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Provably recursive functions in fragments of Peano arithmetic

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§ 1. Introduction.

In this paper, we will make a proof-theoretic study of Paris-Harrington's independence results for Peano arithmetic [10]. First, we will give a characterization of provably recursive functions in fragments of Peano arithmetic. Then, we will analyze the combinatorial statements by Paris and Harrington, making use of our characterization.

Let PA be Peano's first-order arithmetic and PA* be the extension of PA obtained by adding all true Π_1-formulas as axioms. The fragments under consideration can be obtained by restricting induction formulas of the mathematical induction to formulas containing at most k quantifiers, for a given k.

Some basic facts on ordinal recursive functions and Wainer's hierarchy [17] will be presented in § 2. In § 3, we will prove our theorem on the characterization of provably recursive functions. By our theorem, the relation between α-ordinal recursive functions ($\alpha<\varepsilon_0$) and provably recursive functions in fragments of PA* will be clarified. Another characterization of these functions will be also stated in our theorem, in terms of the provability of some bounded formulas or some Δ_1-formulas in fragments of PA. We emphasize here that our theorem can be shown by using a purely proof-theoretic method.

This characterization enables us to analyze the combinatorial statements by Paris and Harrington which are shown to be independent of Peano arithmetic. This will be done in § 4, by making use of the estimation of rapidly growing functions associated with these combinatorial statements, due to Ketonen and Solovay [3]. Indeed, we can give an alternative proof of a result by Paris [9], in a proof-theoretic way. We will also mention explicitly how the provability or the unprovability of these statements depends on their representation in formal systems.

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§ 2. Preliminaries.

In the following, we will use small Greek letters $\alpha, \beta, \gamma, \cdots$ for ordinal numbers. Define the ordinal $\omega_n(m)$ for each $m, n<\omega$ by

$$\omega_0(m) = m, \qquad \omega_{n+1}(m) = \omega^{\omega_n(m)}.$$

We will abbreviate $\omega_n(1)$ to ω_n. As usual, ε_0 denotes the first ordinal α such that $\alpha=\omega^\alpha$. In most cases in the following discussion, we will deal with only number-theoretic functions. So, hereafter by functions we mean number-theoretic ones, unless otherwise mentioned.

For each natural number $k>0$, $<_k$ denotes the elementary recursive well-ordering of natural numbers of order-type ω_k, which is defined in § 3 of Wainer [17]. For each x, $\mathrm{ord}_n(x)$ is the ordinal represented by x in the well-ordering $<_n$ and for each $\alpha<\omega_n$, $\mathrm{num}_n(\alpha)$ is the unique natural number x such that $\mathrm{ord}_n(x)=\alpha$. Let α be any ordinal less than ε_0 and n be the smallest natural number such that $\alpha<\omega_n$. Following [17], we will define $U(\alpha)$ to be the smallest class of functions containing all primitive recursive functions, which is closed under substitution and the following (unnested) recursion up to α;

$$f(0, \boldsymbol{z}) = g_1(\boldsymbol{z}),$$
$$f(x+1, \boldsymbol{z}) = g_2(x+1, \boldsymbol{z}, f(h(x+1, \boldsymbol{z}), \boldsymbol{z})),$$

where $h(x, \boldsymbol{z})<_n x$ for each x such that $0<_n x<_n \mathrm{num}_n(\alpha)$ and $h(x, \boldsymbol{z})=0$ otherwise. A function f is said to be *α-ordinal recursive* if f belongs to $U(\alpha)$, and f is said to be *ordinal recursive* if f is ω_n-ordinal recursive for some $n<\omega$.

In [17], Wainer introduced a hierarchy $\{\mathcal{F}_\alpha\}_{\alpha\leqq\varepsilon_0}$ of a certain subclass of recursive functions, as follows. First, a fixed fundamental sequence $\{\alpha\}(n)$ $(n<\omega)$ will be defined for each limit ordinal $\alpha\leqq\varepsilon_0$. Suppose that $\alpha<\varepsilon_0$ and α is of the form $\omega^\beta\cdot(\gamma+1)$. Then, $\{\alpha\}(n)$ is $\omega^\beta\cdot\gamma+\omega^\delta\cdot n$ if $\beta=\delta+1$, and is $\omega^\beta\cdot\gamma+\omega^{\{\beta\}(n)}$ if β is a limit ordinal. When $\alpha=\varepsilon_0$, $\{\varepsilon_0\}(n)$ is defined to be ω_n for each n. For each unary function f, f^m is defined inductively as $f^1(x)=f(x)$, $f^{k+1}(x)=f(f^k(x))$. Now, the functions F_α $(\alpha\leqq\varepsilon_0)$ are defined inductively as follows;

$$F_0(x) = x+1,$$
$$F_1(x) = (x+1)^2,$$
$$F_{\beta+1}(x) = F_\beta^{x+1}(x) \qquad \text{if } \beta>0,$$
$$F_\sigma(x) = F_{\{\sigma\}(x)}(x) \qquad \text{if } \sigma \text{ is a limit ordinal.}$$

For each $\alpha\leqq\varepsilon_0$, let $\mathcal{F}_\alpha$ be the smallest class of functions containing F_α, the zero function, addition and projection functions, which is closed under substitution and limited primitive recursion. Suppose that f is any unary function and g is any n-ary function. Then, we say that g *is dominated by* f if there exists a natural number k such that

$$g(x_1, \cdots, x_n) < f(\max\{x_1, \cdots, x_n\})$$

whenever $k \leqq \max\{x_1, \cdots, x_n\}$. The following propositions are proved by Wainer (For the details of Wainer's hierarchy, see [**16**], [**17**].)

PROPOSITION 2.1. *For each* $\alpha \leqq \varepsilon_0$,
1) F_α *is strictly increasing,*
2) *if* $\beta < \alpha$ *then* F_β *is dominated by* F_α,
3) *If* $\beta < \alpha$ *then* F_β *is elementary recursive in* F_α *(in the Csillag-Kalmar sense).*

PROPOSITION 2.2. *For each* $\alpha \leqq \varepsilon_0$,
1) *if* $\beta < \alpha$ *then every function in* $\mathcal{F}_\beta$ *is dominated by* F_α,
2) *if* $\beta < \alpha$ *then* $\mathcal{F}_\beta \subsetneqq \mathcal{F}_\alpha$,
3) $\mathcal{F}_\alpha$ *is equal to the class of functions elementary recursive in* F_α.

As for the elementary recursive functions, see Chapter 2 of Monk [**7**]. We have also the following, by using Proposition 2.17 of [**7**].

