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Traces on group extensions and C^*-crossed products

Dedicated to Professor O. Takenouchi on his 60th birthday

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Introduction.

This paper is devoted to the study of representations, ideals, weights, and especially traces on a C^*-crossed product induced from those on a smaller C^*-crossed product. The main result is the characterization of a trace on a C^*-crossed product $C_r^*(A, G, \alpha)$ to be induced from a trace of a sub-crossed product $C_r^*(A, N, \alpha)$ with N a closed normal subgroup of G.

Let (A, G, α) be a C^*-dynamical system. When N is a closed normal subgroup, starting from certain relatively invariant traces on $C_r^*(A, N, \alpha)$, we have induced traces on $C_r^*(A, G, \alpha)$ [**16**], [**4**]. They have some special properties. When G/N is abelian, they are invariant under the dual action [**16**]. When G/N is not abelian, we have no more a convenient dual action, and we replace the role of the dual action by coaction. As our main result it will be shown that induced traces are characterized by the invariance under the coaction of G/N dual to the original action of G. The statement established may be regarded as an analogue of Takesaki's imprimitivity theorem with respect to traces.

In the theory of ideals of C^*-crossed products, Effros and Hahn have made a famous conjecture. A complete answer to this conjecture has been established in [**3**]. A further conjecture was made by P. Green that Effros-Hahn conjecture would hold for traces. Precisely stating, this conjecture is as follows. "Let (A, G, α) be a C^*-dynamical system, and the action of G on $\mathrm{Prim}(A)$ be free. Then all the traces on $C_r^*(A, G, \alpha)$ are induced from A". When G is a discrete amenable group, this was proved in [**4**]. But When G is not discrete, by the absence of conditional expectations, the situation becomes difficult. We believe that the result we obtained here makes a step toward the solution of the problem.

We are given much freedom by the C^*-induction in the studies of induced representations, and the imprimitivity theorem becomes a stronger weapon when it is formulated in the C^*-theoretic framework [**18**]. We use it to obtain a duality for induced representation in non-abelian cases.

In this paper, we only study, separable C^*-algebras, 2nd countable locally compact groups, separable Hilbert spaces and von Neumann algebras with separable preduals.

Let $\mathfrak{H}$ be a Hilbert space. We denote by $\mathfrak{B}(\mathfrak{H})$ all bounded linear operators on $\mathfrak{H}$ and by $\mathfrak{C}(\mathfrak{H})$ all compact linear operators. Let S be a subset of $\mathfrak{B}(\mathfrak{H})$. We denote by $C^*(S)$ the C^*-algebra generated by S. Let π be a unitary representation of a locally compact group, or a $*$-representation of a C^*-algebra. Denote by $W^*(\pi)$ the von Neumann algebra generated by the range of π.

Let A and B be two C^*-algebras, and ω be an element of the dual Banach space B^* of B. We denote by E_A^ω the slice map from $A\otimes B$ to A associated with ω. Let ϕ be a weight on a C^*-algebra A. Put $n_\phi=\{x\in A:\phi(x^*x)\leqq\infty\}$, $m_\phi=n_\phi^*n_\phi$ and $N_\phi=\{x\in A:\phi(x^*x)=0\}$.

For a C^*-dynamical system (A, G, α) (a W^*-dynamical system (M, G, β)), we denote by $C^*(A, G, \alpha)$ (resp. $W^*(M, G, \beta)$) the corresponding C^*-crossed product, and by $C_r^*(A, G, \alpha)$ the corresponding reduced C^*-crossed product.

Let δ be a coaction of a locally compact group G on a C^*-algebra A (W^*-algebra M). We denote by $\hat{C}^*(A, G, \delta)$ (resp. $\hat{W}^*(M, G, \delta)$) the corresponding C^*-cocrossed product (resp. W^*-cocrossed product). Let δ be an action of a Kac C^*-algebra (Kac algebra) on a C^*-algebra A (resp. W^*-algebra M). For $k\in A^*$, we put $\delta_k(x)=E_A^k(\delta(x))$ for $x\in A$.

Let G be a locally compact group. We denote by $L(G)$ the von Neumann algebra generated by the left regular representation of G, and by $A(G)$ the Fourier algebra of G. We use only left Haar measures and denote by Δ_G the modular function of G. Let N be a closed normal subgroup of G. For $s\in G$, we denote by $\dot{s}$ the quotient image of s in G/N.

We refer to [**12**], [**14**] and [**20**] as general references.

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1. Induction of weights.

Let (A, G, α) be a C^*-dynamical system, N be a closed normal subgroup of G. Let $B=C_r^*(A, N, \alpha)$ and π be a G-invariant representation of B. Put $M=W^*(\pi)$. Denote the action of G on M induced from α by β. Let $C=C_r^*(A, G, \alpha)$. We consider the induced representation $\tilde{\pi}=\mathrm{Ind}_{N\uparrow G}\pi$ of C. Put $P=W^*(\tilde{\pi})$. Put $L^2(\mathfrak{H}_\pi, G, \pi)=\{\xi : G\to\mathfrak{H}_\pi$ measurable maps, $\xi(sn)=\pi(n)^*\xi(s)$ for $s\in G$, $n\in N$, and $\int_{G/N}\|\xi(s)\|^2d\dot{s}<\infty\}$. This is the underlying space of the induced representation $\tilde{\pi}$. Let

$$\pi_\beta(x)\xi(s) = \beta_s^{-1}(x)\xi(s),$$
$$\tilde{\lambda}(t)\xi(s) = \xi(t^{-1}s) \qquad \text{for } x \in M,\ s, t \in G \text{ and } \xi \in L^2(\mathfrak{H}_\pi, G, \pi).$$

The representation $\tilde{\pi}$ is the integrated form of the covariant representation $(\tilde{\lambda}, \pi_\beta)$. We have $P = W^*(\tilde{\lambda}, \pi_\beta)$, and this algebra is a typical example of the extended covariance algebra in [**19**].

Let ϕ be a faithful normal semifinite weight on M. We may assume that M is acting on the standard representation space $\mathfrak{H}_\phi$ of ϕ. Let J, σ^ϕ, Δ_ϕ be the modular objects of ϕ, and W be a cone preserving unitary representation implementing the action β. Let

$K(M, G, \pi) = \{$Maps f from G to M : (1) σ-strongly $*$-continuous, (2) $f(sn) = \pi(n)^* f(s)$ for $s \in G$, $n \in N$, (3) $\|f(s)\|$ has a compact support as a function on $G/N\}$.

For f, g in $K(M, G, \pi)$, we define $f * g$ and f^* by

$$(f*g)(s) = \int_{G/N} \beta_{s^{-1}t}(f(t))g(t^{-1}s)d\dot{t},$$
$$f^*(s) = \Delta_{G/N}(\dot{s})^{-1}\beta_s^{-1}(f(s^{-1})^*).$$

For $f \in K(M, G, \pi)$, we put,

$$\|f\|_2 = \left\{\int_{G/N} \phi(f(s)^* f(s))d\dot{s}\right\}^{1/2}.$$

DEFINITION 1. Let $\mathfrak{A}$ be the set of all functions in $K(M, G, \pi)$ satisfying $\|f\|_2 < \infty$ and $\|f^*\|_2 < \infty$.

Then, by the estimate of the norm $\|\cdot\|_2$, we can show that $\mathfrak{A}$ is a $*$-algebra.

