D_\alpha$.

Because $I_0(M)$ is noncompact, there is a sequence $\{\gamma_i\}$, $\gamma_i \in I_0(M)$ such that a subsequence of $\{\gamma_i(p)\}$ is unbounded. In particular, $d(\gamma_i(p), M_\alpha^\alpha(\phi)) > D_\alpha$ for i large. Then the sequence $\{\eta_i \mid \eta_i = \gamma_i$ if $\phi(\gamma_i(p)) > \alpha$, $\eta_i = \gamma_i^{-1}$ if $\phi(\gamma_i(p)) < \alpha\}$ has the properties;

(a) $\phi(\eta_i(p)) > \alpha$, by the previous argument,

(b) $\lim d(\eta_i(p), M_\alpha^\alpha(\phi)) = +\infty$, by the earlier observation that $d(\eta_i(p), M_\alpha^\alpha(\phi)) \geq d(\gamma_i(p), M_\alpha^\alpha(\phi)) - D_\alpha$ if $\eta_i = \gamma_i^{-1}$ (and i is large enough) and of course also if $\eta_i = \gamma_i$.

(2) *If* $\alpha, \beta \in \phi(M)$, *then the* $(n-1)$-*volume* $\mathrm{Vol}(M_\alpha^\alpha(\phi))$ *of* $M_\alpha^\alpha(\phi)$ *equals the* $(n-1)$-*volume* $\mathrm{Vol}(M_\beta^\beta(\phi))$ *of* $M_\beta^\beta(\phi)$.

PROOF OF (2). Suppose without loss of generality that $\beta < \alpha$. Then, by Lemma 1 there is a distance nonincreasing map of $M_\alpha^\alpha(\phi)$ onto $M_\beta^\beta(\phi)$ that is the restriction to $M_\alpha^\alpha(\phi)$ of a retraction of $M_\beta(\phi)$ onto $M_\beta^\beta(\phi)$. In particular, since both $M_\beta^\beta(\phi)$ and $M_\alpha^\alpha(\phi)$ represent the same generator of the $(n-1)$-homology of $M_\beta(\phi)$, it follows that the image of $M_\alpha^\alpha(\phi)$ under the retraction must be (all of) $M_\beta^\beta(\phi)$. Thus

$$\mathrm{Vol}(M_\beta^\beta(\phi)) \leq \mathrm{Vol}(M_\alpha^\alpha(\phi)).$$

To establish the reverse inequality, choose a point $p \in M_\beta^\beta(\phi)$ and, by step (1), an $\eta \in I_0(M)$ such that $\phi(\eta(p)) > \alpha$ and $d(\eta(p), M_\alpha^\alpha(\phi)) > D_\beta$, where $D_\beta =$ the diameter of $M_\beta^\beta(\phi)$. Then every point $q \in \eta(M_\beta^\beta(\phi))$ can be connected to $\eta(p)$ by a curve in M of length $\leq D_\beta$ and this curve cannot intersect $M_\alpha^\alpha(\phi)$ because $d(\eta(p), M_\alpha^\alpha(\phi)) > D_\beta$. Hence $\eta(M_\beta^\beta(\phi))$ must be contained in $M_\alpha(\phi)$. Because $\eta(M_\beta^\beta(\phi))$ is not homologous to zero in $M_\alpha(\phi)$, the image of $\eta(M_\beta^\beta(\phi))$ under a retraction of $M_\alpha(\phi)$ onto $M_\alpha^\alpha(\phi)$ must be all of $M_\alpha^\alpha(\phi)$. In particular, this holds for the distance-nonincreasing retraction obtained from Lemma 1. Thus

$$\mathrm{Vol}(M_\beta^\beta(\phi)) = \mathrm{Vol}(\eta(M_\beta^\beta(\phi))) \geq \mathrm{Vol}(M_\alpha^\alpha(\phi)).$$

Hence $\mathrm{Vol}(M_\beta^\beta(\phi)) = \mathrm{Vol}(M_\alpha^\alpha(\phi))$.

(3) *For each $\alpha \equiv \phi(M)$, the Lipschitz submanifold $M_\alpha^\alpha(\phi)$ is absolutely $(n-1)$-volume minimizing among all homologically nontrivial rectifiable $(n-1)$-currents.*

PROOF OF (3). Suppose $C \subset M$ is homologically nontrivial. Then there is a compact subset K in M such that $C \cap K$ also represents a nontrivial $(n-1)$-homology class in M. (This is so by the observation that the homology class of a cycle C_1 in $N \times R$ is the limit of the classes $(i_a)_*(C_1 \cap N \times [-a, a])$, as $a \to \infty$, where i_a is the injection $N \times [-a, a] \to N \times R$.) In particular, $C \cap M_\beta(\phi)$ is homologically nontrivial for some $\beta \in \phi(M)$. By Lemma 1, there is a distance nonincreasing retraction of $M_\beta(\phi)$ onto $M_\beta^\beta(\phi)$; the image of C under this retraction is an $(n-1)$-current C_1 in $M_\beta^\beta(\phi)$ that is homologically nontrivial. In particular, the $(n-1)$-mass, or $(n-1)$-volume, of C_1 is at least $\mathrm{Vol}(M_\beta^\beta(\phi))$, i. e., $\mathrm{Vol}(C_1) \geq \mathrm{Vol}(M_\beta^\beta(\phi))$. Because the retraction is distance nonincreasing, $\mathrm{Vol}(C) \geq \mathrm{Vol}(C_1)$. But, by Step (2), $\mathrm{Vol}(M_\beta^\beta(\phi)) = \mathrm{Vol}(M_\alpha^\alpha(\phi))$, so $\mathrm{Vol}(C) \geq \mathrm{Vol}(C_1) \geq \mathrm{Vol}(M_\beta^\beta(\phi)) = \mathrm{Vol}(M_\alpha^\alpha(\phi))$.