PROPOSITION 2.3. *Each* $\mathcal{F}_\alpha$ *is closed under bounded minimalization. More precisely, let* $f(\boldsymbol{x}, z)$ *be a function defined by the condition that* $f(\boldsymbol{x}, z)$ *is equal to the smallest* y *such that* $y < z$ *and* $g(\boldsymbol{x}, y) = 0$ *if there exists such a* y, *and is equal to* 0 *otherwise. Then, if* $g(\boldsymbol{x}, y)$ *belongs to* $\mathcal{F}_\alpha$ *then so does* $f(\boldsymbol{x}, z)$.

The following result by Wainer [**17**] shows a relation between ordinal recursive functions and Wainer's hierarchy.

PROPOSITION 2.4. *For each ordinal* α *such that* $0 < \alpha < \varepsilon_0$,

$$U(\omega^\alpha) = \bigcup_{\beta < \alpha \cdot \omega} \mathcal{F}_\beta .$$

In particular, if $n \geqq 1$ *then*

$$\bigcup_{m < \omega} U(\omega_n(m)) = \bigcup_{\beta < \omega_n} \mathcal{F}_\beta .$$

It can be shown that the class $\bigcup_{m<\omega} U(\omega_1(m))$, or equivalently the class $\bigcup_{\beta<\omega} \mathcal{F}_\beta$, is the class of all primitive recursive functions. In [**3**], Ketonen and Solovay have introduced a sequence $\{G_\alpha\}_{\alpha \leqq \varepsilon_0}$ of functions similarly as Wainer's $\{F_\alpha\}_{\alpha \leqq \varepsilon_0}$. (In [**3**], G_α is written as F_α. To avoid the confusion of them and Wainer's F_α, we will write the former as G_α's throughout this paper.) The functions G_α ($\alpha \leqq \varepsilon_0$) are defined inductively as follows;

$$G_0(x) = x+1 ,$$
$$G_{\beta+1}(x) = G_\beta^{x+1}(x) \qquad \text{if } \beta \geqq 0 ,$$

$$G_\sigma(x) = G_{\{\sigma\}(x)}(x) \qquad \text{if } \sigma \text{ is a limit ordinal.}$$

Then, we can show easily the following lemma, by using the transfinite induction on α.

LEMMA 2.5. *For each* $\alpha<\varepsilon_0$ *and each* $x<\omega$,

$$G_\alpha(x) \leqq F_\alpha(x) \leqq G_{\alpha+1}(x).$$

§3. Provably recursive functions in fragments PA_k and PA_k^*.

In this section, we will introduce some fragments of Peano arithmetic and will study provably recursive functions in them. Our formal system PA of Peano arithmetic is similar to the system PA in Takeuti [**15**]. The language of our PA contains also function symbols for primitive recursive functions (see Proposition 10.6 of [**15**]). Our system PA is obtained from Gentzen's sequential calculus LK, by adding 1) the axioms for equality, the axioms for successor and defining equations for each primitive recursive function, as its mathematical initial sequents and 2) a rule of inference called "*ind*" which represents the mathematical induction.

In principle, we will distinguish between informal objects and their formal expressions, although in some cases we will use the same symbols for them only for the brevity's sake. For example, $\bar{f}$ denotes the function symbol for a primitive recursive function f and $\bar{k}$ denotes the numeral for a natural number k, and so on. Sometimes, we will introduce some predicate symbols for some primitive recursive relations and add their defining axioms as initial sequents. For example, if $R(\boldsymbol{x})$ is a primitive recursive relation and $f(\boldsymbol{x})$ is the characteristic function of $R(\boldsymbol{x})$ then we introduce the predicate symbol $\bar{R}$ and add the defining axiom $\rightarrow \bar{R}(\boldsymbol{x}) \equiv \bar{f}(\boldsymbol{x})=0$.

We will suppose that our language contains also the inequality $<$ as a predicate symbol. Moreover, we will make use of the following primitive recursive functions and relations. (We will use the same symbols for the formal symbols corresponding to them.) Let J be the function defined by $J(x, y)=[((x+y)^2+3x+y)/2]$. Then J is a bijection from $\boldsymbol{N}\times\boldsymbol{N}$ to $\boldsymbol{N}$. We can define two projection functions K and L related to J, satisfying that

1) $$J(K(z), L(z)) = z,$$

2) $$K(J(x, y)) = x \quad \text{and} \quad L(J(x, y)) = y.$$

(As for the detail of these functions, see Chapter 3 of Davis [**1**].) Next, for each $n\geqq 1$, T_n denotes so-called Kleene's ($n+2$-ary) T-predicate and U denotes the unary function associated with T-predicates (see [**5**].) By Kleene's normal form theorem, if e is a Gödel number of an n-ary recursive function $f(x_1, \cdots, x_n)$

then

$$f(m_1, \cdots, m_n) = U(\mu y T_n(e, m_1, \cdots, m_n, y))$$

for every natural number $m_1, \cdots, m_n$. Sometimes, we will omit the subscript n of T_n.

Next, we will define Π_m-formulas and Σ_m-formulas for each $m \geqq 0$. Any quantifier-free formula is both a Π_0-formula and a Σ_0-formula. A formula A is a Π_{n+1}-formula if it is of the form $\forall x_1 \cdots \forall x_k B$ with a Σ_n-formula B, and A is a Σ_{n+1}-formula if it is of the form $\exists x_1 \cdots \exists x_k C$ with a Π_n-formula C.

The formal system PA* is obtained from PA by adding all sequents of the form $\rightarrow C$, where C is any true Π_1-formula, as its new initial sequents. For each $k \geqq 0$, the formal system PA_k (or PA_k^*) is obtained from PA (or PA*) by restricting the induction formulas of the mathematical induction "ind" to formulas containing at most k quantifiers.

We can show that any *bounded formula*, i.e., a formula containing only bounded quantifiers, is equivalent to a quantifier-free formula in PA_1 by introducing predicate symbols for appropriate primitive recursive relations and that for each $n \geqq 1$ if A is a Σ_n-formula (or a Π_n-formula) then $\forall x(x<y \supset A)$ (or $\exists x (x<y \wedge A)$) is equivalent to a Σ_n-formula (or a Π_n-formula, respectively) in PA_n (see [**11**] and [**13**]). From this it follows that the provability in PA_k or PA_k^* $(k>0)$ is unchanged, even if we replace the restriction of induction formulas by Π_k-formulas (or, Σ_k-formulas, or formulas with at most k nested unbounded quantifiers).

An n-ary recursive function f is said to be *provably recursive* in PA_k (PA_k^*, PA and PA*) if there exists a Gödel number e of f such that

$$\forall x_1 \cdots \forall x_n \exists y T_n(\bar{e}, x_1, \cdots, x_n, y)$$

is provable in PA_k (PA_k^*, PA and PA*, respectively), where T_n is the Π_0-formula representing Kleene's T-predicate. (To specify our definition, we will take the Gödel numbering introduced in [**1**]. So, precisely speaking, e means the Gödel number of a Turing machine which computes the function f.)