LEMMA 2. *$\mathfrak{A}$ is a left Hilbert algebra with the above algebraic structure and the inner product inherited from $\|\cdot\|_2$.*

For $f \in K(M, G, \pi)$ and $\xi \in L^2(\mathfrak{H}_\phi, G, \pi)$, we put

$$\nu(f)\xi(s) = \int_{G/N} \beta_{s^{-1}t}(f(t))\xi(t^{-1}s)d\dot{t}.$$

Then, $f \to \nu(f)$ is a $*$-representation of $K(M, G, \pi)$ on $L^2(M, G, \pi)$ and it is an extension of the left representation of $\mathfrak{A}$.

DEFINITION 3. The canonical weight $\tilde{\phi}$ associated with $\mathfrak{A}$ on P is called the induced weight of ϕ.

For elements in $\mathfrak{A}$, the value of $\tilde{\phi}$ is

$$\tilde{\phi}(\nu(f)^*\nu(f)) = \int_{G/N} \phi(f(s)^* f(s))d\dot{s}. \qquad (*)$$

The unitary involution $\tilde{J}$ and the modular automorphism $\sigma^{\tilde{\phi}}$ of $\tilde{\phi}$ are given by the formulae in the following lemma.

LEMMA 4.

(1) $(\tilde{J}\xi)(s)=\Delta_{G/N}(\dot{s})^{-1/2}W(s)^*J\xi(s^{-1})$ *for* $\xi\in L^2(\mathfrak{H}_\phi, G, \pi)$.

(2) $\sigma_t^{\tilde{\phi}}(\pi_\beta(x))=\pi_\beta(\sigma_t^\phi(x))$ *for* x *in* M.

The identity in Lemma 4 (2) determines $\sigma^{\tilde{\phi}}$ uniquely by σ-weak continuity.

LEMMA 5. *Let* $\tilde{\phi}'$ *be a f.n.s. weight on* $W^*(\tilde{\pi})$ *and satisfy*

(1) $\tilde{\phi}'(\nu(f)^*\nu(f))=\int_{G/N}\phi(f(s)^*f(s))d\dot{s}$,

(2) $\sigma^{\tilde{\phi}'}=\sigma^{\tilde{\phi}}$.

Then, two weights $\tilde{\phi}'$ *and* $\tilde{\phi}$ *are identical.*

PROOF. Since $\sigma^{\tilde{\phi}'}=\sigma^{\tilde{\phi}}$, by Theorem 5.9 in [**15**] it suffices to show that $\mathfrak{A}$ is $\sigma^{\tilde{\phi}}$ invariant. By the KMS condition for $\tilde{\phi}$, $\tilde{\phi}$ is $\sigma^{\tilde{\phi}}$ invariant. Since $\sigma^{\tilde{\phi}}$ is a $*$-automorphism, it conserves the relation $\|f\|_2<\infty$ and $\|f^*\|_2<\infty$. By Lemma 2-6 of [7], $s\to(D\phi\circ\beta_s:D\phi)_t$ is σ-strongly continuous. These arguments show the invariance of $\mathfrak{A}$. q. e. d.

2. Coactions and operator valued weights.

We first define the canonical coaction δ of G/N on $W^*(\tilde{\pi})$. Consider the Hilbert space tensor product $L^2(\mathfrak{H}_\phi, G, \pi)\otimes L^2(G/N)$. An element of this space can be considered as a function ξ from $G\times(G/N)$ to $\mathfrak{H}_\phi$ satisfying $\xi(sn, \dot{t})=\pi(n)^*\xi(s, \dot{t})$ for $n\in N$, $s\in G$, $\dot{t}\in G/N$. Define a unitary operator W on $L^2(\mathfrak{H}_\phi, G, \pi)\otimes L^2(G/N)$ by

$$(W\xi)(s, \dot{t})=\xi(s, \dot{s}\dot{t}).$$

For $x\in P$, define δ by $\delta(x)=W^*(x\otimes 1)W$.

LEMMA 6. *The map* δ *is a coaction of* G/N *on* P.

We call δ the canonical coaction of G/N. According to Proposition II. 2 of [**2**], we can define an operator valued weight T from P to the fixed point algebra P^δ as follows. Let ψ be the Haar weight on $L(G/N)$. Put $F=\{\phi\in L(G/N)_* : 0\leqq\phi\leqq\psi\}$. Put $E(x)=\sup_{\phi\in F}E_P^\phi(x)$ for $x\in P\otimes L(G/N)^+$. Then E is an operator valued weight from $P\otimes L(G/N)$ to P. We put, for $x\in P^+$, $T(x)=E(\delta(x))$.

LEMMA 7. *Let* f *be in* $K(M, G, \pi)$. *Then, we have,*

$$T(\nu(f)^*\nu(f)) = \int_{G/N} \pi_\beta(f(s)^*f(s))d\dot{s}.$$

In particular, T is semifinite.

We omit the proof.

LEMMA 8. *The fixed point algebra P^δ is equal to $\pi_\beta(M)$.*

PROOF. We may assume that P acts on the standard representation space given by $\tilde{\phi}$. For $k \in L^\infty(G/N)$, we put $\rho(k)\xi(s) = k(\dot{s})\xi(s)$ for $\xi \in L^2(\mathfrak{H}_\phi, G, \pi_\phi)$. Let x be in P^δ. By the argument similar to that in Proposition 2-3 of [11], we have $x\rho(k) = \rho(k)x$ for all k in $L^\infty(G/N)$. This shows that P^δ is in the commutant of $L^\infty(G/N)$. By using Lemma 4 (1), we have,

$$\tilde{J}(\pi_\beta(M))\tilde{J} = JMJ = M' \qquad \text{and} \qquad \tilde{J}(\rho(L^\infty(G/N)))\tilde{J} = \rho(L^\infty(G/N)).$$

By the second identity, we have $\tilde{J}(\rho(L^\infty(G/N)'))\tilde{J} = \rho(L^\infty(G/N))'$. Since $x \in \rho(L^\infty(G/N))'$, we have $\tilde{J}x\tilde{J} \in \rho(L^\infty(G/N))' \cap P' = M'$. This shows that $x \in \pi_\beta(M)$.

q. e. d.

By these two lemmas, T is a faithfull normal semifinite operator valued weight from P to $\pi_\beta(M)$. By [6], it gives a map from f. n. s. weights on M to those on P by $\bar{\phi} = \phi \circ \pi_\beta^{-1} \circ T$ for a f. n. s. weight ϕ on M.

PROPOSITION 9. *For a f. n. s. weight ϕ on M, $\bar{\phi}$ is the induced weight of ϕ defined in* § 1.

PROOF. By the normality of ϕ, $\tilde{\phi}$ and $\bar{\phi}$ are identical on $\nu(\mathfrak{A})$. By Theorem 4.7 of [6], we get

$$\sigma_t^{\bar{\phi}}(\pi_\beta(x)) = \sigma_t^{\phi \circ \pi_\beta^{-1}}(\pi_\beta(x)) = \pi_\beta(\sigma_t^\phi(x)) \qquad \text{for } x \in M.$$

For a positive definite function k on G/N, we have the identity $E_P^k(\tilde{\lambda}(s)x\tilde{\lambda}(s)^*) = \tilde{\lambda}(s)E_P^{k(\dot{s}\cdot\dot{s}^{-1})}(x)\tilde{\lambda}(s)^*$ for $x \in P$, $s \in G$. Since E is given by the supremum of E_P^k's, we have, $T(\tilde{\lambda}(s)x\tilde{\lambda}(s)^*) = \Delta_{G/N}(\dot{s})\tilde{\lambda}(s)T(x)\tilde{\lambda}(s)^*$ for $x \in P$, $s \in G$. Using this and by the general arguments, we have,

$$\sigma_t^{\bar{\phi}}(\tilde{\lambda}(s)) = \tilde{\lambda}(s)\pi_\beta((D\phi \circ \beta_s : D\phi)_t).$$

By Lemma 5, we have $\bar{\phi} = \tilde{\phi}$. q. e. d.