(4) *For each $\alpha \equiv \phi(M)$, the set $M_\alpha^\alpha(\phi)$ is a compact C^∞ totally geodesic submanifold of M.*

PROOF OF (4). The property of being C^∞ totally geodesic is local so it is enough to establish the property in a neighborhood of each point $p \in M_\alpha^\alpha(\phi)$. If $M_\alpha^\alpha(\phi)$ is C^∞ in a neighborhood of p, then it is totally geodesic in that neighborhood: To see this, note that, because $M_\alpha^\alpha(\phi)$ is absolutely $(n-1)$-volume minimizing in its homology class (Step (3)), its second fundamental form has trace zero at each point in a neighborhood of which $M_\alpha^\alpha(\phi)$ is C^∞. On the other hand, because $M_\alpha^\alpha(\phi)$ is the boundary of a convex set, its second fundamental form (at C^∞ points of $M_\alpha^\alpha(\phi)$) is nonnegative definite. Hence the second fundamental form vanishes, and the submanifold is totally geodesic. Thus it is enough to show $M_\alpha^\alpha(\phi)$ is C^∞ in a neighborhood of each $p \in M_\alpha^\alpha(\phi)$.

In [6], it is shown that there is a local coordinate system $(x_1, \cdots, x_n)$ centered at $p \in M_\alpha^\alpha(\phi)$ such that $M_\alpha^\alpha(\phi) \cap (\text{the domain of the coordinate system})$ is the graph of a necessarily smooth function f over $\{(x_1, \cdots, x_n) \mid x_n = 0\}$. Thus coordinate system can be chosen so that its range is R^n. Then

$$M_\alpha^\alpha(\phi) \cap (\text{domain of the coordinates})$$

$$= \{(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) \mid (x_1, \cdots, x_{n-1}) \in R^{n-1}\}.$$

For convenience, suppose it is so chosen. Because ϕ is a locally Lipschitz continuous function, there is, for each fixed compact subset of R^{n-1}, a constant C such that the function f is Lipschitz continuous with the Lipschitz constant C on the compact subset.

Now set $Q=\{(x_1, \cdots, x_{n-1})\in R^{n-1} \mid f$ is C^∞ in a neighborhood of $(x_1, \cdots, x_{n-1})\}$. By the fundamental minimal hypersurface regularity result of [3], the set $\{(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) \mid (x_1, \cdots, x_{n-1})\in R^{n-1}-Q\}$ has Hausdorff dimension at most $n-7$. Hence the Hausdorff dimension of $R^{n-1}-Q$ is at most $n-7$. In particular, Q is dense in R^{n-1}.

Choose a point $q_0\in Q$, $q_0=(a_1, \cdots, a_{n-1})$ such that the point $q_1=(a_1, \cdots, a_{n-1}, f(a_1, \cdots, a_{n-1}))\in M_\alpha^\alpha(\phi)$ has the properties: (a) $d(q_1, p)<(1/10)$(injectivity radius at q_1 for the M-metric); (b) $\{x\in M \mid d(q_1, x)\leqq 2d(q_1, p)\}$ is contained in the domain of the $(x_1, \cdots, x_n)$ local coordinate system and (c) the projection of $\{(x_1, \cdots, x_n) \mid (x_1, \cdots, x_n)=\exp_{q_1}v, v\in T_{q_1}M_\alpha^\alpha(\phi), \|v\|<3d(q_1, p)\}$ onto R^{n-1} is a diffeomorphism. This choice of q_0 is possible because of the density of Q in R^{n-1}; because of the continuity of the map $Q\to M_\alpha^\alpha(\phi)$ given by $(x_1, \cdots, x_{n-1})\to$ $(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))$; and finally because of the Lipschitz continuity of f which implies that the tangent spaces of $M_\alpha^\alpha(\phi)$ are bounded away from containing the x_n direction. This last property is used to ensure property (c). With q_0 so chosen, each geodesic segment of length $2d(q_1, p)$ emanating from q_1 projects by suppression of the x_n-coordinate onto a curve segment in R^{n-1} emanating from q_0. These projections are pairwise disjoint except for their common initial point q_0. The set of such geodesic segments each with the property that the projection lies entirely in Q is open and dense in the set of all such geodesic segments; here the set of geodesic segments is topologized by identifying it with the set of unit vectors in $T_{q_1}M_\alpha^\alpha(\phi)$ and so with S^{n-2}, $n=$ $\dim M$. The openness is clear. To see the density, let $T=$ the set of geodesic segments that have a projection that intersects $R^{n-1}-Q$. Then, because R^n-Q has Hausdorff $(n-6)$-measure zero, so does T. In particular, T, which is a subset of S^{n-2}, cannot be open in S^{n-2}.

Suppose γ is a geodesic segment of length $2d(q_1, p)$ from q_1 tangent to $M_\alpha^\alpha(\phi)$ and that $\gamma\notin T$. Let

$$\gamma_0 = \{(x_1, \cdots, x_{n-1})\in R^{n-1} \mid (x_1, \cdots, x_{n-1}, x_n)\in \gamma \text{ for some unique } x_n\}.$$

If $(x_1, \cdots, x_{n-1})\in \gamma_0$, then $M_\alpha^\alpha(\phi)$ is C^∞ totally geodesic in a neighborhood of $(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))$. It follows that for all $(x_1, \cdots, x_{n-1})$ in some neighborhood V of γ_0, the subset $\{(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) \mid (x_1, \cdots, x_{n-1})$ $\in V\}$ of $M_\alpha^\alpha(\phi)$ coincides with a subset of $\{\exp_{q_1}w \mid w\in T_{q_1}M_\alpha^\alpha(\phi), \|w\|<3d(p, q_1)\}$. In particular, this subset of $M_\alpha^\alpha(\phi)$ contains γ. The density of the complement of T implies (by taking closures) that $M_\alpha^\alpha(\phi)$ contains $\{\exp_{q_1}w \mid w\in T_{q_1}M_\alpha^\alpha(\phi), \|w\|<2d(p, q_1)\}$. It follows that $M_\alpha^\alpha(\phi)$ is C^∞ totally geodesic in that neighborhood.

