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Countable models and unions of theories

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§ 0. Introduction.

This paper is a natural continuation of [7], in which we gave a partial positive solution to the following conjecture:

CONJECTURE. There is no (1-st order) stable theory which has only n (>1) countable models.

More precisely, in [7], we showed that we cannot construct a theory with n (>1) countable models by imitating the construction of Ehrenfeucht's example. In the present paper, we shall prove the above conjecture for those theories which can be conceived as limits of certain theories.

In § 1, we shall explain some necessary definitions and conventions. Some basic facts are given without proofs.

In § 2, we shall deal with those theories which are unions of ω-categorical theories. The main theorem of § 2 is the following:

THEOREM A. *Let T be the union of ω-categorical theories T_n ($n<\omega$) such that $T_n \subset T_{n+1}$ ($n<\omega$). If T has only n (>1) countable models, then T has a (definable) dense order.*

Theorem A is proven by combining the methods used in [4] and [7]: The most important fact used in the proof is that T has a definable dense order if and only if there is a subtheory T_0 of T formulated in some finite language which has a definable dense order. From Theorem A, we can deduce that no ω-categorical theories without dense orders can be extended to theories which have finitely many (>1) countable models by adding axioms for constant symbols.

In § 3, we shall prove the following result for those theories which are unions of pseudo-superstable theories (see Definition 2.1).

THEOREM B. *Let T be the union of pseudo-superstable theories T_n ($n<\omega$) such that $T_n \subset T_{n+1}$ ($n<\omega$). Then $I(\omega, T)=1$ or $I(\omega, T)\geqq\omega$.*

This result can be proven by a close examination of Pillay's proof (see [5]) of Lachlan's theorem.

§1. Preliminaries.

Our notations and conventions are fairly standard. T, T_n $(n<\omega)$ denote complete theories formulated in some *countable* languages. The language of T is denoted by $L(T)$. M, M_n $(n<\omega)$ denote *countable* models of such theories. $A, B, \cdots$ denote subsets of the big model. We use $\bar{a}, \bar{b}, \cdots$ to denote finite tuples of elements in the big model. $p, q, \cdots$ will be used to denote complete types. The set of pure types of T is denoted by $S(T)$. The set of types over A is denoted by $S(A)$. $I(\omega, T)$ is the number of countable models of T. Finite numbers are denoted by $m, n, \cdots$. Cardinals are denoted by $\kappa, \lambda, \cdots$.

Let $\varphi(\bar{x}, \bar{y})$ be a formula with $lh(\bar{x})=lh(\bar{y})$. Then $\varphi^n(\bar{x}, \bar{y})$ denotes the n-times iteration of φ, i.e., $\varphi^n(\bar{x}, \bar{y})=\exists\bar{x}_0, \cdots, \bar{x}_{n-1}[\varphi(\bar{x}, \bar{x}_0)\wedge\cdots\wedge\varphi(\bar{x}_{n-1}, \bar{y})]$. We assume that the reader is familiar with the notion of forking. (In §2, we do not use forking.) We use the notation $p \supset_{\mathrm{nf}} q$ $(p \supset_{\mathrm{f}} q)$ to denote the relation that p is a non-forking (forking) extension of q. As usual, we say two tuples $\bar{a}$ and $\bar{b}$ are independent over A, if $\mathrm{tp}(\bar{a}/A^\frown\bar{b})$ does not fork over A. A set $I=\{\bar{a}_i\}_{i<\kappa}$ is said to be independent over A, if whenever $\{i_0, \cdots, i_m\}$ and $\{j_0, \cdots, j_n\}$ are disjoint subsets of κ then $\bar{a}_{i_0}{}^\frown\cdots{}^\frown\bar{a}_{i_m}$ and $\bar{a}_{j_0}{}^\frown\cdots{}^\frown\bar{a}_{j_n}$ are independent over A.

DEFINITION 1.1. (i) A type $r(\bar{x})\in S(A)$ is said to be isolated if there is a formula $\varphi(\bar{x})$ in $r(\bar{x})$ which generates $r(\bar{x})$.