Suppose that a von Neumann algebra M admits a coaction δ of a group H. A f. n. s. weight ϕ on M is called δ-invariant if $\langle \delta(x), \phi \otimes \omega \rangle = \langle x \otimes 1, \phi \otimes \omega \rangle$ for all $x \in M^+$ and all $\omega \in L(H)_*$. Let j be the canonical involution of $L(H)$. A weight ϕ is called (δ, j) invariant if it is δ-invariant and it satisfies $\langle (y^* \otimes 1)\delta(x), \phi \otimes k \rangle = \langle \delta(y^*)(x \otimes 1), \phi \otimes (k \circ j) \rangle$ for any x, y in n_ϕ and k in $A(H)$. For $k \in A(H)$, put $k^0 = k^* \circ j$. $A(H)$ has a natural Banach $*$-algebra structure by pointwise multiplication and involution ^{0}. Then the map $k\in A(H)\to\delta_k$ is a $*$-representation in algebraic sense.

LEMMA 10 (Theorem 0-2-12 of [21]). *A f.n.s. weight ϕ is (δ, j) invariant if and only if $k\to\delta_k$ can be extended to a bounded $*$-representation of $A(H)$ on the GNS representation space $\mathfrak{H}_\phi$ of ϕ.*

Let δ be the canonical coaction of G/N on P, j be the involution of $L(G/N)$.

LEMMA 11. *For a f.n.s. weight ϕ on M, the induced weight $\tilde{\phi}$ on P is (δ, j) invariant.*

PROOF. The GNS Hilbert space of $\tilde{\phi}$ is $L^2(\mathfrak{H}_\phi, G, \pi)$. In this situation, δ_k is clearly a bounded $*$-representation. q.e.d.

PROPOSITION 12. *A f.n.s. weight $\tilde{\phi}$ on P is (δ, j) invariant if and only if $\tilde{\phi}$ is induced from a f.n.s. weight ϕ on M.*

PROOF. "If" part is already proved. Let $\tilde{\phi}$ be (δ, j) invariant. We use the canonical argument in [7]. Take a f.n.s. weight ψ on M. Denote by $\tilde{\psi}$ the induced weight of ψ. By a general result of operator valued weights ([6]), $(D\tilde{\phi}:D\tilde{\psi})_t$ is contained in $\pi_\beta(M)$. Put $u_t=\pi_\beta^{-1}((D\tilde{\phi}:D\tilde{\psi})_t)$. This u_t is a unitary cocycle with respect to σ^ψ. By the converse of Connes' theorem, there exists a f.n.s. weight ϕ on M such that $(D\phi:D\psi)_t=u_t$. Let $\bar{\phi}$ be the induced weight from ϕ. Then $(D\bar{\phi}:D\tilde{\psi})_t=\pi_\beta((D\phi:D\psi)_t)=(D\tilde{\phi}, D\tilde{\psi})_t$, and this shows that $\tilde{\phi}=\bar{\phi}$.
q.e.d.

3. Duality for induced representations.

Let $\dot{\lambda}$ be the representation of G on $C_0(G/N)$ or $L^\infty(G/N)$ by the left translation.

LEMMA 13. *Two C^*-algebras $\hat{C}_r^*(C_r^*(A, G, \alpha), G/N, \delta)$ and $C_r^*(A\otimes C_0(G/N), G, \alpha\otimes\dot{\lambda})$ are spatially isomorphic.*

PROOF. We assume that A is represented on $\mathfrak{H}$ faithfully. The representation space of the first algebra is $L^2(\mathfrak{H}, G\times(G/N))$ and that of the second is $L^2(\mathfrak{H}, (G/N)\times G)$. We define a unitary operator U from the latter to the former as follows.

$$(U\xi)(t, \dot{s}) = \xi(\dot{t}\dot{s}, t) \qquad \text{for } \xi\in L^2(\mathfrak{H}, (G/N)\times G),\ t\in G \text{ and } \dot{s}\in G/N.$$

Then AdU gives the desired spatial isomorphism between the two C^*-algebras.
q.e.d.

REMARK. When (M, G, α) is a W^*-dynamical system and N is a normal subgroup, we have a spatial isomorphism between $\hat{W}^*(W^*(M, G, \alpha), G/N, \delta)$ and

$W^*(M\otimes L^\infty(G/N),\ G,\ \alpha\otimes\dot{\lambda})$ similarly.

By this lemma, we may consider the representation theory of $C_r^*(A\otimes C_0(G/N),\ G,\alpha\otimes\dot{\lambda})$ instead of $\hat{C}_r^*(C_r^*(A, G, \alpha), G/N, \delta)$. Put $P=C_c(A, G\times(G/N))$, $Q=C_c(A, G)$ and $R=C_c(A,\ N\times(G/N))$. We shall give pre C^*-algebra structures to P, Q and R, by the following natural inclusions, $P\subset C^*(A\otimes C_0(G/N),\ G,\ \alpha\otimes\dot{\lambda})$, $Q\subset C^*(A,\ G,\ \alpha)$ and $R\subset C^*(A,\ N,\ \alpha)\otimes C_0(G/N)$. We put,

$$\text{for } \xi_1\in Q\,,\quad (T_1\xi_1)(n)=\xi_1(n)\quad \text{for } n\in N\,,$$

$$\text{for } \xi_2\in P,\quad (T_2\xi_2)(t)=\int_{G/N}\xi_2(t,\ \dot{s})ds\quad \text{for } t\in G\,,$$

$$\text{for } \xi_3\in R\,,\quad (S_1\xi_3)(n)=\int_{G/N}\xi_3(n,\ \dot{s})d\dot{s}\quad \text{for } n\in N\,,$$

$$\text{for } \xi_4\in P,\quad (S_2\xi_4)(n,\ \dot{s})=\xi_4(n,\ \dot{s})\quad \text{for } n\in N,\ \dot{s}\in G/N.$$

Then we have $T_1\xi_1\in C_c(A,\ N)$, $T_2\xi_2\in Q$, $S_1\xi_3\in C_c(A,\ N)$ and $S_2\xi_4\in R$.

LEMMA 14. *Four linear maps T_1, T_2, S_1 and S_2 are all generalized conditional expectations.*

PROOF. We have only to consider T_2. For $\xi\in P$, put $(\gamma_t\xi)(s,\ \dot{u})=\xi(s,\ \dot{u}\dot{t})$. Since the right and the left translations commute, γ_t is an automorphism of P. We have $T_2(\xi)=\int\gamma_{\dot{t}}(\xi)d\dot{t}$. Checking the conditions in the Definition 4-12 of [**18**], we have, the conditions (1), (2), (3) and (4). Since P^2 is dense in P with respect to the inductive limit topology, (5) follows. (6) is trivial. q. e. d.

LEMMA 15. *These generalized conditional expectations give inductions of the corresponding "reduced" crossed products.* (*For example, T_1 gives the induction from $C_r^*(A,\ N,\ \alpha)$ to $C_r^*(A,\ G,\ \alpha)$.*)

PROOF. This follows from the fact that these inductions transform each "regular" representation of a subgroup crossed product to a "regular" representation of a total crossed product. q. e. d.

DEFINITION 16. We call the induction given by T_2 the induction of a coaction δ.

LEMMA 17. *The composition T_1T_2 and S_1S_2 are also generalized conditional expectations and they are equal.*

PROOF. This is a direct consequence of the 2nd version of Theorem 5-11 of [**18**] (Stage theorem). q. e. d.