2. Proof of Lemma 1.

To prove Lemma 1, it is enough to construct, for each $\alpha \in \phi(M)$, a distance-nonincreasing retraction of $M_\alpha^{\alpha+1}(\phi)$ onto $M_\alpha^\alpha(\phi)$, for then a distance-nonincreasing retraction of $M_\alpha(\phi)$ onto $M_\alpha^\alpha(\phi)$ can be found by iteration. Specifically, if a distance-nonincreasing retraction $\eta_\alpha : M_\alpha^{\alpha+1}(\phi) \to M_\alpha^\alpha(\phi)$ is given for each $\alpha \in \phi(M)$, then a distance-nonincreasing retraction $\psi_\alpha : M_\alpha(\phi) \to M_\alpha^\alpha(\phi)$ can be constructed as follows: For each $p \in M_\alpha(\phi)$, let $k(p)$ be the largest integer such that $\phi(p) \geq \alpha + k(p)$. Define

$$\psi_\alpha(p) = (\eta_\alpha \circ \cdots \circ \eta_{\alpha+k(p)-1} \circ \eta_{\alpha+k(p)})(p) .$$

It is easily checked that $\psi_\alpha : M_\alpha(\phi) \to M_\alpha^\alpha(\phi)$ is a distance-nonincreasing retraction.

It is also enough to find the distance-nonincreasing retractions η_α, and hence the ψ_α, for the case $\alpha = \inf_M \phi$. (Of course in the present paper's applications, the case $\alpha \in \phi(M)$, $\alpha = \inf_M \phi$ does not occur: that $\inf_M \phi \notin \phi(M)$ is a hypothesis of the theorem. But Lemma 1 holds also in the case $\alpha = \inf_M \phi \in \phi(M)$.) To check that the $\alpha \neq \inf_M \phi$ case implies that $\alpha = \inf_M \phi$ case, suppose $\inf_M \phi \in \phi(M)$ and suppose given, for each $\alpha \in \phi(M) - \{\inf_M \phi\}$, a distance-nonincreasing retraction $\psi_\alpha : M_\alpha(\phi) \to M_\alpha^\alpha(\phi)$. The Arzela-Ascoli Theorem implies that there is a sequence $\{\alpha_j \in \phi(M) \mid j=1, 2, \cdots\}$ such that $\lim \alpha_j = \inf_M \phi$ and such that $\{\psi_{\alpha_j} \mid j=1, 2, \cdots\}$ converges uniformly on compact subsets of $\{x \in M \mid \phi(x) > \inf_M \phi\}$ to $\{x \in M \mid \phi(x) = \inf_M \phi\}$, and this map extends uniquely to be a continuous map on all of M; and this extension is a distance-nonincreasing retraction of $M = M_\alpha(\phi)$ onto $M_\alpha^\alpha(\phi)$, $\alpha = \inf_M \phi$.

Thus there remains only to construct a distance-nonincreasing retraction of $M_\alpha^{\alpha+1}(\phi)$ onto $M_\alpha^\alpha(\phi)$ for each $\alpha \in \phi(M) - \{\inf_M \phi\}$. For this construction, note first that there is a (large) positive integer N_0 such that, if $n \geq N_0$, then the subdivision α, $\alpha+(1/n)$, $\alpha+(2/n)$, $\cdots$, $\alpha+1$ of $[\alpha, \alpha+1]$ has the following property: If $p \in M$ is such that $\phi(p) \in [\alpha+(k/n), \alpha+((k+1)/n)]$, $k \in \{0, \cdots, n-1\}$, then there is a unique point q with $\phi(q) = \alpha+(k/n)$ and $d(p, q) = d(p, M_{\alpha+(k/n)}^{\alpha+(k/n)}(\phi))$. The existence of such an N_0 is proved in detail in [4]. The idea of the proof is as follows: By local strict convexity of the function $q \to d(p, q)^2$, one sees that if $p \in M_\beta(\phi)$ is close enough to the compact convex set $M_\beta^\beta(\phi)$, $\beta \in \phi(M)$, then there is a unique point $q \in M_\beta^\beta(\phi)$ with $d(p, q) = d(p, M_\beta^\beta(\phi))$. The closeness required can be taken uniformly in β for β varying over $[\alpha, \alpha+1]$. To see that $p \in M_{\alpha+k/n}^{\alpha+(k+1)/n}(\phi)$ implies that p is sufficiently close to $M_{\alpha+k/n}^{\alpha+k/n}(\phi)$ requires an estimate from below on the rate of change of ϕ along shortest geodesic connections from higher ϕ-levels to lower ϕ-levels. Specifically, one can show (and it is shown in [4]) that the following estimate holds: Let $\alpha > \inf_M \phi$ and let $\varepsilon \in (0, \alpha - \inf_M \phi)$. Set $\delta = \max\{d(x, y) \mid x, y \in M, \phi(x) = \alpha - \varepsilon, \phi(y) = \alpha\}$. Then $\delta > 0$;

and if $\beta_1 > \beta_2 = \alpha$, and if $\phi(z) = \beta_1$ then, with $\Delta = \varepsilon/\delta$,

$$d(z, M_{\beta_2}^{\beta_1}(\phi)) \leq (\beta_1 - \beta_2)/\Delta.$$

From this estimate, it follows that if N_0 is sufficiently large and if $n \geq N_0$, then the distance from $p \in M_{\alpha+k/n}^{\alpha+(k+1)/n}(\phi)$ to $M_{\alpha+k/n}^{\alpha+k/n}(\phi)$ is small, in particular so small that a unique point q with the indicated properties exists. It is easy to see that under these circumstances, q depends continuously on p (cf. [4]).