(ii) Let $p(\bar{x})$ and $q(\bar{x})$ be types over A. A type $r(\bar{x}, \bar{y})\in S(A)$ which is an extension of $p(\bar{x})\cup q(\bar{y})$ is said to be semi-isolated over $\bar{x}$, if there is a formula $\varphi(\bar{x}, \bar{y})$ in $r(\bar{x}, \bar{y})$ such that $p(\bar{x})\cup\{\varphi(\bar{x}, \bar{y})\}$ generates $q(\bar{y})$.

(iii) A type $q(\bar{x}, \bar{y})$ is said to be an order expression if it is isolated over the first variables $\bar{x}$ and not isolated over the second variables $\bar{y}$.

(iv) A type $q(\bar{x}, \bar{y})$ is said to be a weak order expression if it is semi-isolated over the first variables $\bar{x}$ and not semi-isolated over the second variables $\bar{y}$.

DEFINITION 1.2 (Benda). A type $p(\bar{x})$ is said to be a powerful type of T if whenever a model of T realizes it then M is weakly saturated, i.e., M realizes all pure types of T.

The following facts will be easily proven:

FACT (i). *If $I(\omega, T)<\omega$, then a powerful type of T exists.*

FACT (ii). *Let $1<I(\omega, T)<\omega$. Then for each powerful type $p(\bar{x})\in S(T)$, there is an order expression $q(\bar{x}, \bar{y})$ which extends $p(\bar{x})\cup p(\bar{y})$.*

§2. Union of ω-categorical theories.

As is stated in the introduction, it is not known whether there is a stable theory T with n (>1) countable models. In this section we shall prove that if a theory T is a union of ω-categorical theories without definable dense orders, then $I(\omega, T)=1$ or $I(\omega, T)\geqq\omega$. First we shall prove the following easy lemma:

LEMMA 2.1. *If $p(\bar{x})\cup\{\varphi(\bar{x}, \bar{y})\}$ generates an order expression, then every type $q(\bar{x}, \bar{y})$ which extends $p(\bar{x})\cup\{\varphi^n(\bar{x}, \bar{y})\}$ is a weak order expression.*

PROOF. Let $q(\bar{x}, \bar{y})$ be a type which extends $p(\bar{x})\cup\{\varphi^{n+1}(\bar{x}, \bar{y})\}$. Choose three realizations $\bar{a}_i$ $(i<3)$ of p such that

(1) $\models \varphi(\bar{a}_0, \bar{a}_1)$; $\models \varphi^n(\bar{a}_1, \bar{a}_2)$;

(2) $\mathrm{tp}(\bar{a}_0, \bar{a}_2) = q(\bar{x}, \bar{y})$.

By way of a contradiction, suppose that $q(\bar{x}, \bar{y})$ is not a weak order expression. Then there is a formula $\theta(\bar{y}, \bar{x})$ in $q(\bar{x}, \bar{y})$ such that $p(\bar{y})\cup\{\theta(\bar{y}, \bar{x})\}$ proves $p(\bar{x})$. Let $\psi(\bar{u}, \bar{x})$ be the formula $\exists\bar{y}(\varphi^n(\bar{u}, \bar{y})\wedge\theta(\bar{y}, \bar{x})\wedge\varphi(\bar{x}, \bar{u})]$. On the other hand, by the definition of ψ, it is clear that $p(\bar{u})\cup\{\psi(\bar{u}, \bar{x})\}$ generates $\mathrm{tp}(\bar{a}_0, \bar{a}_1)$. Thus $\mathrm{tp}(\bar{a}_0, \bar{a}_1)$ is not an order expression. This is a contradiction.

The following theorem is a strengthening of the main theorem in [7].

THEOREM 2.2. *Let T_n $(n<\omega)$ be ω-categorical theories such that $T_n\subset T_{n+1}$ for all $n<\omega$. Let T be the union of all T_n $(n<\omega)$. If T has only n countable models $(1<n<\omega)$, then there is a formula $\varphi(\bar{x}, \bar{y})\in L(T)$ which defines a dense order. (So T has the strict order property.)*