By Corollary 2.8 in P. Green's paper [**5**], there exists an isomorphism U from $C^*(A\otimes C_0(G/N),\ G,\ \alpha\otimes\dot{\lambda})$ to $C^*(A,\ N,\ \alpha)\otimes\mathfrak{C}(L^2(G/N))$. Let τ be a representation of $C^*(A,\ N,\ \alpha)$. On the space of $\mathrm{Ind}_{N\uparrow G}\tau$, there exists a canonical diagonal representation η of $C_0(G/N)$. Let $\tilde{\tau}$ be a representation of $C^*(A\otimes C_0(G/N), G, \alpha\otimes\dot{\lambda})$ on $L^2(\mathfrak{H}_\tau, G, \tau)$ given by

$$\tilde{\lambda}(t)\xi(s)=\xi(t^{-1}s), \qquad \tau_\alpha(x)\xi(s)=\alpha_s^{-1}(x)\xi(s), \qquad \eta(k)\xi(s)=k(\dot{s})\xi(s)$$

for $t\in S$, $x\in A$ and $k\in C_0(G/N)$. We call $\tilde{\tau}$ the combination of $\mathrm{Ind}_{N\uparrow G}\tau$ and η. By the Remark 2-5 of [**5**], $\tilde{\tau}\circ U^{-1}$ is transformed to $\tau\otimes\mathrm{Id}_{\mathfrak{C}(L^2(G/N))}$ on $\mathfrak{H}_\tau\otimes L^2(G/N)$. This is a canonical argument of Borel section. We call this correspondence "the transfer principle" for induced representations.

LEMMA 18. *$C_r^*(A\otimes C_0(G/N),G,\alpha\otimes\dot{\lambda})$ is isomorphic to $C_r^*(A,N,\alpha)\otimes\mathfrak{C}(L^2(G/N))$ and the isomorphism is given by U.* (We again denote it by U.)

PROOF. We apply "the transfer principle" to the regular representation of $C^*(A, N, \alpha)$ which gives $C_r^*(A, N, \alpha)$. q. e. d.

Let π be a G-invariant representation of $B=C_r^*(A, N, \alpha)$. Put $\tilde{\pi}=\mathrm{Ind}_{N\uparrow G}\pi$. We induce $\tilde{\pi}$ to $\hat{C}_r^*(C_r^*(A, G, \alpha), G/N, \delta)$ by T_2, and put $\tilde{\tilde{\pi}}=\mathrm{Ind}_{T_2}\tilde{\pi}$.

PROPOSITION 19. *This induced representation $\tilde{\tilde{\pi}}$ is unitarily equivalent to a multiple of $\pi\otimes\mathrm{Id}_{\mathfrak{C}(L^2(G/N))}$ as a representation of $C_r^*(A, N, \alpha)\otimes\mathfrak{C}(L^2(G/N))$.*

PROOF. Consider $\tilde{\tilde{\pi}}$ by using S_1 and S_2. Let π_μ be the canonical representation of $C_0(G/N)$ on $L^2(G/N)$. Let $\pi_{\dot{x}}$ be the representation of $C_0(G/N)$ given by $\pi_{\dot{x}}(f)=f(\dot{x})$ for $f\in C_0(G/N)$. By the G-invariance of π, there exists a unitary representation V of G satisfying $\beta_s(\pi(x))=V_s\pi(x)V_s^*$ for all $s\in G$ and $x\in C_r^*(A,N,\alpha)$. Then, inducing π by S_1 is simply taking the tensor product with π_μ. And so, $\mathrm{Ind}_{S_2\circ S_1}\pi=\mathrm{Ind}_{N\uparrow G}\pi\otimes\pi_\mu$, and this is naturally equivalent to $\int_{G/N}^{\oplus}(\mathrm{Ind}_{N\uparrow G}\pi\otimes\pi_{\dot{x}})d\dot{x}$. The representation space $\mathfrak{H}_1$ of the latter representation is the set $\{\xi : G\times(G/N)\to\mathfrak{H}_\pi$ measurable, $\xi(sn, \dot{t})=\pi(n)^*\xi(s, \dot{t})$ for $s\in G$, $n\in N$, $\dot{t}\in G/N$, $\int_{G/N}\|\xi(s)\|^2d\dot{s}<\infty\}$. Let σ be a Borel section from G/N to G. We put $(W\xi)(s, \dot{t})=V_{\sigma(\dot{t})}\xi(s\sigma(\dot{t}), \dot{t})\Delta_{G/N}(\dot{t})^{-1/2}$. Then W is a unitary map from $\mathfrak{H}_1$ to $L^2(G/N)\otimes L^2(\mathfrak{H}_\pi, G, \pi)$ and we have,

$$W(\mathrm{Ind}_{N\uparrow G}\pi\otimes\pi_\mu)W^*=\mathrm{Ind}_{N\uparrow G}\pi\otimes\pi_{\dot{e}}\otimes I_{L^2(G/N)}.$$

This representation is obtained as the combination of $\mathrm{Ind}_{N\uparrow G}\pi$ and the diagonal multiplication η. By "the transfer principle", this representation is $\pi\otimes\mathrm{Id}_{\mathfrak{C}(L^2(G/N))}$ on $L^2(G/N)$ as a representation of $C_r^*(A, N, \alpha)\otimes\mathfrak{C}(L^2(G/N))$. q. e. d.

Now, we see that $\mathrm{Ind}_{N\uparrow G}(\pi\otimes\pi_\mu)$ is quasi equivalent to $\mathrm{Ind}_{N\uparrow G}(\pi\otimes\pi_{\dot{e}})$. We denote by $\overline{W}$ the corresponding $*$-isomorphism between the von Neumann algebras generated by them.

REMARK 20. As a corollary of this proposition, we have Imai-Takai duality [**9**]. When G/N is abelian, this proposition is a special case of [**17**].

4. Duality for induced weights.

At first, we need one technical result. Let M be a von Neumann algebra, $(A, \delta_A, j_A, \phi_A)$ and $(B, \delta_B, j_B, \phi_B)$ be two Kac algebras. Suppose that there exist actions δ_1 and δ_2 of A and B on M respectively. Let σ be the flip map from $A\otimes B$ to $B\otimes A$ by $\sigma(x\otimes y)=y\otimes x$.

DEFINITION 21. Two actions δ_1 and δ_2 are called to commute if $(\delta_1\otimes\iota_B)\delta_2=(\iota\otimes\sigma)(\delta_2\otimes\iota_A)\delta_1$ holds, where ι_A and ι_B are the identity maps on A and on B respectively.

The von Neumann tensor product $A\otimes B$ has a natural Kac algebra structure by $(\iota_A\otimes\sigma\otimes\iota_B)(\delta_A\otimes\delta_B)=\delta$, $j_A\otimes j_B=j$ and $\phi_A\otimes\phi_B=\phi$ as its Kac algebra objects.

LEMMA 22. *If two actions δ_1 and δ_2 commute, then $\delta_1\times\delta_2=\tilde{\delta}=(\delta_1\otimes\iota_B)\delta_2$ defines an action of $A\otimes B$ on M.*

PROOF. The linear map $\delta_1\times\delta_2$ is a normal $*$-isomorphism. The associativity follows from those of δ_1 and δ_2, and the commutativity follows from a straightforward calculation. q. e. d.

LEMMA 23. *If δ_1 and δ_2 commute, $\delta_{1h}\delta_{2k}=\delta_{2k}\delta_{1h}$ for $h\in A_*$ and $k\in B_*$, and equal to $\tilde{\delta}_{h\otimes k}$.*

PROOF. We fix $\omega\in M_*$. Then we have,

$$\langle\delta_1\times\delta_{2\,h\otimes k}(x),\ \omega\rangle=\langle\delta_2(x),\ \omega\circ\delta_{1h}\otimes k\rangle=\langle\delta_{1h}\delta_{2k}(x),\ \omega\rangle.$$

This shows that $\delta_1\times\delta_{2\,h\otimes k}=\delta_{1h}\delta_{2k}$. Similarly we have another identity.

q. e. d.