In the following, we will try to characterize the class of provably recursive functions in PA_k and PA_k^*. A formula A is called a *Δ_1-formula* in PA_m (or PA_m^*), if there exist a Σ_1-formula B and a Π_1-formula C, each of which is equivalent to A in PA_m (or PA_m^*). Let $R(x_1, \cdots, x_n, y)$ be a formula with $n+1$ free variables $x_1, \cdots, x_n, y$, such that

$$\forall x_1 \cdots \forall x_n \exists y R(x_1, \cdots, x_n, y)$$

is true. Then, we can define an n-ary function $f(x_1, \cdots, x_n)$ by the condition that

$f(m_1, \cdots, m_n)$ is the least natural number k such that
$R(\bar{m}_1, \cdots, \bar{m}_n, \bar{k})$ is true.

Such a function $f(x_1, \cdots, x_n)$ will be denoted by $\mu yR(x_1, \cdots, x_n, y)$.

Now, we will state our main theorem. The rest of this section will be devoted to showing this theorem.

THEOREM 3.1. *Let* $n \geqq 1$. *Then, the following four conditions are equivalent;*
1) f *is provably recursive in* PA_n,
2) *there exists a bounded formula* $R(\boldsymbol{x}, y)$ *and a primitive recursive function* $g(\boldsymbol{x}, z)$ *such that* $f(\boldsymbol{x}) = g(\boldsymbol{x}, \mu yR(\boldsymbol{x}, y))$ *and* $\forall \boldsymbol{x} \exists yR(\boldsymbol{x}, y)$ *is provable in* PA_n,
3) *there exists a* Δ_1*-formula* $S(\boldsymbol{x}, y)$ *in* PA_n *such that* $f(\boldsymbol{x}) = \mu yS(\boldsymbol{x}, y)$ *and* $\forall \boldsymbol{x} \exists ! yS(\boldsymbol{x}, y)$ *is provable in* PA_n,
4) f *is* $\omega_n(m)$*-ordinal recursive for some* $m < \omega$.

Moreover, the equivalence of these conditions still holds if we replace PA_n *by* PA_n^*.

In the above theorem, $\exists ! zB(z)$ is the abbreviation of the formula

$$\exists zB(z) \wedge \forall u \forall v (B(u) \wedge B(v) \supset u = v).$$

Clearly, the equivalence of 1) and 4) of Theorem 3.1 gives a refinement of the result on the relation between provably recursive functions in Peano arithmetic and ordinal recursive functions, shown by Kreisel [**6**] and Kino [**4**]. In the following, i′) will denote the condition obtained from the condition i) in Theorem 3.1 by replacing PA_n by PA_n^*, for $i = 1, 2, 3$. Clearly, 1) implies 1′), 1) implies 2) and 1′) implies 2′) by the definition.

LEMMA 3.2. *Let* $n \geqq 1$. *Suppose that* $R(\boldsymbol{x}, y)$ *is a bounded formula such that* $\forall \boldsymbol{x} \exists yR(\boldsymbol{x}, y)$ *is provable in* PA_n *and that* $f(\boldsymbol{x}) = g(\boldsymbol{x}, \mu yR(\boldsymbol{x}, y))$ *for a primitive recursive function* g. *Then, there exists a* Δ_1*-formula* $S(\boldsymbol{x}, z)$ *in* PA_n *such that* $f(\boldsymbol{x}) = \mu zS(\boldsymbol{x}, z)$ *and* $\forall \boldsymbol{x} \exists ! zS(\boldsymbol{x}, z)$ *is provable in* PA_n. *Thus,* 2) *in Theorem* 3.1 *implies* 3). *Similarly,* 2′) *implies* 3′).

PROOF. We will define formulas $S(\boldsymbol{x}, z)$ and $S'(\boldsymbol{x}, y)$ by

$$S(\boldsymbol{x}, z) \equiv \exists y (R(\boldsymbol{x}, y) \wedge \forall u < y \neg R(\boldsymbol{x}, u) \wedge z = g(\boldsymbol{x}, y))$$

and

$$S'(\boldsymbol{x}, z) \equiv \forall y ((R(\boldsymbol{x}, y) \wedge \forall u < y \neg R(\boldsymbol{x}, u)) \supset z = g(\boldsymbol{x}, y)).$$

Clearly, $S(\boldsymbol{x}, z)$ is equivalent to a Σ_1-formula and $S'(\boldsymbol{x}, z)$ is equivalent to a Π_1-formula. Moreover, $S(\boldsymbol{x}, z) \equiv S'(\boldsymbol{x}, z)$ is provable in PA_n, by using the assumption that $\forall \boldsymbol{x} \exists yR(\boldsymbol{x}, y)$ is provable in PA_n and the least number principle for bounded formulas, which is provable in PA_n (see [**11**]). Therefore, $S(\boldsymbol{x}, z)$ is a Δ_1-formula in PA_n. It can be easily shown that $f(\boldsymbol{x}) = \mu zS(\boldsymbol{x}, z)$ and $\forall \boldsymbol{x} \exists ! zS(\boldsymbol{x}, z)$ is provable in PA_n. Thus, we have our lemma.

It is obvious that 3) implies 3′). So, we will show next that 3′) implies 4).

LEMMA 3.3. *Let* $n \geqq 1$. *Suppose that* $R(\boldsymbol{x}, y)$ *is a* Π_0*-formula and* $A_1, \cdots, A_t$ *be true* Π_1*-formulas such that the sequent*

$$A_1, \cdots, A_t \rightarrow \forall \boldsymbol{x} \exists y R(\boldsymbol{x}, y)$$

is provable in PA_n. *Then, the function defined by* $f(\boldsymbol{x}) = \mu y R(\boldsymbol{x}, y)$ *is* $\omega_n(m)$*-ordinal recursive for some* $m < \omega$.

PROOF. By using functions K, L and J, we can assume that the sequence $\boldsymbol{x}$ of variables consists of only one variable x and that the conjunction of $A_1, \cdots, A_t$ is equivalent to a formula of the form $\forall y B(y)$ with a Π_0-formula $B(y)$. By the assumption,

$$\forall y B(y) \supset \forall x \exists y R(x, y) \tag{1}$$

is provable in PA_n, and hence

$$\exists y (\neg B(y) \vee R(x, y)) \tag{2}$$

is also provable in it. Now, by Remark 12.14 of Takeuti [**15**] or results in §4 of Shirai [**14**], we can assume that the proof of (2) in PA_n consists only of prenex formulas with at most n quantifiers. More precisely, we can obtain a proof P of (2) in the formal system PA_n in the sense of [**15**]. Then, as in p. 112 of [**15**], we can assign an ordinal less than $\omega_n(m)$ for some m to the proof P. Hence the function g defined by

$$g(x) = \mu y (\neg B(y) \vee R(x, y)) \tag{3}$$

is $\omega_n(m)$-ordinal recursive by Corollary 12.16 of [**15**]. On the other hand, since $\forall y B(y)$ is true, it can be shown that $f(x) = g(x)$. Hence, we have our lemma.