With N_0 as in the previous paragraph, define, for each $n > N_0$, a map $\eta_n : M_\alpha^{\alpha+1}(\phi) \to M_\alpha^\alpha(\phi)$ as follows: If $p \in M_\alpha^{\alpha+1}(\phi)$ and $\phi(p) \in [\alpha+k/n, \alpha+(k+1)/n]$, then let $q_k(p) =$ the unique point of $M_{\alpha+k/n}^{\alpha+k/n}(\phi)$ closest to p, $q_{k-1}(p) =$ the unique point of $M_{\alpha+(k-1)/n}^{\alpha+(k-1)/n}(\phi)$ closest to $q_k(p)$, $\cdots$, $q_0(p) =$ the unique point of $M_\alpha^\alpha(\phi)$ closest to $q_1(p)$. Then set $\eta_n(p) = q_0(p)$.

The maps η_n are (continuous) retractions of $M_\alpha^{\alpha+1}(\phi)$ onto $M_\alpha^\alpha(\phi)$. In euclidean space, the maps η_n would each be distance-nonincreasing by the convexity of the function ϕ and hence of all the sets $M^\beta(\phi)$, $\beta \in \phi(M)$. However the possible curvature of M makes it further possible that the η_n are not distance-nonincreasing. However, the maps η_n are, for all n sufficiently large, Lipschitz continuous; and, moreover, they have Lipschitz constants C_n such that $\lim \sup_{n \to +\infty} C_n \leq 1$: These facts will be established momentarily. Assuming these facts for the moment, one can find a distance-nonincreasing retraction $M_\alpha^{\alpha+1}(\phi)$ onto $M_\alpha^\alpha(\phi)$ by applying the Arzela-Ascoli Theorem to the sequence $\{\eta_n\}$: There is a uniformly convergent subsequence, and the limit of this subsequence is a retraction of $M_\alpha^{\alpha+1}(\phi)$ onto $M_\alpha^\alpha(\phi)$ that is necessarily distance-nonincreasing by virtue of the fact that $\lim \sup_{n \to +\infty} C_n \leq 1$.

Thus, to complete the proof of Lemma 1, there remains only to establish the already stated facts about the Lipschitz continuity of the η_n. These facts will be established by an argument closely related to the argument used in [4] to establish that the diameter of the sets $M_\alpha^\alpha(\phi)$ is a nondecreasing function of α. The presently required argument depends on two observations from Riemannian geometry:

OBSERVATION (1): *Suppose $\gamma : [a, b] \to M$ is a geodesic segment without self-intersections (in a complete Riemannian manifold) and that $p \in M - \gamma([a, b])$. Suppose also that $q \in \gamma([a, b])$ has the property that $d(p, q) = d(p, \gamma([a, b])) > 0$. Then one of the three possibilities occurs: (a) $q \in \gamma((a, b))$ and every minimal geodesic from p to q meets γ at a right angle; (b) $q = \gamma(a)$ and, for each minimal geodesic from p to q, the tangent T at q of the geodesic satisfies $\langle T, \dot\gamma(a) \rangle \geq 0$; (c) $q = \gamma(b)$ and, for each minimal geodesic from p to q, the tangent T of the geodesic at q satisfies $\langle T, \dot\gamma(b) \rangle \leq 0$.*

The observation (1) is an immediate consequence of the standard formula

for first variation of arc length; the situation is illustrated in Figure 1.

OBSERVATION (2): *Suppose $q \in M$ and that $\varepsilon > 0$ is so small that the ball of radius 5ε around q is strongly convex and so small that, for each arc-length-parameter geodesic segment $\gamma : (-5\varepsilon, 5\varepsilon) \to M$ with $\gamma(0) = q$, the exponential map of the normal bundle of γ is a diffeomorphism of the 5ε neighborhood of the 0-section onto an open subset of M. Suppose also that $0 < \varepsilon_1 < \varepsilon$, $0 < \varepsilon_2 < \varepsilon$, $0 < \delta < \min\{\varepsilon_1, \varepsilon_2\}$ and that $\gamma_1 : [-\delta, \delta] \to M$, $\gamma_2 : [0, \varepsilon_1] \to M$ and $\gamma_3 : [0, \varepsilon_2] \to M$ are arc-length-parameter geodesic segments with $\gamma_1(0) = q$, $\gamma_2(0) = \gamma_1(-\delta)$, $\gamma_3(0) = \gamma_1(\delta)$, $\langle \dot{\gamma}_2(0), \dot{\gamma}_1(-\delta) \rangle = 0$ and $\langle \dot{\gamma}_3(0), \dot{\gamma}_1(\delta) \rangle \geq 0$. (Figure 2.) Then there is a constant C that depends only on the supremum of the absolute values of the sectional curvature of M on the 10ε-ball around q such that*

$$d(\gamma_2(\varepsilon_1), \gamma_3(\varepsilon_2)) \geq 2\delta - C\delta(\max\{\varepsilon_1, \varepsilon_2\})^2.$$
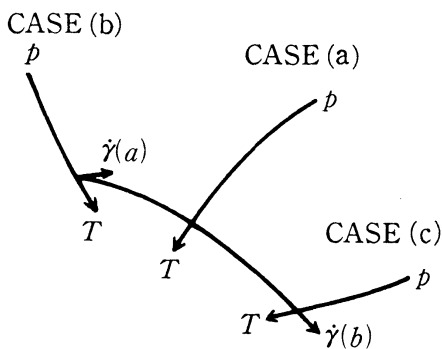