LEMMA 24. *Suppose that δ_1 and δ_2 commute, and one of A and B is of the form $L^\infty(G)$ or $L(H)$. Then $\delta_1\times\delta_2(x)=x\otimes 1_A\otimes 1_B$ holds if and only if $\delta_1(x)=x\otimes 1_A$ and $\delta_2(x)=x\otimes 1_B$ for $x\in M$.*

PROOF. "If" part is trivial. We assume that $\tilde{\delta}(x)=x\otimes 1_A\otimes 1_B$. This is equivalent to $\delta_1\times\delta_{2\,h\otimes k}(x)=\langle h,\ 1\rangle\langle k,\ 1\rangle x$ for every $h\in A_*$ and $k\in B_*$. We have $\delta_{1h}\delta_{2k}(x)=\langle h,\ 1\rangle\langle k,\ 1\rangle x$. Suppose that A is of the form $L^\infty(G)$. Let $h\to\delta_e$. Then $\delta_{1h}(\delta_{2k}(x))\to\delta_{2k}(x)$ and $\langle h,\ 1\rangle\to 1$, so we have $\delta_{2k}(x)=\langle k,\ 1\rangle x$ and $\delta_{1h}(x)=\langle h,\ 1\rangle x$. Other cases are analogous. q. e. d.

This lemma shows that $M^{\delta_1\times\delta_2}=(M^{\delta_1})^{\delta_2}=(M^{\delta_2})^{\delta_1}$. We consider three operator valued weights $T^{\delta_1\times\delta_2}$, T^{δ_1} and T^{δ_2} given by the actions $\delta_1\times\delta_2$, δ_1 and δ_2 respectively (Proposition II. 2 [2]).

LEMMA 25. $T^{\delta_1\times\delta_2}=T^{\delta_1}T^{\delta_2}=T^{\delta_2}T^{\delta_1}$.

PROOF. Let $F_A=\{h\in A_*^+ : h\leqq\phi_A\}$ and $F_B=\{k\in B_*^+ : k\leqq\phi_B\}$. Then for every $x\in M^+$, $T^{\tilde{\delta}}(x)=\sup_{h\in F_A, k\in F_B}\tilde{\delta}_{h\otimes k}(x)=\sup_{h\in F_A, k\in F_B}\delta_{1h}(\delta_{2k}(x))=\sup_{k\in F_B}T^{\delta_1}(\delta_{2k}(x))=T^{\delta_1}(T^{\delta_2}(x))$. We used the fact that T^{δ_1} is normal. We get another identity similarly. q. e. d.

Let (A, G, α) be a C^*-dynamical system, and N be a closed normal subgroup of G. We chose a G-invariant representation π of $C_r^*(A, N, \alpha)$. Put $M=W^*(\pi)$, and denote by β the action of G on M induced from α. Let $\tilde{\pi}=\mathrm{Ind}_{N\uparrow G}\pi$ and $P=W^*(\tilde{\pi})$.

Since $W^*(\mathrm{Ind}_{T_2}\tilde{\pi})=\hat{W}^*(P, G/N, \delta)\,(=X)$, by Lemma 13, X is generated by $\mathrm{Ind}_{N\uparrow G}\pi\otimes\pi_\mu(C_r^*(A\otimes C_0(G/N), G, \alpha\otimes\dot{\lambda}))$. The space of this representation is $L^2(G/N)\otimes L^2(\mathfrak{H}_\phi, G, \pi)$. We show that there exists an action of G/N and a coaction of G/N on X. For this, we define two unitary operators W^1 and W^2. For $\xi_1\in L^2(G/N)\otimes L^2(\mathfrak{H}, G, \pi)\otimes L^2(G/N)$,

$$(W^1\xi_1)(\dot{t}, s, \dot{x}) = \Delta_{G/N}(\dot{x})^{1/2}\xi_1(\dot{t}\dot{s}, s, \dot{x}) \qquad \text{for } \dot{t}, \dot{x}\in G/N,\ s\in G.$$

For $\xi_2\in L^2(G/N)\otimes L^2(\mathfrak{H}, G, \pi)\otimes L^2(G/N)$,

$$(W^2\xi_2)(\dot{t}, s, \dot{x}) = \xi_2(\dot{t}, s, \dot{s}\dot{x}) \qquad \text{for } \dot{t}, \dot{x}\in G/N,\ s\in G.$$

For $x\in X$, we put $\gamma(x)=W^1(x\otimes 1_{L^2(G/N)})W^{1*}$ and $\delta(x)=W^{2*}(x\otimes 1_{L^2(G/N)})W^2$.

LEMMA 26. *The map γ is an action of $L^\infty(G/N)$ on X, and δ is an action of $L(G/N)$ on X. Moreover γ and δ commute.*

PROOF. This is shown by a simple calculation. q. e. d.

By the general theory, there exist three operator valued weights T^γ, T^δ and $T^{\gamma\times\delta}$ on X. By Lemma 26, $X^{\gamma\times\delta}$ is equal to the image of M in X. Let ϕ be a f. n. s. weight on M, and $\tilde{\phi}$ be the induced weight on P. We induce $\tilde{\phi}$ to X by T^γ, and denote it by $\tilde{\tilde{\phi}}$. By definition, $\tilde{\tilde{\phi}}$ is induced by $T^\delta\circ T^\gamma$. The above lemma allows us to consider $\tilde{\tilde{\phi}}$ as induced by $T^\gamma\circ T^\delta$ from M. Denote by ϕ_μ the trace on $C_0(G/N)$ given by a left Haar measure of G/N. The induced weight $\bar{\phi}$ on $M\otimes L^\infty(G/N)$ $(=X^\delta)$ is simply $\phi\otimes\phi_\mu$. By these arguments $\tilde{\tilde{\phi}}=\mathrm{Ind}_{N\uparrow G}\phi\otimes\phi_\mu$. The GNS representation associated with $\bar{\phi}$ is $\pi_\phi\otimes\pi_\mu$ and that of $\tilde{\tilde{\phi}}$ is $\mathrm{Ind}_{N\uparrow G}(\pi_\phi\otimes\pi_\mu)$. The following proposition is the direct consequence of Proposition 19 (cf. [**13**]).

LEMMA 27. *The $*$-isomorphism $\overline{W}$ defined after the proof of Proposition* 19 *and the canonical Borel section argument give a $*$-isomorphism between $W^*(\mathrm{Ind}_{N\uparrow G}(\pi_\phi\otimes\pi_\mu))$ and $M\otimes\mathfrak{B}(L^2(G/N))$.*

Let Tr be the ordinary trace on $\mathfrak{B}(L^2(G/N))$. By $\overline{W}$, X is transferred to an operator algebra on $L^2(\mathfrak{H}_\phi, G, \pi)$. Let σ be a Borel section from G/N to G and

$\varepsilon(s)=\sigma(\dot{s})^{-1}$. We define a unitary map V from $L^2(\mathfrak{H}_\phi, G, \pi)$ to $L^2(\mathfrak{H}_\phi, G/N)$. This map is defined in [17]. For $\xi\in L^2(\mathfrak{H}_\phi, G, \pi)$, $(V\xi)(\dot{s})=\pi(\varepsilon(s))\xi(s)$. $\mathrm{Ad}\,V$ is the so called "canonical cross section argument". Put $X_0=\overline{W}(X)$. Let ϕ^1 be the transfer of $\tilde{\tilde{\phi}}$ to X_0 by $\overline{W}$, and ϕ^2 be the transfer of $\phi\otimes\mathrm{Tr}$ to X_0 by $\mathrm{Ad}\,V^*$.