A similar result as the above lemma is shown in [**12**] without the proof. We can derive the following easily from Lemma 3.3.

COROLLARY 3.4. *Let* $n \geqq 1$. *Suppose that* $R(\boldsymbol{x}, y)$ *is a* Σ_1*-formula such that* $\forall \boldsymbol{x} \exists ! y R(\boldsymbol{x}, y)$ *is provable in* PA_n^*. *Then the function* f *defined by* $f(\boldsymbol{x}) = \mu y R(\boldsymbol{x}, y)$ *is* $\omega_n(m)$*-ordinal recursive for some* $m < \omega$. *Thus, the condition* 3′) *implies* 4).

Now, it remains to show that 4) in Theorem 3.1 implies 1). We will take the canonical, primitive recursive well-ordering $\prec$ on natural numbers which is of order-type ε_0. For each natural number x, define $\mathrm{ord}(x)$ to be the ordinal represented by x in the above ordering $\prec$ and for each ordinal $\alpha < \varepsilon_0$, define $\mathrm{num}(\alpha)$ to be the natural number x such that $\mathrm{ord}(x) = \alpha$. We will introduce a relation $\mathrm{Lim}(x)$ by the condition that

$$\mathrm{Lim}(x) \quad \text{if and only if} \quad \mathrm{ord}(x) \text{ is a limit number.}$$

We introduce also two functions fs and pr by

$$\mathrm{fs}(x,y)=\begin{cases}\mathrm{num}(\{\mathrm{ord}(x)\}(y)) & \text{if } \mathrm{ord}(x) \text{ is a limit number,}\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathrm{pr}(x)=\begin{cases}\mathrm{num}(\beta) & \text{if } \mathrm{ord}(x)=\beta+1,\\ 0 & \text{otherwise.}\end{cases}$$

We can assume that $\mathrm{Lim}(x)$ is a primitive recursive relation and both fs and pr are primitive recursive functions.

By using the ordering $\prec$, we can define another ordering $\prec^*$ on natural numbers by the condition that

$$u\prec^* v \text{ if and only if either } L(u)\prec L(v) \text{ or } (L(u)=L(v) \text{ and } K(u)<K(v)),$$

where $<$ is the usual order relation on the set of natural numbers. It is easy to see that $\prec^*$ is also a primitive recursive well-ordering of order-type ε_0. We assume that our language contains these symbols as $\prec$, $\prec^*$, Lim, fs and pr. (For the sake of brevity, we will use the same symbols for both informal objects and their formal expressions.) As usual, $x\preccurlyeq y$ is an abbreviation of the formula $x\prec y\vee x=y$.

In [**14**], Shirai obtained the provability and the unprovability results of transfinite induction in fragments of Peano arithmetic, by examining into the Gentzen's proof [**2**] in detail. For our present purpose, we will refer to his results in the following specialized form.

PROPOSITION 3.5. *Let* $\alpha<\omega_n$ *for* $n\geqq 2$. *Then,*

1) $$\forall y[\forall x(x\prec^* y\supset \varepsilon(x))\supset\varepsilon(y)] \quad\supset\quad \forall u\forall v(v\preccurlyeq\overline{\mathrm{num}(\alpha)}\supset\varepsilon(J(u,v)))$$

is provable in PA_{n-1}, *where* $\varepsilon(z)$ *is a new predicate symbol,*

2) *in particular, if* $A(z)$ *is a* Π_2*-formula then*

$$\forall y[\forall x(x\prec^* y\supset A(x))\supset A(y)] \quad\supset\quad \forall u\forall v(v\preccurlyeq\overline{\mathrm{num}(\alpha)}\supset A(J(u,v)))$$

is provable in PA_n.

Notice here that the set $\{x\ ;\ L(x)\preccurlyeq\mathrm{num}(\alpha)\}$ is an initial segment of the well-ordering $\prec^*$, which is of order-type $\omega\cdot\alpha$ $(<\omega_n)$, when $n\geqq 2$ and $\alpha<\omega_n$. We can show easily the following lemma (cf. the proof of §3 of [**4**]).

LEMMA 3.6. *For* $n\geqq 1$, *the class of all provably recursive functions in* PA_n *contains the zero function, addition and projection functions and is closed under substitution and primitive recursion.*

The following lemma says that Kleene's iteration theorem can be proved in PA_1.

LEMMA 3.7. *For each natural number* m, *there exists a primitive recursive function* s^m *such that*

$$\forall x_1 \cdots \forall x_n \exists y T_{m+n}(\bar{c}, \bar{k}_1, \cdots, \bar{k}_m, x_1, \cdots, x_n, y)$$
$$\equiv \forall x_1 \cdots \forall x_n \exists y T_n(\overline{s^m(c, k_1, \cdots, k_m)}, x_1, \cdots, x_n, y)$$

is provable in PA_1 *for every natural number* $c, k_1, \cdots, k_m$.

PROOF. Firstly, we will check that the proof of Kleene's iteration theorem in Chapter 9 of Davis [**1**], for example, can be carried out in the primitive recursive way. Here, we assume the familiarity with notations and the terminology of [**1**]. For the sake of brevity, suppose that $m=n=1$. As stated in the proof of Theorem 1.1 in Chapter 9 of [**1**], when a natural number k is given, we can construct a Turing machine Z_k from a given Turing machine Z such that $\Psi_{Z_k}^{(1)}(x)=\Psi_Z^{(2)}(k, x)$, in a uniform and primitive recursive way. Now, define s^1 to be the primitive recursive function which computes the Gödel number $s^1(c, k)$ of Z_k from the Gödel number c of Z.

Suppose that a sequence $\alpha_1, \cdots, \alpha_p$ is a *computation* of Z with $\alpha_1=q_1(\overline{k, x})$, where $(\overline{k, x})$ denotes the tape expression $\bar{k}B\bar{x}$ (see Definition 2.2 of Chapter 1 in [**1**]). Define $\tilde{\alpha}_i$ to be the *instantaneous description* obtained from α_i by replacing each occurrence of q_j by q_{j+k+2}. Moreover, let $\beta_1, \cdots, \beta_s$ be the sequence of instantaneous descriptions such that $\beta_1=q_1\bar{x}$, $\beta_i \rightarrow \beta_{i+1}$ (Z_k) for $1 \leqq i < s$ and $\beta_s= q_{k+3}(\overline{k, x})$, where $s=2k+4$. (See the proof of Theorem 1.1 of Chapter 9 in [**1**].) Then, $\beta_1, \cdots, \beta_{s-1}, \tilde{\alpha}_1 (=\beta_s), \tilde{\alpha}_2, \cdots, \tilde{\alpha}_p$ becomes a computation of Z_k such that $\langle \tilde{\alpha}_p \rangle = \langle \alpha_p \rangle$. (Here, $\langle \alpha \rangle$ denotes the number of occurrences of 1 in α.)