PROOF. By Fact (i), there is a powerful type $p(\bar{x})$ of T. Using Fact (ii), choose a number $n<\omega$, and a consistent formula $\varphi(\bar{x}, \bar{y})\in L(T_n)$ such that $p(\bar{x})\cup\{\varphi(\bar{x}, \bar{y})\}$ generates an order expression $q(\bar{x}, \bar{y})$ which extends $p(\bar{x})\cup p(\bar{y})$. Since T_n is ω-categorical, we can choose $m<\omega$ such that $\{\varphi^0, \varphi^1, \cdots, \varphi^{m-1}\}$ is a maximal enumeration of pairwise non-equivalent formulas in $\{\varphi^i\}_{i<\omega}$. For each i, $j<\omega$, let $F(i, j)$ be the set

$$\{k<m:\ \exists\bar{z}[\varphi^i(\bar{x}, \bar{z})\wedge\varphi^j(\bar{z}, \bar{y})]\leftrightarrow\varphi^k(\bar{x}, \bar{y})\}.$$

Using this F, define D_i $(i<\omega)$ by the following recursion:

$$D_0=m;\qquad D_{i+1}=\bigcup\{F(j, k):\ j,\ k\in D_i\}.$$

It is clear that, for each $i<\omega$, $D_{i+1}\subset D_i$ and $D_i\neq\emptyset$. Since m is finite, $D=\bigcap_{i<\omega}D_i$ is a non-empty subset of m. For this D, we put $\theta(\bar{x}, \bar{y})=\bigvee_{i\in D}\varphi^i(\bar{x}, \bar{y})$. The following will be easily seen:

(1) $p(\bar{x})\vdash\exists\bar{y}\theta(\bar{x}, \bar{y})$;

(2) $p(\bar{x})\vdash\theta(\bar{x}, \bar{y})\rightarrow\neg\theta(\bar{y}, \bar{x})$;

(3) $p(\bar{x})\vdash\theta(\bar{x}, \bar{y})\leftrightarrow\exists\bar{z}[\theta(\bar{x}, \bar{z})\wedge\theta(\bar{z}, \bar{y})]$.

(1) is ovbious. By Lemma 2.1, every type $q(\bar{x}, \bar{y})$ which extends $p(\bar{x})\cup\{\theta(\bar{x}, \bar{y})\}$ becomes a weak order expression. So clearly (2) holds. (3) is easily proven by

using the definitions of D and F. Next we choose formulas $\psi(\bar{x}), \psi_0(\bar{x}) \in p(\bar{x})$ such that

(1)′ $\psi(\bar{x}) \vdash \exists \bar{y} \theta(\bar{x}, \bar{y})$;

(2)′ $\psi(\bar{x}) \vdash \theta(\bar{x}, \bar{y}) \to \neg\theta(\bar{y}, \bar{x})$;

(3)′ $\psi(\bar{x}) \vdash \theta(\bar{x}, \bar{y}) \leftrightarrow \exists \bar{z}[\theta(\bar{x}, \bar{z}) \wedge \theta(\bar{z}, \bar{y})]$;

(4) $\psi_0(\bar{x}) \vdash \theta(\bar{x}, \bar{x}_0) \wedge \theta(\bar{x}_0, \bar{x}_1) \wedge \cdots \wedge \theta(\bar{x}_i, \bar{y}) \to \psi(\bar{y}) \quad (i<\omega)$.

(4) is possible, since $\theta(\bar{x}, \bar{y})$ is transitive. Letting $\varphi(\bar{x}, \bar{y})$ be the formula $\theta(\bar{x}, \bar{y}) \wedge \exists \bar{z}[\psi_0(\bar{z}) \wedge \theta(\bar{z}, \bar{x})]$, we prove that φ defines a dense order. By (1)′ and (3)′, the sentence $\exists \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y})$ is true in T. The antisymmetry of φ is clear by (2)′. The transitivity of φ is also clear by (3)′. Now it is enough to see that the order defined by φ is dense. Suppose that $\varphi(\bar{a}, \bar{b})$ holds in the big model. By (3)′, we can choose a tuple $\bar{d}$ for which both $\theta(\bar{a}, \bar{d})$ and $\theta(\bar{d}, \bar{b})$ holds in the big model. By the choice of $\bar{a}$, $\exists \bar{z}[\psi_0(\bar{z}) \wedge \theta(\bar{z}, \bar{a})]$ clearly holds. Hence $\theta(\bar{a}, \bar{d})$ holds. Using (3)′, we can also see that $\theta(\bar{d}, \bar{b})$ holds. Thus φ defines a dense order in T.