LEMMA 28. *A f.n.s. weight ϕ^2 is induced from $M\otimes L^\infty(G/N)$.*

PROOF. By Proposition 12, it suffices to show that ϕ^2 is (δ, j) invariant for the canonical coaction of G/N. Denote by δ^0 the natural coaction of G/N on $\mathfrak{B}(L^2(G/N))$. The coaction δ is transferred to $I\otimes\delta^0$ on $M\otimes\mathfrak{B}(L^2(G/N))$ by V. Since Tr on $\mathfrak{B}(L^2(G/N))$ is (δ^0, j^0) invariant, $\phi\otimes\mathrm{Tr}$ is $(I\otimes\delta^0, j^0)$ invariant. This shows the (δ, j) invariance of ϕ^2. q.e.d.

We define a f.n.s. weight ϕ_0^2 on $M\otimes L^\infty(G/N)$ by $\phi^2=\mathrm{Ind}_{N\uparrow G}\phi_0^2$. For simplicity, put $\Pi=\pi_\phi\otimes\pi_{\dot{e}}$, and $\tilde{\Pi}=\mathrm{Ind}_{N\uparrow G}\Pi$.

Let $f\in K(M\otimes L^\infty(G/N), G, \Pi)$ and $\xi\in L^2(\mathfrak{H}_\phi, G, \Pi)$. Then we have,

$$\tilde{\Pi}(f^{*}*f)\xi(t)=\int_{G/N}\int_{G/N}\beta_t^{-1}(f(u, \dot{s})^*)\beta_{t^{-1}s}(f(us, \dot{s}^{-1}\dot{t}))\xi(\dot{s}^{-1}\dot{t})d\dot{u}d\dot{s}.$$

We realize $\tilde{\Pi}(f^{*}*f)$ as an operator on $L^2(\mathfrak{H}_\phi, (G/N), \Pi)$.

$$\begin{aligned}V^*\tilde{\Pi}(f^{*}*f)V\xi(t)&=\int_{G/N}\int_{G/N}\pi(\varepsilon(t))\beta_t^{-1}((f(u, \dot{t})^*)\beta_{t^{-1}s}(f(us, \dot{s}^{-1}\dot{t}))]\\&\qquad\pi(\varepsilon(s^{-1}t))^*\xi(\dot{s}^{-1}\dot{t})d\dot{u}d\dot{s}\\&=\int_{G/N}\Big(\int_{G/N}\Delta_{G/N}(\dot{s})^{-1}\pi(\varepsilon(t))\beta_t^{-1}(f(u, \dot{t})^*)\beta_s^{-1}(f(uts^{-1}, \dot{s}))\\&\qquad\pi(\varepsilon(s))^*d\dot{u}\Big)\xi(\dot{s})d\dot{s}.\end{aligned}$$

Hence, $V^*\tilde{\Pi}(f^{*}*f)V$ is represented as an operator valued integral kernel. Denote it by $K(\dot{s}, \dot{t})$. Suppose that there exists a compact subset Y of G/N such that K is supported in $Y\times Y$ and $K|_{Y\times Y}$ is σ-weakly continuous. We call this "continuity assumption". Then, by the well known fact, we have,

$$\begin{aligned}(\phi\otimes\mathrm{Tr})(V^*\tilde{\Pi}(f^{*}*f)V)&=\int_{G/N}\phi(K(\dot{t}, \dot{t}))d\dot{t}\\&=\int_{G/N}\Big(\int_{G/N}\Delta_{G/N}(\dot{t})^{-1}\phi(\beta_{\sigma(\dot{t})}^{-1}(f(u, \dot{t})^*f(u, \dot{t})))d\dot{t}\Big)d\dot{u}.\end{aligned}$$

Seeing this, we define a f.n.s. weight ϕ_0' on $M\otimes L^\infty(G/N)$ by

$$\phi_0'(T)=\int_{G/N}\Delta_{G/N}(\dot{x})^{-1}\phi(\beta_{\sigma(\dot{x})}^{-1}(T_{\dot{x}}))d\dot{x},$$

where $T=\int_{G/N}^{\oplus}T_{\dot{x}}d\dot{x}$ is in $(M\otimes L^\infty(G/N))^+$. Let Δ_ϕ be the modular operator of ϕ, and put $\Delta_\phi^{it}(s)=\pi(\varepsilon(s))^*\Delta_\phi^{it}\pi(\varepsilon(s))$ and $(\tilde{\Delta}_\phi^{it}\xi)(s)=\Delta_\phi^{it}(s)\xi(s)$ for $\xi\in L^2(\mathfrak{H}_\phi, G, \Pi)$. Then the modular automorphism group σ^{ϕ^2} of ϕ^2 is given by $\mathrm{Ad}\,\tilde{\Delta}_\phi^{it}$. Then we have

$$(\tilde{\Delta}_{\phi}^{it}\pi_{\beta\otimes\lambda}(x\otimes k)\tilde{\Delta}_{\phi}^{-it}\xi)(s)=\beta_s^{-1}(\beta_{\sigma(\dot{s})}\sigma_t^{\phi}\beta_{\sigma(\dot{s})}^{-1}(x))k(\dot{s})\xi(\dot{s}).$$

This shows that $\sigma^{\phi_0^2}$ is given by $\int_{G/N}^{\oplus}\beta_{\sigma(\dot{s})}\sigma_t^{\phi}\beta_{\sigma(\dot{s})}^{-1}d\dot{s}$. Clearly, we have $\sigma_t^{\phi_0'}=\int_{G/N}^{\oplus}\beta_{\sigma(\dot{s})}\sigma_t^{\phi}\beta_{\sigma(\dot{s})}^{-1}d\dot{s}$. We have $\sigma^{\phi_0^2}=\sigma^{\phi_0'}$. Let F be a measurable map from G to $W^*(\pi_{\phi})$ such that $F(sn)=\pi_{\phi}(n)^*F(s)$, and $\dot{s}\to\|F(s)\|$ is bounded and has a compact support. Then,

LEMMA 29 (Lemma 3.1.3 in [**17**]). *The map* $s\in G\to F^{*}*F(s)$ *is* σ*-weakly continuous.*

Let D be the linear span of the set $\{D(\dot{x})=\beta_{\sigma(\dot{x})}(T)1_C(\dot{x})\Delta_{G/N}(\dot{x}) : T\in m_{\phi}, C$ is a compact subset of G/N such that $\sigma|_C$ is continuous$\}$ in $M\otimes L^{\infty}(G/N)$. Then, by the σ^{ϕ}-invariance of m_{ϕ}, D is a $\sigma^{\phi_0'}$-invariant weakly dense subalgeba of $M\otimes L^{\infty}(G/N)$.