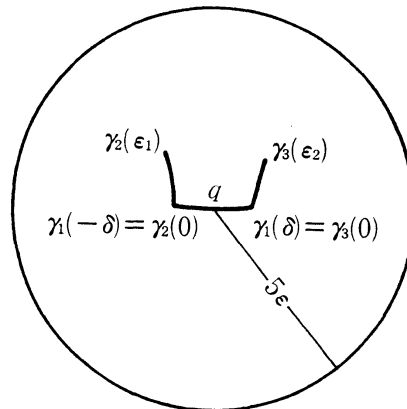Figure 1.   Figure 2.

This observation (2) can be thought of in the following terms; in euclidean space, the inequality would hold with $C = 0$ because of the hypothesis about the angles between γ_1, γ_2 and γ_1, γ_3. In the case of general Riemannian manifolds, the smallness restrictions on ε mean that one is dealing with local geometry only, so the only change from euclidean space is an error term estimatable by curvature bounds. For the actual proof, one proceeds as follows:

Let γ_4 be a minimal geodesic segment from $\gamma_2(\varepsilon_1)$ to $\gamma_3(\varepsilon_2)$. By the assumptions on angles between γ_1, γ_2 and γ_1, γ_3 and the choice of ε, there are points a_1 and a_2 of γ_4 such that the portion of γ_4 between a_1 and a_2 is (up to parametrization) the exponentiation of a C^∞ normal vector field along γ_1 with the normal vector field of length everywhere less than 5ε. Call this vector field $V(t) \in TM_{\gamma_1(t)}$, where t runs over $[-\delta, \delta]$. The length of the curve $t \to \exp_{\gamma_1(t)} V(t)$ is less than or equal to the length of γ_4, which latter equals $d(\gamma_2(\varepsilon_1), \gamma_3(\varepsilon_2))$. To estimate the length of $t \to \exp_{\gamma_1(t)} V(t)$, define, for each t, a Jacobi field $J_t(s)$

along the geodesic $s \to \exp_{\gamma_1(t)} sV(t)$, $s \in [0, 1]$, by

$$J_t(s) = \frac{\partial}{\partial t} \exp_{\gamma_1(t)} sV(t) .$$

Then $J_t(0) = \dot{\gamma}_1(t)$ and $J_t(1)$ is the tangent vector of $t \to \exp_{\gamma_1(t)} V(t) =$ the tangent vector of γ_4 at $\exp_{\gamma_1(t)} V(t)$. Then the length of $t \to \exp_{\gamma_1(t)} V(t) = \int_{-\delta}^{\delta} \|J_t(1)\| dt$. By the estimating from the Jacobi equation, one sees that

$$\|J_t(1)\| \geqq \|J_t(0)\| - C_1 \|V(t)\|^2 ,$$

where C_1 is a constant depending on the supremum of the absolute values of sectional curvatures along $s \to \exp_{\gamma_1(t)} sV(t)$, this curve being contained in the 10ε-ball around q by the triangle inequality. Note that, also by the triangle inequality, $\|V(t)\| \leqq 4\max(\varepsilon_1, \varepsilon_2)$. Thus

$$\text{length of } \gamma_4 \geqq \text{length of } (t \to \exp_{\gamma_1} V(t))$$
$$\geqq \int_{-\delta}^{\delta} \|J_t(0)\| dt - C_1 \int_{-\delta}^{\delta} \|V(t)\|^2 dt$$
$$\geqq 2\delta - C\delta (\max(\varepsilon_1, \varepsilon_2))^2 ,$$

where C depends only on sectional curvature absolute value supremum on the ball of radius 10ε around q.

Note that in observation (2) the value of C and the smallness of ε can both be chosen uniformly relative to variation of q over compact subset of M.

The observations just given apply in particular to the following situation: Suppose $\beta \in \phi(M)$ and p_1 and p_2 are points of $M_\beta(\phi)$ that are close to $M_\beta^\beta(\phi)$ and close to each other. Let $p_1' =$ the unique point of $M_\beta^\beta(\phi)$ closest to p_1, and $p_2' =$ the unique point of $M_\beta^\beta(\phi)$ closest to p_2. Set $\omega = \max(d(p_1, p_1'), d(p_2, p_2'))$. Then

$$d(p_1, p_2) \geqq d(p_1', p_2') - C\omega^2 d(p_1', p_2') ,$$

where C depends only on sectional curvature bounds. This inequality follows from observations (1) and (2). Observation (1) is to be applied with the geodesic segment γ there being the minimal one between p_1' and p_2' here, and this same minimal segment is to play the role of γ_1 in observation (2). The convexity of $M^\beta(\phi)$ implies that the whole segment from p_1' to p_2' lies in $M^\beta(\phi)$ and hence that (for instance) p_1' is the closest point of the segment to p_1; thus observation (1) does indeed apply and sets up the hypotheses for observation (2). The inequality for $d(p_1, p_2)$ just given implies immediately that closest point projection on $M_\beta^\beta(\phi)$ takes an arc of length ζ that is within ε (ε small) of $M_\beta(\phi)$ to an arc of length $\leqq \zeta(1 + C\varepsilon^2)$, where C depends only on sectional curvature bounds in a neighborhood of $M_\beta^\beta(\phi)$.