Conversely, suppose that a sequence $\gamma_1, \cdots \gamma_r$ is a computation of Z_k with $\gamma_1=q_1\bar{x}$. Then, $\gamma_i=\beta_i$ for $1 \leqq i \leqq 2k+4$, $r=2k+3+p$ and γ_{2k+3+j} is of the form $\tilde{\alpha}_j$ for $1 \leqq j \leqq p$ such that $\alpha_1, \cdots, \alpha_p$ is a computation of Z with $\alpha_1=q_1(\overline{k, x})$ and that $\langle \alpha_p \rangle = \langle \gamma_r \rangle$. We remark that the computation $\beta_1, \cdots, \beta_{s-1}, \tilde{\alpha}_1, \cdots, \tilde{\alpha}_p$ is obtained from the computation $\alpha_1, \cdots, \alpha_p$ in a primitive recursive way, and vice versa.

It should be remarked here that the primitive recursive arithmetic can be developed in the fragment PA_1 (see the definition of our formal systems, in the first part of this section). Thus, by arithmetizing the above facts, we have that

(1) $$T_2(\bar{c}, \bar{k}, x, y) \supset T_1(\overline{s^1(c, k)}, x, u*\tilde{y})$$

and

(2) $$T_1(\overline{s^1(c, k)}, x, z) \supset \exists y(z=u*\tilde{y} \wedge T_2(\bar{c}, \bar{k}, x, y))$$

are both provable in PA_1, where u is the Gödel number of the sequence $\beta_1, \cdots, \beta_{s-1}$, y and $\tilde{y}$ are Gödel numbers of $\alpha_1, \cdots, \alpha_p$ and $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_p$, respectively and $u*\tilde{y}$ gives the Gödel number of the sequence $\beta_1, \cdots, \beta_{s-1}, \tilde{\alpha}_1 \cdots, \tilde{\alpha}_p$.

(As for the details of the arithmetization, see Chapter 4 §1 of [1].) From (1) and (2), it follows that

$$\forall x \exists y T_2(\bar{c}, \bar{k}, x, y) \equiv \forall x \exists y T_1(\overline{s^1(c, k)}, x, y)$$

is provable in PA_1.

Next, we will introduce a ternary function h by $h(u, v, x) = F^{u+1}_{\mathrm{ord}(v)}(x)$, where F_α's are functions defined by Wainer (see §2). Clearly, h is a recursive function. Let e be a Gödel number of h. (Imagine such a Turing machine Z that Z computes the function h following the inductive definition of F_α's, and take the Gödel number of such a Z for e.) In the following, we will write s^2 simply as s. By Lemma 3.7, we have that

$$\forall x \exists y T_3(\bar{e}, \bar{m}, \bar{n}, x, y) \equiv \forall x \exists y T_1(\overline{s(e, m, n)}, x, y)$$

is provable in PA_1 for every natural number m, n. Clearly, $s(e, m, n)$ is a Gödel number of the function $h_{m,n}(x)$ $(=h(m, n, x) = F^{m+1}_{\mathrm{ord}(n)}(x))$.

Now, we will show the following lemma.

LEMMA 3.8. *Let* $\alpha < \omega_n$ *for* $n \geqq 2$. *Then*

$$v \preccurlyeq \mathrm{num}(\alpha) \supset \forall x \exists y T(\bar{e}, u, v, x, y)$$

is provable in PA_n.

PROOF. Let $W(z)$ be the Π_2-formula $\forall x \exists y T(\bar{e}, K(z), L(z), x, y)$. We will first show that the formula

$$\forall u (u \prec^* v \supset W(u)) \supset W(v) \tag{1}$$

is provable in PA_1. Suppose first that $K(v)=0$. If $L(v)$ is either 0 or equal to num(1) then $W(v)$ is provable in PA_1 and hence (1) is also provable in it. Next, we assume that

$$K(v)=0 \wedge \overline{\mathrm{num}(1)} \prec L(v) \wedge \neg \mathrm{Lim}(L(v)). \tag{2}$$

Then, $J(x, \mathrm{pr}(L(v))) \prec^* v$ is provable in PA_1. Therefore,

$$\forall u (u \prec^* v \supset W(u)) \rightarrow W(J(x, \mathrm{pr}(L(v))))$$

is provable in PA_1. On the other hand,

$$\exists y T(\bar{e}, x, \mathrm{pr}(L(v)), x, y) \rightarrow \exists y T(\bar{e}, K(v), L(v), x, y)$$

i. e.,

$$W(J(x, \mathrm{pr}(L(v)))) \rightarrow W(v)$$

is also provable in PA_1, since we can effectively construct *the computation for the input* $(K(v), L(v), x)$ from *the computation for the input* $(x, \mathrm{pr}(L(v)), x)$. Hence, (1) is provable in PA_1 under the assumption (2). Similarly, we can show that (1) is provable in PA_1 under the assumption that $K(v)=0 \wedge \mathrm{Lim}(L(v))$ or $K(v)>0$. Combining these facts, we can deduce that (1) is provable in PA_1. Now, taking $W(z)$ for $A(z)$ in Proposition 3.5 2), we obtain that

$$v \preccurlyeq \overline{\mathrm{num}(\alpha)} \supset \forall x \exists y T(\bar{e}, u, v, x, y)$$

is provable in PA_n.

LEMMA 3.9. 1) *For* $n \geqq 1$, *if* $\alpha<\omega_n$ *then* F_α *is provably recursive in* PA_n.
2) *For* $n \geqq 1$, *if a function* f *is* $\omega_n(m)$*-ordinal recursive for some* $m<\omega$ *then* f *is provably recursive in* PA_n.

PROOF. 1) If $n=1$ then F_α is primitive recursive. Hence it is provably recursive in PA_1 (see Lemma 3.6). Suppose that $n>1$. Then by Lemmas 3.7 and 3.8, $\forall x \exists y T_1(s(\bar{e}, 0, \overline{\mathrm{num}(\alpha)}), x, y)$ is provable in PA_n and $s(e, 0, \mathrm{num}(\alpha))$ is a Gödel number of F_α (since $\mathrm{ord}(\mathrm{num}(\alpha))=\alpha$). Thus, F_α is provably recursive in PA_n.
2) From 1) and Lemma 3.6, it follows that every function in the class $\mathcal{F}_\alpha$ is provably recursive in PA_n, if $\alpha<\omega_n$. On the other hand, $\bigcup_{m<\omega} U(\omega_n(m))= \bigcup_{\beta<\omega_n} \mathcal{F}_\beta$ by Proposition 2.4, so we can derive that every $\omega_n(m)$-ordinal recursive function is provably recursive in PA_n.