From this theorem we can deduce the following corollaries:

COROLLARY 2.3. *Let T_0 be an ω-categorical theory without a definable dense order. Let T be an extension of T_0 by axioms for constants. Then $I(\omega, T)=1$ or $I(\omega, T) \geqq \omega$.*

Using the above theorem, in every model of T, we can choose a copy $(\bar{a}_t)_{t \in \boldsymbol{Q}}$ of the set of rationals. Thus we have the following corollary:

COROLLARY 2.4. *Suppose that T has n (>1) countable models. If T is the union of T_n $(n<\omega)$ as in the above theorem, then each model M of T has 2^ω elementary extensions up to isomorphism over M.*

PROOF. Applying Theorem 2.2 to T, we have a formula $\varphi(\bar{x}, \bar{y})$ and a sequence $(\bar{a}_t)_{t \in \boldsymbol{Q}}$ in M such that for all rationals t and s,

$$M \models \forall \bar{x}[\varphi(\bar{x}, \bar{a}_t) \to \varphi(\bar{x}, \bar{a}_s)] \quad \text{iff} \quad t \geqq s.$$

For each real number $r \in \boldsymbol{R}$, let $p_r(\bar{x}) \in S(M)$ be an arbitrary completion of the following:

$$\{\varphi(\bar{x}, \bar{a}_t) : t \in \boldsymbol{Q} \text{ and } t>r\} \cup \{-\varphi(\bar{x}, \bar{a}_t) : t \in \boldsymbol{Q} \text{ and } t<r\}.$$

Then p_r and $p_{r'}$ are different types in $S(M)$, if $r \neq r'$. So there are 2^ω non-equivalent types in $S(M)$. This implies that there are 2^ω different elementary extensions of M up to isomorphism over M.

§3. Union of pseudo-superstable theories.

In this section, we shall introduce the notion of pseudo-superstability so that a superstable theory is pseudo-superstable. It is well-known that if T is superstable then there is a type whose (local) weight is one. However, so far as the author knows, this fact has not been generalized to a non-superstable theory. We shall make a change in the definition of weight and prove the existence of a weight one type for a pseudo-superstable theory. Its proof is essentially a modification of Pillay's proof of Lachlan's theorem concerning the number of countable models.

DEFINITION 3.1. A theory T is said to be pseudo-superstable, if there do not exist tuples $\bar{a}$, $\bar{b}_i$ $(i<\omega)$ and a set A such that

(a) $\{\bar{b}_i\}_{i<\omega}$ is independent over A;

(b) $\bar{a}$ and $\bar{b}_i$ are not independent over A, for each $i<\omega$.

It is not hard to see that if T is superstable then T is pseudo-superstable. The following example shows the converse does not hold.

EXAMPLE. Let T be the theory of refining equivalence relations $E_n(x, y)$ $(n<\omega)$ such that each E_n-class is divided into infinitely many E_{n+1}-classes. Then T is not superstable but pseudo-superstable. (See [8], for reference.)

DEFINITION 3.2. (i) Let S be a subset of $S(A)$. A set $R\subset S\times S$ of types is said to be a transitive forking class (on S) if (a) $\mathrm{tp}(\bar{a}^\frown\bar{b}/A)\in R$ implies $\mathrm{tp}(\bar{a}/\bar{b}^\frown A)\supset_f \mathrm{tp}(\bar{a}/A)$, and (b) $\mathrm{tp}(\bar{a}^\frown\bar{b}/A)$, $\mathrm{tp}(\bar{b}^\frown\bar{d}/A)\in R$ implies $\mathrm{tp}(\bar{a}^\frown\bar{d}/A)\in R$.

(ii) Let T be stable and $R\subset S\times S$ a transitive forking class. Let p be a type in $S(A)$. The R-weight $w_R(p)$ of p in R is the maximum cardinal κ such that for every $\lambda<\kappa$, there are a realization $\bar{a}$ of p and realizations $\bar{b}_i$ $(i\leqq\lambda)$ of types in S with the following properties:

(a) $\{\bar{b}_i\}_{i\leqq\lambda}$ are independent over A;

(b) $q_i(\bar{x}, \bar{y})=\mathrm{tp}(\bar{a}, \bar{b}_i/A)$ belongs to R, for each $i\leqq\lambda$.