Now consider the previously discussed situation: $n \geq N_0$, $[\alpha, \alpha+1]$ partitioned into $[\alpha, \alpha+1/n]$, $[\alpha+1/n, \alpha+2/n]$, $\cdots$, $[\alpha+(n-1)/n, \alpha+1]$. If $p_1, p_2 \in M_\alpha^{\alpha+1}(\phi)$ and $\phi(p_1) \geq \phi(p_2)$ and if γ is a minimal geodesic segment joining p_1 to p_2, then $\gamma \subset M^{\phi(p_1)}(\phi)$. Also, for each $n \geq N_0$, the arc obtained by applying η_n to those portions of γ lying in $M_\alpha^{\alpha+1}(\phi)$ and fixing the remainder of γ (the remainder lies in $M^\alpha(\phi)$) joins $\eta_n(p_1)$ and $\eta_n(p_2)$. Thus $d(p_1, p_2) \geq$ the length of $(\eta_n \circ \gamma)$, where $\eta_n \circ \gamma$ is interpreted as indicated. The map η_n is obtained as the composition of n closest-point projections $M^{\alpha+1}(\phi) \to M^{\alpha+1-(1/n)}(\phi)$, $M^{\alpha+1-(1/n)}(\phi) \to M^{\alpha+1-(2/n)}(\phi)$, $\cdots$, $M^{\alpha+(1/n)}(\phi) \to M^\alpha(\phi)$. Moreover, as noted earlier,

$$\max\{d(z, M^{\alpha+(k/n)}(\phi)) \mid z \in M^{\alpha+(k+1)/n}(\phi)\} \leq 1/(n\Delta) ,$$

where Δ is as defined previously. Applying n times the previously obtained inequality on the arc-length of closest point projections of arcs yields that

$$(\text{length of } \eta_n \circ \gamma) \leq \text{length}(\gamma) \cdot (1+(C/n\Delta)^2)^n ,$$

where C is a constant independent of n (C depends only on the curvature of M in a neighborhood of $M_\alpha^{\alpha+1}(\phi)$).

Now $\lim_{n \to +\infty}(1+(C/n\Delta)^2)^n = 1$. Thus the required estimate on the Lipschitz constants of the η_n holds, and the proof of Lemma 1 is complete.

3. Proof of Lemma 2.

An n-dimensional Riemannian manifold M is locally a metric product of an interval and an $(n-1)$-manifold if and only if there is locally a C^∞ unit parallel vector field. The existence of such a vector field if M is such a local product is clear; that M is a local metric product whenever such a vector field exists is a special case of the (local) de Rham Decomposition Theorem ([6]; this special case is alternatively easily established directly). For the implication that M is a local metric product, it is not actually necessary to assume that a C^∞ parallel vector field exists. In fact, if a local unit vector field exists that is locally Lipschitz continuous and that has zero covariant derivative wherever its covariant derivative exists, then it follows that the vector field is actually C^∞ and parallel, and hence that M is locally a metric product as before. To see this, suppose V is such a vector field defined in an ε-ball around $p \in M$ with $\varepsilon <$ the injectivity radius of M at p. Because V is Lipschitz continuous, V is covariant differentiable almost everywhere (a. e.) relative to the measure on M determined by the (or, equivalently, any other) Riemannian metric of M. By Fubini's Theorem, the covariant derivative of V exists a. e. along almost all of the arc-length-parameter geodesic segments $\gamma: (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, where "almost all" is in the sense of $(n-1)$-Lebesgue measure on the unit sphere in TM_p, which contains

the tangent vectors at p of such geodesic segments. Suppose $\gamma:(-\varepsilon, \varepsilon)\to M$ is one of the segments along which DV exists (and equals 0) a.e. so that in particular $D_{\dot{\gamma}}V|_{\gamma(t)}$ exists and is 0 for almost all $t\in(-\varepsilon, \varepsilon)$. Let $P_1, \cdots, P_n$ be a parallel orthonormal frame along γ, and write $V(\gamma(t))=\sum_{j=1}^{n}a_j(t)P_j(t)$. Then each $a_j(t)$ is differentiable a.e. on $(-\varepsilon, \varepsilon)$; and, because V and hence the a_j are Lipschitz continuous, $a_j(t)=a_j(0)+\int_0^t\left(\dfrac{d}{du}a_j(u)\right)du$. Then, because $D_{\dot{\gamma}}V|_{\gamma(t)}$ exists and is 0 a.e. on $(-\varepsilon, \varepsilon)$, it follows that $(d/dt)a_j=0$ a.e. on $(-\varepsilon, \varepsilon)$ so that $a_j(t)=a_j(0)$, $t\in(-\varepsilon, \varepsilon)$. Hence V is C^∞ (and parallel) along γ. Because the set of such γ is dense in the set of all $(-\varepsilon, \varepsilon)$ geodesic segments at p, it follows by continuity that V is C^∞ and parallel along every $(-\varepsilon, \varepsilon)$ geodesic segment centered at p and hence that V is C^∞ on the ε-ball round p except possibly at p. Repeating the argument with p replaced by another point near p yields that V is C^∞ across p as well. Thus V is C^∞ and parallel.

The remarks of the previous paragraph are motivation for the proof technique for Lemma 2: It will be shown that the unit normals to the sublevels (in the direction of increase of ϕ) form a locally Lipschitz continuous vector field, and it will then be shown that this vector field has zero covariant derivative a.e. along every geodesic segment. Then, as in the previous paragraph, it follows that the vector field is C^∞ parallel. The passage from local product to global product structure is trivial here, because the normals have a globally consistent orientation (direction of increase of ϕ). Henceforth this (oriented) unit normal vector field will be denoted by N.