Thus, we have completed our proof of Theorem 3.1. The following corollary will be often used in the next section (cf. Corollary 9 in [8]).

COROLLARY 3.10. *Suppose that* $R(\boldsymbol{x}, y)$ *is a* Π_0*-formula such that* $\forall \boldsymbol{x} \exists y R(\boldsymbol{x}, y)$ *is true and* f *is a function satisfying* $f(\boldsymbol{x})=\mu y R(\boldsymbol{x}, y)$. *For* $n \geqq 1$, f *is* $\omega_n(m)$*-ordinal recursive for some* $m<\omega$ *if and only if* $\forall \boldsymbol{x} \exists y R(\boldsymbol{x}, y)$ *is provable in* PA_n^*.

PROOF. The if-part can be easily derived from Theorem 3.1. Suppose that f is $\omega_n(m)$-ordinal recursive. By Theorem 3.1, for some Gödel number e of f $\exists y T(\bar{e}, \boldsymbol{x}, y)$ is provable in PA_n. Since $f(\boldsymbol{x})=\mu y R(\boldsymbol{x}, y)$, $\forall \boldsymbol{x} \forall z(T(\bar{e}, \boldsymbol{x}, z) \supset R(\boldsymbol{x}, U(z)))$ is a true Π_1-formula and hence it is provable in PA_0^*. Thus, $\exists y T(\bar{e}, \boldsymbol{x}, y) \supset \exists y R(\boldsymbol{x}, y)$ is also provable in PA_0^*. Hence, $\forall \boldsymbol{x} \exists y R(\boldsymbol{x}, y)$ is provable in PA_n^*.

We notice here that PA_n^* can not be replaced by PA_n in the above corollary.

§4. Undecidable combinatorial statements in fragments of Peano arithmetic.

Using Theorem 3.1, we will analyze the combinatorial statements, which are shown to be unprovable in Peano arithmetic by Paris and Harrington [**10**]. Then, we will give an alternative proof of a result by Paris in [**9**] (Theorems 4.5 and 4.7). While Paris used a model-theoretic method, our method is of a purely proof-theoretic character. Moreover, we will point out that the provability and the unprovability of statements treated in it depend on how to express them in formal systems. We owe our proof much to the close examination of rapidly growing functions related to that combinatorial statement, due to Ketonen and Solovay [**3**].

We will give some definitions. For a set A of natural numbers and a natural number n, define $A^{[n]}$ to be the set of all subsets of A of cardinality n. Let f be a function from $A^{[n]}$ to a set X. Then, a subset H of A is *homogeneous* for f if f is constant on $H^{[n]}$. A set H of natural numbers is *large*, if H is non-empty and H has at least s elements where s is the smallest element of H. For any natural numbers k, m, $[k, m]$ means the set of natural numbers $\{x\,;\,k\leqq x\leqq m\}$. For any natural numbers c, k, m, n, the expression

$$[k, m] \underset{*}{\rightarrow} (n+1)_c^n$$

means that for every function f from $[k, m]^{[n]}$ to the set $\{0, 1, \cdots, c-1\}$, there exists a large, homogeneous set H of cardinality at least $n+1$. We remark that $[k, m]\underset{*}{\rightarrow}(n+1)_c^n$ is a primitive recursive relation with respect to c, k, m, n. Moreover, the following can be shown by using infinite Ramsey theorem.

PROPOSITION 4.1. *For each natural number c, k, n, there is a natural number m such that $[k, m]\underset{*}{\rightarrow}(n+1)_c^n$ holds.*

By the above remark and proposition, we can define a recursive function $\sigma_{n,c}$ by

$$\sigma_{n,c}(k) = \mu y([k, y]\underset{*}{\rightarrow}(n+1)_c^n).$$

By the definition, the following lemma can be easily shown.

LEMMA 4.2. *If $c\leqq c'$ and $k\leqq k'$ then $\sigma_{n,c}(k)\leqq\sigma_{n,c'}(k')$.*

In [**3**], Ketonen and Solovay obtained a sharp estimation of functions $\sigma_{n,c}$ and using it, they gave an alternative proof of Paris-Harrington's theorem which says that

$$\forall w\forall x\forall z\exists y([x, y]\underset{*}{\rightarrow}(w+1)_z^w) \tag{1}$$

is not provable in Peano arithmetic. On the other hand, it is pointed out that

$$\forall x\forall z\exists y([x, y]\underset{*}{\rightarrow}(n+1)_z^n) \tag{2}$$

is provable in Peano arithmetic for each natural number n (see [8] and [**10**]). To clarify this situation, we will investigate the provability of the formula (2) in fragments of Peano arithmetic, by utilizing results in [**3**]. The next two propositions proved in [**3**] are essential in the following discussion. Recall here that G_α's are functions introduced in § 2.

PROPOSITION 4.3 *Let $n\geqq 2$, $c\geqq 2$ and $k\geqq 4$. Then,*

$$\sigma_{n,c}(k) \leqq G_{\omega_{n-2}(c+5)}(k).$$

PROPOSITION 4.4. *Let $n \geqq 2$. For any weakly monotone increasing function f, f is dominated by $\sigma_{n,c}$ for some c if and only if f is dominated by G_α for some $\alpha < \omega_{n-1}$.*

We will call the relation $[x, y] \rightarrow_* (w+1)^w_z$ (with respect to x, y, z, w), *the Ramsey relation*, and the relation $\sigma_{w,z}(x) = y$ (with respect to x, y, z, w), *the strong Ramsey relation*. In the following, we will give an alternative proof of a result by Paris [**9**] on the provability and the unprovability of some combinatorial statements in fragments of PA (or PA*).