Let $f(s,\dot{x})=\pi(\varepsilon(s))^*k(\dot{s})\beta_{\sigma(\dot{x})}(T)1_C(\dot{x})\Delta_{G/N}(\dot{x})^{1/2}$, where $k\in C_c(G/N)$, $T\in m_{\phi}$. Then $f\in K(M\otimes L^{\infty}(G/N), G, \Pi)$. Then the integral kernel K_0 of $\tilde{\Pi}(f^{*}*f)$ is given by

$$K_0(\dot{s},\dot{t})=\Delta_{G/N}(\dot{t}\dot{s}^{-1})^{1/2}1_C(\dot{t})1_C(\dot{s})\pi(\varepsilon(t))\beta_{t^{-1}\sigma(\dot{t})}(T^*)\int_{G/N}k(\dot{u})k(\dot{u}\dot{t}\dot{s}^{-1})$$
$$\beta_t^{-1}(\pi(\varepsilon(u))\beta_s^{-1}(\pi(uts^{-1}))^*)d\dot{u}\cdot\beta_{s^{-1}\sigma(\dot{s})}(T)\pi(\varepsilon(s))^*.$$

The right hand side is well defined on G/N because of the covariance relation between β and π. Put $F(s)=\pi(\varepsilon(s))^*k(\dot{s})$. Then

$$K_0(\dot{s},\dot{t})=\Delta_{G/N}(\dot{t}\dot{s}^{-1})^{1/2}1_C(\dot{t})1_C(\dot{s})\pi(\varepsilon(t))\beta_{t^{-1}\sigma(\dot{t})}(T^*)\beta_s^{-1}(F^{*}*F)(ts^{-1})$$
$$\cdot\beta_{s^{-1}\sigma(\dot{s})}(T)\pi(\varepsilon(s))^*.$$

By the condition of C this integral kernel satisfies "continuity assumption". We have,

$$\phi^2(\tilde{\Pi}(f^{*}*f))=\int_{G/N}|k(\dot{u})|^2d\dot{u}\int_{G/N}\phi(T^*T)1_C(\dot{t})d\dot{t}<\infty.$$

Hence for $D(x)=\beta_{\sigma(\dot{x})}(T)1_C(\dot{x})\Delta_{G/N}(\dot{x})$,

$$\phi_0^2(D)=\int_{G/N}\phi(T^*T)1_C(\dot{t})d\dot{t}<\infty.$$

By Theorem 5.9 of [**15**], we have $\phi_0^2=\phi_0'$.

We conclude that ϕ^2 is induced from ϕ_0'. Since $(D(\phi\otimes\phi_{\mu}):D\phi_0^2)_t=\int_{G/N}^{\oplus}\Delta_{G/N}(\dot{x})^{it}(D\phi:D\phi\circ\beta_{\sigma(\dot{x})}^{-1})_t d\dot{x}$ and $(D\phi':D\phi^2)_t=\pi_{\beta\otimes\lambda}((D(\phi\otimes\phi_{\mu}):D\phi_0^2)_t)$, we have the following

PROPOSITION 30.

$$(D\phi^1:D\phi^2)_t\xi(s)=\Delta_{G/N}(\dot{s})^{it}(D\phi\circ\beta_s:D\phi\circ\beta_{\varepsilon(s)})_t\xi(s),\qquad\text{for }\xi\in L^2(\mathfrak{H}_{\phi},G,\pi).$$

REMARK 31. When $N=\{e\}$, it reduces to the duality of Stratila-Voiculescu-Zsido [**21**], and when G/N is abelian, reduces to that of N. V. Pedersen [**17**]. We use the technique developed there.

COROLLARY 32. *When ϕ is a $\Delta_{G/N}$-relatively invariant f. n. s. trace on M, $\tilde{\tilde{\phi}}$ is transformed to $\phi\otimes\mathrm{Tr}$ on $M\otimes\mathfrak{B}(L^2(G/N))$.*

5. Induced traces on C^*-crossed products.

Let (A, G, α) be a C^*-dynamical system, and N be a closed normal subgroup of G. Let ϕ_0 be a lower semicontinuous semifinite (l. s.) trace on $C_r^*(A, N, \alpha)$ which is $\Delta_{G/N}$-relatively invariant under the original action of G. Then the GNS representation π_{ϕ_0} is G-invariant, and so we can construct an induced weight ϕ on $C_r^*(A, G, \alpha)$. This is the composition of $\mathrm{Ind}_{N\uparrow G}\pi_{\phi_0}$ and the induced weight on $W^*(\mathrm{Ind}_{N\uparrow G}\pi_{\phi_0})$ in W^*-sense. By $\Delta_{G/N}$-relative invariance condition, ϕ is a trace. Since G/N is not necessarily abelian, we use invariance under coaction as in W^*-case.

For the definition and simple properties of coaction in C^*-sense, we refer to [**10**]. There exists a coaction δ of G/N on $C_r^*(A, G, \alpha)$ dual to α. The involution of the Kac C^*-algebra $C_r^*(G/N)$ (cf. [**24**]) is denoted by j. We define (δ, j)-invariance of l. s. weights on C^*-algebras under actions of Kac C^*-algebras as in W^*-case.

LEMMA 33. *The induced trace ϕ is (δ, j)-invariant under the coaction δ of G/N.*

PROOF. The GNS representation π_ϕ associated with ϕ is the induced representation $\mathrm{Ind}_{N\uparrow G}\pi_{\phi_0}$. This is the situation in § 1. The von Neumann algebra $W^*(\pi_\phi)$ is acting on $L^2(\mathfrak{H}_\phi, G, \pi_{\phi_0})$, and there exists a canonical coaction $\tilde{\delta}$ of G/N on $W^*(\pi_\phi)$ corresponding to δ. Then, by Lemma 11, the normal extension $\tilde{\phi}$ of ϕ to $W^*(\pi_\phi)$ is $(\tilde{\delta}, \tilde{j})$-invariant. Since π_ϕ is an induced representation from N, there exists a coaction $\bar{\delta}$ of G/N on $\pi_\phi(C_r^*(A, G, \alpha))$ corresponding to δ on $C_r^*(A, G, \alpha)$ via π_ϕ. The trace $\bar{\phi}$ on $\pi_\phi(C_r^*(A, G, \alpha))$ obtained by the restriction from $\tilde{\phi}$ is clearly $(\bar{\delta}, \bar{j})$-invariant. By the definition of π_ϕ, $\bar{\delta}_k$ and δ_k, we have $\bar{\delta}_k(\pi_\phi(x))=\pi_\phi(\delta_k(x))$ for $k\in A(G/N)$ and $x\in C_r^*(A, G, \alpha)$. Hence, ϕ is (δ, j)-invariant. q. e. d.

Let ϕ be a l. s. trace on $C_r^*(A, G, \alpha)$ which is (δ, j)-invariant. Let π_ϕ be the GNS representation associated with ϕ, and $\mathfrak{H}_\phi$ be its representation space. Put $B=C_r^*(A, G, \alpha)$ and $M=W^*(\pi_\phi)$.

LEMMA 34. *Pointwise multiplication of $C_0(G/N)$ on $C_r^*(A, G, \alpha)$ extends to a bounded $*$-representation of $C_0(G/N)$ on $\mathfrak{H}_\phi$.*

PROOF. By Proposition 1-7 in [**1**], ϕ is supremum of positive linear functionals. Using this fact, we can conclude that $k\in A(G/N)\to\delta_k$ is a bounded representation of $A(G/N)$ on $\mathfrak{H}_\phi$ analogously to the W^*-situation (Lemma 0.2.3 in [**21**]). $*$-preserving property of this representation follows from j-invariance of ϕ. The enveloping C^*-algebra of $A(G/N)$ is $C_0(G/N)$. Hence, the above representation extends to $C_0(G/N)$. q. e. d.