If T is stable then $w_R(p)$ is no more than $\kappa(T)$. Moreover, if T is pseudo-superstable then $w_R(p)$ is not greater than ω. Let $S\subset S(A)$ be a set of non-principal types then

$$R=\{q(\bar{x}, \bar{y})\supset p(\bar{x})\cup r(\bar{y}) : q \text{ is semi-isolated over } \bar{x}\,;\ p, r\in S\}$$

is a transitive forking class on S. The following Proposition can be proven by essentially the same argument as that of Lemma 6 in [5].

PROPOSITION 3.3. *Let T be pseudo-superstable and $R\subset S\times S$ a transitive forking class. Then there is a type $p\in S$ with $w_R(p)=1$.*

PROOF. By way of a contradiction, assume that there are no such types. We shall construct a sequence $\{\bar{a}_i\}_{i<\omega}$ of realizations of types in S such that

(1) both $\mathrm{tp}(\bar{a}_{2i}, \bar{a}_{2i+1}/A)$ and $\mathrm{tp}(\bar{a}_{2i}, \bar{a}_{2(i+1)}/A)$ are members of R;

(2) $\{\bar{a}_{2j+1}\}_{j\leq i}\cup\{\bar{a}_{2(i+1)}\}$ are independent over A.

The construction can be done inductively. We assume that we have already defined $\{\bar{a}_j\}_{j\leq 2i}$. We must define $\bar{a}_{2i+1}$ and $\bar{a}_{2(i+1)}$. Note that $\bar{a}_{2i}$ realizes a type in S. By our assumption, there are two realizations $\bar{b}$ and $\bar{d}$ of types in S such that

(1)′ both $\mathrm{tp}(\bar{a}_{2i}, \bar{b}/A)$ and $\mathrm{tp}(\bar{a}_{2i}, \bar{d}/A)$ are members of R;

(2)′ $\bar{b}$ and $\bar{d}$ are independent over A.

Next choose two tuples $\bar{a}_{2i+1}$ and $\bar{a}_{2(i+1)}$ such that

(3) $\mathrm{tp}(\bar{a}_{2i+1}{}^\frown\bar{a}_{2(i+1)}/A{}^\frown\{\bar{a}_{2j+1}\}_{j<i}{}^\frown\{\bar{a}_{2i}\})\supset_{\mathrm{nf}}\mathrm{tp}(\bar{b}{}^\frown\bar{d}/A{}^\frown\{\bar{a}_{2i}\})$.

We prove that these $\bar{a}_{2i+1}$ and $\bar{a}_{2(i+1)}$ satisfy the conditions stated in (1) and (2) above. Since (1) is clear, we prove (2) only. By (3) and forking symmetry, we have

$$\mathrm{tp}(\{\bar{a}_{2j+1}\}_{j<i}/A{}^\frown\bar{a}_{2i+1}{}^\frown\bar{a}_{2(i+1)}{}^\frown\bar{a}_{2i})\supset_{\mathrm{nf}}\mathrm{tp}(\{\bar{a}_{2j+1}\}_{j<i}/A{}^\frown\bar{a}_{2i}).$$

By the induction hypothesis, $\mathrm{tp}(\{\bar{a}_{2j+1}\}_{j<i}/A{}^\frown\bar{a}_{2i})$ does not fork over A. Thus we have

(4) $\mathrm{tp}(\{\bar{a}_{2j+1}\}_{j<i}/A{}^\frown\bar{a}_{2i+1}{}^\frown\bar{a}_{2(i+1)})\supset_{\mathrm{nf}}\mathrm{tp}(\{\bar{a}_{2j+1}\}_{j<i/A})$.

Since $\{\bar{a}_{2j+1}\}_{j<i}$ are independent over A, (4) shows that $\{\bar{a}_{2j+1}\}_{j\leq i}$ is also independent over A. Again by (4),

$$\mathrm{tp}(\bar{a}_{2i+1}{}^\frown\bar{a}_{2(i+1)}/A{}^\frown\{\bar{a}_{2j+1}\}_{j<i})\supset_{\mathrm{nf}}\mathrm{tp}(\bar{a}_{2i+1}{}^\frown\bar{a}_{2(i+1)}/A).$$

Thus we have

$$\mathrm{tp}(\bar{a}_{2(i+1)}/A{}^\frown\{\bar{a}_{2j+1}\}_{j\leq i})\supset_{\mathrm{nf}}\mathrm{tp}(\bar{a}_{2(i+1)}/A{}^\frown\bar{a}_{2i+1}).$$