Before doing so, we must pay attention to the fact that there will be many ways of expressing the (strong) Ramsey relation by formulas. For a fixed n, we say that a formula $P(x, z, y)$ *represents the Ramsey relation*, when for every c, k, m, $P(\bar{k}, \bar{c}, \bar{m})$ is true if and only if $[k, m] \rightarrow_* (n+1)^n_c$. Similarly, $Q(x, z, y)$ *represents the strong Ramsey relation*, when for every c, k, m, $Q(\bar{k}, \bar{c}, \bar{m})$ is true if and only if $\sigma_{n,c}(k) = m$. Since both the Ramsey and the strong Ramsey relations are primitive recursive, there exist quantifier-free formulas which represent them. Suppose that a formula $P(x, z, y)$ represents the Ramsey relation. Let us define $Q(x, z, y)$ by

$$Q(x, z, y) \equiv P(x, z, y) \wedge \forall y'(y' < y \supset \neg P(x, z, y')).$$

Then, $Q(x, z, y)$ represents the strong Ramsey relation. Moreover, if $P(x, z, y)$ is quantifier-free then so does $Q(x, z, y)$, and in this case $\exists y P(x, z, y) \supset \exists ! y Q(x, z, y)$ is provable in PA_1, by [**11**]. Now, we will prove the following.

THEOREM 4.5. *For $n \geqq 2$ and any Σ_0-representation of the Ramsey relation, $\forall x \forall z \exists y([x, y] \rightarrow_* (n+1)^n_z)$ is provable in PA^*_n, but not in PA^*_{n-1}. More precisely, if $P(x, z, y)$ is a bounded formula which represents the Ramsey relation $[x, y] \rightarrow_* (n+1)^n_z$, then the formula $\forall x \forall z \exists y P(x, z, y)$ is provable in PA^*_n, but not in PA^*_{n-1}.*

PROOF. First, we will show that $\forall x \forall z \exists y P(x, z, y)$ is provable in PA^*_n. We define a unary function δ_n by

$$(1) \qquad \delta_n(x) = \sigma_{n, L(x)}(K(x)).$$

By Lemma 4.2, Proposition 4.3 and Lemma 2.5, for each x

$$(2) \qquad \delta_n(x) = \sigma_{n, L(x)}(K(x)) \leqq \sigma_{n,x}(x) \leqq \sigma_{n,x+2}(x+4) \leqq G_{\omega_{n-2}(x+7)}(x+4)$$
$$\leqq G_{\omega_{n-2}(x+7)}(x+7) \leqq G_{\omega_{n-1}}(x+7) \leqq F_{\omega_{n-1}}(x+7),$$

since $K(x) \leqq x$ and $L(x) \leqq x$ hold. Since $F_{\omega_{n-1}}(x+7)$ is obtained from $F_{\omega_{n-1}}$ and a primitive recursive function $x+7$ by the substitution, it belongs to $\mathcal{F}_{\omega_{n-1}}$ Next define a formula P^* by

(3) $$P^*(w, y) \equiv P(K(w), L(w), y).$$

As remarked in § 3, there exists a function symbol $\bar{f}$, which represents a primitive recursive function f, such that

(4) $$\bar{f}(w, y) = 0 \equiv P^*(w, y)$$

is provable in PA_1. Clearly, $\delta_n(w)=\mu y(f(w, y)=0)$ holds. By (2), it holds that

(5) $$\delta_n(w) = \mu y \leqq F_{\omega_{n-1}}(w+7) \ (f(w, y) = 0).$$

This means that δ_n can be obtained from $F_{\omega_{n-1}}(x+7)$ and a primitive recursive function $f(w, y)$ by using bounded minimalization and substitution. Hence, δ_n belongs also to $\mathcal{F}_{\omega_{n-1}}$, by using Proposition 2.3. So, $\forall w \exists y(\bar{f}(w, y)=0)$ is provable in PA_n^*, by Proposition 2.4 and Corollary 3.10. Using (3) and (4), we have that $\forall x \forall z \exists y P(x, z, y)$ is also provable in PA_n^*.

Next, suppose that $\forall x \forall z \exists y P(x, z, y)$ is provable in PA_{n-1}^*. Then $\forall u \exists y P(u, u, y)$ is also provable in PA_{n-1}^*. Since $n \geqq 2$, we can assume that $P(u, u, y)$ is a Π_0-formula. Let us define a function γ_n by $\gamma_n(u)=\mu y P(u, u, y)$, i. e. $\gamma_n(u)=\sigma_{n,u}(u)$. Then, γ_n is $\omega_{n-1}(m)$-ordinal recursive for some $m<\omega$ by Corollary 3.10. So, γ_n belongs to $\mathcal{F}_\beta$ for some $\beta<\omega_{n-1}$ by Proposition 2.4. By Proposition 2.2 1), γ_n is dominated by $F_{\beta+1}$, and therefore it is dominated by $G_{\beta+2}$ by Lemma 2.5. Thus, γ_n is dominated by $\sigma_{n,c}$ for some c by Proposition 4.4. Here we can assume that $c \geqq 2$, by Lemma 4.2. Hence, there exists a k such that for every $u \geqq k$

(6) $$\sigma_{n,u}(u) = \gamma_n(u) < \sigma_{n,c}(u).$$

Let d be $\max\{c+1, k\}$. Then, by (6)

(7) $$\sigma_{n,d}(d) < \sigma_{n,c}(d).$$

Thus, we are led to a contradiction, by (7) and Lemma 4.2. Therefore, $\forall x \forall z \exists y P(x, z, y)$ is not provable in PA_{n-1}^*.

We notice here that $\forall x \forall z \exists y([x, y] \overset{*}{\rightarrow} (n+1)_z^n)$ is not provable in PA_n for some Σ_0-representation of the Ramsey relation, contrary to Theorem 4.5. For, let $P(x, z, y)$ be any Σ_0-formula representing the Ramsey relation $[x, y] \overset{*}{\rightarrow} (n+1)_z^n$ and $\mathrm{Prov}_n(u, v)$ be a Σ_0-formula representing the provability predicate for PA_n in the canonical way. More precisely, $\mathrm{Prov}_n(\ulcorner P \urcorner, \ulcorner A \urcorner)$ means the provability of a formula A in PA_n with a proof P. Then,

$$P(x, z, y) \wedge \neg \mathrm{Prov}_n(x, \ulcorner 0=1 \urcorner)$$

is also a Σ_0-formula representing $[x, y] \overset{*}{\rightarrow} (n+1)_z^n$, since for each m $\neg \mathrm{Prov}_n$ $(\bar{m}, \ulcorner 0=1 \urcorner)$ is true. On the other hand, since

$$\forall x \forall z \exists y \, (P(x, z, y) \wedge \neg \mathrm{Prov}_n(x, \ulcorner 0=1 \urcorner))$$

implies the consistency of PA_n, it is not provable in PA_n. Apparently, this fact seems to conflict with a result in Paris [**9**], which says that $\forall x \forall z \exists y([x, y] \nrightarrow_* (n+1)^n_z)$ is provable in PA_n. But this is not the case. In fact, we can show Paris' result, if we formalize it in such a form as in Theorem 4.7 stated below.