The local Lipschitz continuity of N is easily seen; in fact, the (locally oriented) unit normal field of an arbitrary totally geodesic codimension one foliation is locally Lipschitz continuous. To see this, suppose it false. Then there are two sequences $\{p_i\}$, $\{q_i\}$, $i=1, 2, \cdots$, $p_i\neq q_i$, having the properties that $\lim p_i$ exists, $\lim q_i$ exists, $\lim p_i=\lim q_i$, and $\lim \mathrm{Dis}(N(p_i), N(q_i))/d(p_i, q_i)=+\infty$, where Dis is Riemannian metric distance in the tangent bundle TM of M. Then, since the leaves of the foliation are locally uniformly C^∞, it must be that, for all i sufficiently large, p_i and q_i are in different and hence disjoint local leaves. By the C^∞ character of the exponential map, it follows that the exponentiation of the orthogonal complement of $N(p_i)$ and $N(q_i)$ must intersect in a neighborhood of $\lim p_i$, for each i sufficiently large. This contradicts the total geodesicness of the foliation.

Now let γ be an arc-length-parameter geodesic in M. The function $t\to N(\gamma(t))$ is locally Lipschitz continuous, $t\in\mathrm{domain}\,\gamma$. Here (and henceforth) N is the oriented unit normal vector field to the level surfaces of ϕ. Hence $t\to N(\gamma(t))$ is differentiable a.e. along γ. The goal now is to show that $D_{\dot{\gamma}(t)}N(\gamma(t))=0$ a.e. in the sense that there is a full measure set (i.e., a set with complement of

measure 0) such that for every t_0 in the set, $t \to N(\gamma(t))$ is differentiable at t_0 and $D_{\dot\gamma(t)}N(\gamma(t))|_{t_0}=0$. If ϕ were C^2, then that $D_{\dot\gamma(t)}N(\gamma(t))=0$ could be established by the following computational procedure (this procedure was previously known: see [1]).

For every vector $T \in TM_{\gamma(t)}$ normal to $N(\gamma(t))$, $D_T N=0$ because the levels of ϕ are totally geodesic. Thus $D_{\dot\gamma(t)}N(\gamma(t))=\alpha(t)D_{N(\gamma(t))}N|_{\gamma(t)}$, where $\alpha(t)= \langle N(\gamma(t)), \dot\gamma(t)\rangle$ so that $\langle \dot\gamma(t)-\alpha(t)N(\gamma(t)), N(\gamma(t))\rangle=0$. Thus to show $D_{\dot\gamma(t)}N(\gamma(t))=0$ one need only show $D_N N=0$ at $\gamma(t)$. To see this, note that the second covariant differential $D^2_\phi(aT+bN, aT+bN)$, $T, N \in TM_{\gamma(t)}$, $\langle T, N\rangle=0$ is nonnegative for all $a, b \in \mathbf{R}$. So

$$0 \leq a^2 D^2_\phi(T, T)+2ab D^2_\phi(T, N)+b^2 D^2_\phi(N, N).$$

Now $D^2_\phi(T, T)=0$ because the levels of ϕ are totally geodesic. Hence $D^2_\phi(T, N)=0$ (and $D^2_\phi(N, N)\geq0$). But

$$D^2_\phi(T, N) = N(T\phi)-(D_N T)\phi = -(D_N T)\phi$$

because $T\phi \equiv 0$. So $\langle D_N T, \operatorname{grad}\phi\rangle=0$ for every local vector field T perpendicular to N or equivalently tangent to the levels of ϕ. Hence $\langle D_N T, N\rangle=0$ since $\operatorname{grad}\phi$ is a nonzero multiple of N (for the fact that the multiple is nonzero, note that ϕ without minimum implies that ϕ has no critical points; see, for instance, the more general remarks in [1]). Suppose $T_1, \cdots, T_{n-1}$ are a local orthonormal frame for the tangent spaces of the levels of ϕ along a (local) geodesic tangent to $N(\gamma(t))$: such a choice of $T_1, \cdots, T_{n-1}$ is always possible. Then

$$D_N N = D_N((T_1 \wedge \cdots \wedge T_{n-1})) = \sum_{i=1}^{n-1}(T_1 \wedge \cdots \wedge D_N T_i \wedge \cdots \wedge T_{n-1}).$$

Since $\langle D_N T_i, N\rangle=0$, $D_N T_i=\sum a_i^j T_j$. But $\|T_i\|\equiv1$ implies $a_i^i=0$. Thus

$$T_1 \wedge \cdots \wedge D_N T_i \wedge \cdots \wedge T_{n-1} = 0$$

and $D_N N=0$, as required to show that $D_{\dot\gamma}N(\gamma(t))=0$.

The argument just given applies of course only when D^2_ϕ exists. To treat the general case, ϕ nonsmooth, note first that along each geodesic γ the function $\phi \circ \gamma$ is either constant or, after a change of orientation of γ if necessary, monotone strictly increasing (this is clear because if $\phi \circ \gamma$ has a local minimum at $t=t_0$, then $\dot\gamma(t_0)$ must be tangent to the $\phi \circ \gamma(t_0)$-level set of ϕ and, by total geodesicness, γ is then contained in this level set). If $\phi \circ \gamma$ is constant, then $D_{\dot\gamma}N=0$ again because the level sets of ϕ are totally geodesic. So it suffices now to consider the case of γ such that $\phi \circ \gamma$ is monotone (strictly) increasing.