Since the range of this representation commutes with $\pi_\phi(C_r^*(A, N, \alpha))$, this representation constitutes a transitive system of imprimitivity based on G/N for π_ϕ. By Mackey-Takesaki imprimitivity theorem [**23**], there exists a unique representation π of $C^*(A, N, \alpha)$ such that $\pi_\phi=\mathrm{Ind}_{N\uparrow G}\pi$. On the other hand π is induced from $C_r^*(A\otimes C_0(G/N), G, \alpha\otimes\dot{\lambda})$ in C^*-sense, since π_ϕ is made into a representation of $C_r^*(A\otimes C_0(G/N), G, \alpha\otimes\dot{\lambda})$. By this and Lemma 18, π is a representation of $C_r^*(A, N, \alpha)$. Denote the action of G on $C_r^*(A, N, \alpha)$ by β. Let $s\cdot\pi(x)=\pi(\beta_s(x))$ for $s\in G$, $x\in C_r^*(A, N, \alpha)$. Put $\Pi=\int_G^{\oplus}s\cdot\pi ds$, where ds is a left Haar measure on G. By the argument used in the proof of Proposition 19, $\mathrm{Ind}_{N\uparrow G}s\cdot\pi\cong\mathrm{Ind}_{N\uparrow G}\pi$, and so $\mathrm{Ind}_{N\uparrow G}\Pi$ is unitarily equivalent to $(\mathrm{Ind}_{N\uparrow G}\pi)\otimes I_{L^2(G)}$. Let $\mathrm{Ind}_{N\uparrow G}\Pi=\tilde{\Pi}$. Then $W^*(\pi_\phi)$ is isomorphic to $W^*(\tilde{\Pi})$ by the quasi equivalence between π_ϕ and $\tilde{\Pi}$. We identify these two algebras.

Since $\tilde{\Pi}$ is G invariant, $W^*(\tilde{\Pi})$ can be considered as an extended covariance algebra. Let $\tilde{\phi}$ be the normal extension of ϕ to $W^*(\pi_\phi)$. We consider $\tilde{\phi}$ as a f. n. s. weight on $W^*(\tilde{\Pi})$ by the above isomorphism. Since $W^*(\tilde{\Pi})$ is as in § 2, there exists a canonical coaction $\tilde{\delta}$ of G/N on $W^*(\tilde{\Pi})$, and this corresponds to the original δ on $C_r^*(A, G, \alpha)$ by $\tilde{\Pi}$. Since the GNS representation spaces $\mathfrak{H}_{\pi_\phi}$ and $\mathfrak{H}_{\pi_{\tilde{\phi}}}$ are the same, the pointwise multiplication of $A(G/N)$ on $W^*(\tilde{\Pi})$ (consider this algebra in induced representation sense) gives a bounded $*$-representation of $A(G/N)$. By Lemma 10, $\tilde{\phi}$ is $(\tilde{\phi}, \tilde{j})$-invariant. By Proposition 12, $\tilde{\phi}$ is induced from a f. n. s. weight $\tilde{\phi}_0$ on $W^*(\Pi)$. And by Lemma 4 (2), $\tilde{\phi}_0$ is a trace, and $\tilde{\phi}_0$ is $\Delta_{G/N}$-relatively invariant under the action α of G.

The next problem is whether $\tilde{\phi}_0$ is semifinite or not. For this purpose, we consider the cocrossed products $\hat{C}_r^*(\tilde{\Pi}(B), G/N, \bar{\tilde{\delta}})$ and $\hat{W}^*(M, G/N, \tilde{\delta})$, where $\bar{\tilde{\delta}}$ is the coaction on $\tilde{\Pi}(B)$ given by $\tilde{\delta}$. By Proposition 19 and Lemma 27, $\hat{C}_r^*(\tilde{\Pi}(B), G/N, \bar{\tilde{\delta}})\cong\tilde{\Pi}(C_r^*(A, N, \alpha))\otimes\mathfrak{C}(L^2(G/N))$, and $\hat{W}^*(M, G/N, \tilde{\delta})=W^*(\tilde{\Pi})\otimes\mathfrak{B}(L^2(G/N))$, and moreover, these two decompositions are given by the same unitary operator.

Let $\bar{\phi}$ be $\tilde{\phi}|_{\tilde{\Pi}(B)}$. We consider the induced weight ϕ_1 on $\hat{W}^*(M, G/N, \tilde{\delta})$ by the operator valued weight given by the coaction $\tilde{\delta}$. Let $\bar{\phi}_1$ be the restriction of ϕ_1 to $\hat{C}_r^*(\tilde{\Pi}(B), G/N, \bar{\tilde{\delta}})$.

LEMMA 35. *This restriction $\bar{\phi}_1$ is semifinite. If ϕ is densely defined, so is $\bar{\phi}_1$.*

PROOF. Let $y=\bar{\tilde{\delta}}(x)(1\otimes h)$, where $x\in m_{\bar{\phi}_1}$ and $h\in C_c(G/N)$. Let T be the operator valued weight which gives the induction from $W^*(\tilde{\Pi})$ to $\hat{W}^*(M, G/N, \tilde{\delta})$. Then, y is T integrable in the sense of [13] and $T(y)=\left(\int_{G/N} h(\dot{s})d\dot{s}\right)\cdot x$. This shows that the above y is contained in $m_{\tilde{\phi}_1}$. Since $\hat{C}_r^*(\tilde{\Pi}(B), G/N, \bar{\bar{\delta}})$ is generated by the linear combinations of the products $\bar{\bar{\delta}}(x)(1\otimes h)$ or $(1\otimes h)\bar{\bar{\delta}}(x)$, and $m_{\tilde{\phi}}$ is weakly dense in $\hat{W}^*(M, G/N, \tilde{\delta})$, $m_{\tilde{\phi}_1}$ is weakly dense. Densely defined case is similar. q. e. d.

LEMMA 36. *The trace $\tilde{\phi}_1$ is transformed to $\tilde{\phi}_0\otimes\mathrm{Tr}$ on $W^*(\Pi)\otimes\mathfrak{B}(L^2(G/N))$.*

PROOF. This is just Corollary 32. q. e. d.

LEMMA 37. *The restriction $\bar{\phi}$ of $\tilde{\phi}_0$ to $\Pi(C_r^*(A, N, \alpha))$ is semifinite. If ϕ is densely defined, so is $\bar{\phi}$.*

PROOF. Denote the norm closure of $\{x\in\Pi(C_r^*(A, N, \alpha))\otimes\mathfrak{C}(L^2(G/N)) : \tilde{\phi}_0\otimes\mathrm{Tr}(x^*x)<\infty\}$ by C. Then C is of the form of $D\otimes\mathfrak{C}(L^2(G/N))$, where D is a closed ideal of $\Pi(C_r^*(A, N, \alpha))$. By Lemma 4-2 of [16], $\tilde{\phi}_0|_D$ is densely defined. Since $D\otimes\mathfrak{C}(L^2(G/N))$ is weakly dense in $W^*(\Pi)\otimes\mathfrak{B}(L^2(G/N))$, this shows that $m_{\tilde{\phi}}\cap\Pi(C_r^*(A, N, \alpha))$ is weakly dense in $W^*(\Pi)$. q. e. d.

Let ϕ_0 be $\tilde{\phi}_0\circ\Pi$. It is a l. s. trace on $C_r^*(A, N, \alpha)$, and $\tilde{\phi}_0$ is the canonical normal extension of ϕ_0 to $W^*(\Pi)$. The trace $\tilde{\phi}_0$ is clearly $\Delta_{G/N}$-relatively invariant. From these arguments we have,

LEMMA 38. *The original trace ϕ is induced from a l.s. trace ϕ_0 of $C_r^*(A, N, \alpha)$.*

Now we have the following theorem.

THEOREM. *A lower semicontinuous semifinite trace on $C_r^*(A, G, \alpha)$ is induced from $C_r^*(A, N, \alpha)$ if and only if it is (δ, j)-invariant under the canonical coaction δ of G/N on $C_r^*(A, G, \alpha)$.*

REMARK 39. When $N=\{e\}$ and G is abelian, Theorem reduces to the special case of Theorem 5.1 of [16].

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