We remark also that the second part of Theorem 4.5 can be extended as follows: *For* $n \geqq 2$, $\forall x \forall z \exists ! y(\sigma_{n,z}(x)=y)$ *is not probable in* PA^*_{n-1} *for any* Σ_1-*representation of the strong Ramsey relation.* This can be shown similarly as Theorem 4.5, by using Corollary 3.4 instead of Corollary 3.10 in the proof.

The following result follows immediately from Theorem 4.5 and the above remark (see Theorem 5 in [8]).

COROLLARY 4.6. 1) $\forall w \forall x \forall z \exists y([x, y] \nrightarrow_* (w+1)^w_z)$ *is not provable in* PA^* *for any* Σ_0-*representation of the Ramsey relation.*
2) $\forall w \forall x \forall z \exists ! y(\sigma_{w,z}(x)=y)$ *is not provable in* PA^* *for any* Σ_1-*representation of the strong Ramsey relation.*

On the other hand, it can be easily shown that there exists a Σ_2-formula $P^*(x, z, y, w)$ which represents the (strong) Ramsey relation in PA^*, for which $\forall w \forall x \forall z \exists y P^*(x, z, y, w)$ is provable in PA^*. The following theorem is in some sense stronger, but in another sense more restricted, than the previous theorem.

THEOREM 4.7. *For* $n \geqq 2$, $\forall x \forall z \exists ! y(\sigma_{n,z}(x)=y)$ *is provable in* PA_n, *but not provable in* PA^*_{n-1} *in the following sense: There exists a* Σ_1-*formula* $P(x, z, y, w)$ *such that for each* $n \geqq 2$,
1) $\sigma_{n,z}(x)=\mu y P(x, z, y, \bar{n})$,
2) $\forall x \forall z \exists ! y P(x, z, y, \bar{n})$ *is provable in* PA_n, *but not provable in* PA^*_{n-1}.

PROOF. Our theorem can be shown similarly as Theorem 4.5. First, we define a function g by $g(w)=\mathrm{num}(\omega_{w-1})$. Clearly, g is primitive recursive. Define a function σ^* by

$$\sigma^*(x, z, w) = \sigma_{w,z}(x).$$

Then, similarly as (5) in the proof of Theorem 4.5, we can obtain that

$$\sigma^*(x, z, w) = \mu y \leqq F_{\omega_{w-1}}(J(x, z)+7)\ (f^*(x, z, y, w) = 0),$$

where f^* denotes the characteristic function of the (primitive recursive) Ramsey relation $[x, y] \nrightarrow_* (w+1)^w_z$. Define a function j by

$$j(x, z, w, v) = \mu y \leqq U(v)\ (f^*(x, z, y, w) = 0).$$

Then, j is primitive recursive. Let us take a function $h(u, v, x)=F^{u+1}_{\mathrm{ord}(v)}(x)$ with a Gödel number e (see §3). Since

$$F_{\omega_{w-1}}(J(x, z)+7) = h(0, g(w), J(x, z)+7)$$
$$= U(\mu vT(e, 0, g(w), J(x, z)+7, v)),$$

$\sigma^*(x, z, w)=j(x, z, w, \mu vT(e, 0, g(w), J(x, z)+7, v))$. Now, we will define a Σ_1-formula $P(x, z, y, w)$ by

$$P(x, z, y, w) \equiv \exists v\, (T(\bar{e}, 0, g(w), J(x, z)+7, v) \wedge$$
$$\forall u<v \neg T(\bar{e}, 0, g(w), J(x, z)+7, u) \wedge j(x, z, w, v)=y).$$

Then, we can show that for $n\geqq 2$

(1) $\sigma^*(x, z, w)=\mu yP(x, z, y, w)$,

(2) $P(x, z, y, \bar{n})$ is a Δ_1-formula in PA_n,

(3) $\forall x\forall z\exists! yP(x, z, y, \bar{n})$ is provable in PA_n,

by using Lemma 3.8 (see also the proof of Lemma 3.2). By the remark just above Corollary 4.6, we have also that $\forall x\forall z\exists! yP(x, z, y, \bar{n})$ is not provable in PA^*_{n-1}. Thus, we have our theorem.

We will make another remark. For any natural number c, m, n, s, the expression $m \underset{*}{\rightarrow} (s)^n_c$ means that for every function f from $[0, m-1]^{[n]}$ to the set $\{0, 1, \cdots, c-1\}$, there exists a large, homogeneous set H of cardinality at least s. Then, by using Theorems 4.5 and 4.7, we can show that for $n\geqq 2$, $\forall u\forall z\exists y(y \underset{*}{\rightarrow} (u)^n_z)$ (and, $\forall u\forall z\exists! y(y=\mu v(v\underset{*}{\rightarrow}(u)^n_z))$) is provable in PA^*_n (and PA_n, respectively). In the proof, we use the fact that

$$([x, y] \underset{*}{\rightarrow} (w+1)^w_z) \supset ((y+1) \underset{*}{\rightarrow} (x)^w_z)$$

is provable in PA_0. On the other hand, we can show that neither $\forall w\forall u\forall z\exists y (y\underset{*}{\rightarrow}(u)^w_z)$ nor $\forall w\forall u\forall z\exists y(y=\mu v(v\underset{*}{\rightarrow}(u)^w_z))$ is provable in PA^*, by using Theorem 3.10 in [**3**]. This gives a proof-theoretic demonstration of both Main Theorem and a remark pointed out in the second paragraph of §2 in Paris and Harrington [**10**].

Using the similar argument, we can obtain the following result (see Corollary 28 (ii) in [**9**]).

THEOREM 4.8. *Let $n\geqq 2$. Then for each m, $\forall x\exists y([x, y]\underset{*}{\rightarrow}(n+1)^n_m)$ is provable in PA^*_{n-1} for any Σ_0-representation of the relation $[x, y]\underset{*}{\rightarrow}(n+1)^n_m$ Similarly, there exists a Σ_1-formula $P'(x, z, y)$ such that for each m*

1) $\sigma_{n,m}(x)=\mu yP'(x, \bar{m}, y)$,

2) $\forall x\exists! yP'(x, \bar{m}, y)$ *is provable in* PA_{n-1}.

PROOF. By Proposition 4.3 and Lemma 2.5,

$$\sigma_{n,m}(x) \leqq F_{\omega_{n-2}(m+7)}(x+4)$$

for each x. Thus, similarly as the proof of Theorem 4.5, $\sigma_{n,m}$ belongs to

$\mathcal{F}_{\omega_{n-2}(m+7)}$. Therefore, $\sigma_{n,m}$ is $\omega_{n-1}(m+7)$-ordinal recursive by Proposition 2.4. Thus, $\forall x \exists y([x, y] \nrightarrow_{*} (n+1)^n_m)$ is provable in PA^*_{n-1}, by Corollary 3.10. Similarly, the second part can be shown.

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