Because $t \to (\phi \circ \gamma)(t)$ is convex, the right hand derivative $R(t)= \lim_{h \to 0^+}((\phi \circ \gamma)(t)-(\phi \circ \gamma)(t-h))/h$ exists everywhere. Moreover, $t \to R(t)$ is a non-

decreasing function of t. By a standard theorem of analysis (see, e.g., [7, p. 177]), the (two-sided) derivative of $t \rightarrow R(t)$ exists for almost all t. Also $t \rightarrow N(\gamma(t))$ is Lipschitz continuous a.e. Thus there is a full measure set $\mathcal{R}$ such that, if $t_0 \in \mathcal{R}$, then $t \rightarrow R(t)$ is differentiable at t_0 and $t \rightarrow N(\gamma(t))$ is differentiable at t_0. It will now be shown that $D_{\dot{\gamma}(t)} N(\gamma(t)) = 0$ at such a t_0.

For this purpose, let $\tilde{N}(t) \in TM_{\gamma(t)}$ be a C^∞ unit vector field along γ such that

$$\tilde{N}(t_0) = N(\gamma(t_0)) \quad \text{and} \quad \lim_{t \to t_0} (t-t_0)^{-1}\{\tilde{N}(t) - N(\gamma(t))\} = 0 .$$

Such an $\tilde{N}$ exists because $t \rightarrow N(\gamma(t))$ is differentiable at t_0. Set

$$\tilde{\phi}(t) = \phi(\gamma(t_0)) + (t-t_0) R(t_0) + \frac{1}{2}(t-t_0)^2 \frac{dR}{dt}\bigg|_{t_0} .$$

Then

$$\lim_{t \to t_0} (t-t_0)^{-2}\{\tilde{\phi}(t) - \phi(\gamma(t))\} = 0$$

because $\phi(\gamma(t)) - \phi(\gamma(t_0)) = \int_{t_0}^{t} R(t)dt$ and $R(t)$ is differentiable at t_0. For $\varepsilon > 0$ sufficiently small, the exponential map (of the ε-neighborhood of 0) of the orthogonal complement bundle of $\tilde{N}$ along γ near $\gamma(t_0)$ is a C^∞ diffeomorphism onto a neighborhood of $\gamma(t_0)$; this holds by the inverse function theorem. Specifically, the exponential map of $\{v \in TM_{\gamma(t)} \mid t \in (t_0-\varepsilon, t_0+\varepsilon), \ \|v\| < \varepsilon, \ \langle \tilde{N}(t), v \rangle = 0\}$ given by $v \rightarrow \exp_{\gamma(t)} v$ is a C^∞ diffeomorphism, to be defined by $\mathcal{D}_\varepsilon$, onto a neighborhood of $\gamma(t_0)$ if $\varepsilon > 0$ is sufficiently small. Choose such an ε and let V be the (open) image of the diffeomorphism $\mathcal{D}_\varepsilon$, so that $\gamma(t_0) \in V$. On V, the map defined by $p \rightarrow t \in (t_0-\varepsilon, t_0+\varepsilon)$ such that $\mathcal{D}_\varepsilon^{-1}(p) \in TM_{\gamma(t)}$ is a well-defined C^∞ function. Hence the function $V \rightarrow \boldsymbol{R}$ defined by $p \rightarrow \tilde{\phi}(t)$, t as in the previous sentence, is also a well-defined C^∞ function on V. Denote this function $p \rightarrow \tilde{\phi}(t)$ by $\psi : V \rightarrow \boldsymbol{R}$.

Now $\psi(p) = \psi(\gamma(t_0))$ for each $p \in V$ with $\phi(\gamma(t_0)) = \phi(p)$, by the definition of ψ. Moreover if η is a geodesic with $\eta(0) = \gamma(t_0)$ but with $\langle \eta'(0), N(\gamma(t_0)) \rangle \neq 0$ — or, equivalently, but with $\eta \nsubseteq$ the $\phi(\gamma(t_0))$-level of ϕ — then, from the order of agreement of $\phi(\gamma(t))$, $\tilde{\phi}$ and of $N(\gamma(t))$ and $\tilde{N}(t)$, it follows that

$$\lim_{t \to 0} t^{-2}\{\phi(\eta(t)) - \psi(\eta(t))\} = 0 .$$

The convexity of $t \rightarrow \phi(\eta(t))$ and consequent nonnegativity of its second difference quotients then implies that $(d^2\phi(\eta(t))/dt^2)|_{t=t_0} \geq 0$. In detail,

$$\frac{d^2\phi(\eta(t))}{dt^2}\bigg|_{t=t_0} = \lim_{h \to 0} \frac{\phi(\eta(t_0+h)) + \phi(\eta(t_0-h)) - 2\phi(\eta(t_0))}{h^2}$$

$$= \lim_{h \to 0} \frac{\psi(\eta(t_0+h)) + \psi(\eta(t_0-h)) - 2\psi(\eta(t_0))}{h^2} \geq 0 ,$$

where the two limits are equal by the order of agreement of $\psi(\eta(t))$ and $\phi(\eta(t))$

and the second limit is nonnegative by convexity of ϕ.

Because ϕ is C^∞ and $D^2\phi|_{\gamma(t_0)} \geq 0$, as was just shown, the calculations carried out earlier apply to show that $D_t\tilde{N}(t)|_{t=t_0}=0$. But, by the order of agreement of $\tilde{N}(t)$ and $N(\gamma(t))$ at $t=t_0$, it then follows that $D_{\dot{\gamma}}N|_{\gamma(t_0)}=0$. This holds for all $t_0 \in$ the full measure set $\mathfrak{R}$. Thus that $N(\gamma(t))$ is given along γ by integrating its derivative implies that $N(\gamma(t))$ is a C^∞ parallel vector field along γ. Hence N is a C^∞ parallel vector field on M, and the metric product structure result, Lemma 2, is proved.

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