Since $\bar{b}$ and $\bar{d}$ are independent over A, $\bar{a}_{2i+1}$ and $\bar{a}_{2(i+1)}$ are independent over A. Hence we have

$$\mathrm{tp}(\bar{a}_{2(i+1)}/A{}^\frown(\bar{a}_{2j+1})_{j\leq i})\supset_{\mathrm{nf}}\mathrm{tp}(\bar{a}_{2(i+1)}/A).$$

Thus we have proven the independence of the set $\{\bar{a}_{2j+1}\}_{j\leq i}\cup\{\bar{a}_{2(i+1)}\}$ over A. Now we can easily see that $\{\bar{a}_{2i+1}\}_{i<\omega}$ are independent over A, and each $\mathrm{tp}(\bar{a}_0, \bar{a}_{2i+1}/A)$ belongs to R (by the transitivity of R). Hence $\bar{a}_{2i+1}$ and $\bar{a}_0$ are not independent over A. This is a contradiction, since we are assuming T is pseudo-superstable.

In the proof of the above theorem, if S is finite then we can demand the number of types $\mathrm{tp}(\bar{a}_{2i+j}/A{}^\frown\bar{a}_{2i})$ $(i<\omega;\ j=1, 2)$ to be finite. So we have the following corollary:

COROLLARY 3.4. *Let T_n be pseudo-superstable theories such that $T_n\subset T_{n+1}$*

$(n<\omega)$. *Let T be the union of all T_n $(n<\omega)$. If S is a finite set of non-principal types over A and $R=\{q(\bar{x}, \bar{y})\supset p(\bar{x})\cup r(\bar{y})$: q is semi-isolated over $\bar{x}$ and $p, r\in S\}$, then there is a type $p\in S$ with $w_R(p)=1$.*

PROOF. Since other cases are similar, we assume that $S=\{p\}$. If we assume $w_R(p)\geqq 2$, as in the proof of Proposition 3.3, we have realizations $\{\bar{a}_i\}_{i<\omega}$ of p satisfying the following conditions:

(1) both $\mathrm{tp}(\bar{a}_{2i}, \bar{a}_{2i+1}/A)$ and $\mathrm{tp}(\bar{a}_{2i}, \bar{a}_{2(i+1)}/A)$ are members of R;

(2) $\{\bar{a}_{2j+1}\}_{j\leqq i}\cup\{\bar{a}_{2(i+1)}\}$ are independent over A.

Since $S=\{p\}$, we can assume there are two formulas $\varphi_j\in L(T)$ $(j=1, 2)$ such that for each $i<\omega$,

(3) $\models \varphi_j(\bar{a}_{2i}, \bar{a}_{2i+j})$ $\quad (j=1, 2)$;

(4) every type $q(\bar{x}, \bar{y})\supset p(\bar{x})\cup\{\varphi_j(\bar{x}, \bar{y})\}$ belongs to R $(j=1, 2)$.

Let $\theta_n(\bar{x}, \bar{y})$ be the formula $\exists\bar{z}[\varphi_2^n(\bar{x}, \bar{z})\wedge\varphi_1(\bar{x}_{n-1}, \bar{y})]$. Then every type $q(\bar{x}, \bar{y})$ $\supset p(\bar{x})\cup\{\varphi^i(\bar{x}, \bar{y})\}$ belongs to R $(i<\omega)$. So we can see the following hold in turn:

$$\bar{a} \text{ satisfies } \varphi_i(\bar{a}_0, \bar{x}) \implies \mathrm{tp}(\bar{a}_0, \bar{a}/A)\in R;$$

$$\bar{a} \text{ satisfies } \varphi_i(\bar{a}_0, \bar{x}) \implies \mathrm{tp}(\bar{a}/A^\frown\bar{a}_0) \text{ forks over } A;$$

$$\varphi_i(\bar{a}_0, \bar{x}) \text{ forks over } A \text{ (in } T);$$

Choose a number $n<\omega$ such that $L(T_n)$ contains all φ_i. Then $\varphi_i(\bar{a}_0, \bar{x})$ forks over A (in T_n). Now we can conclude that $\{\bar{a}_{2i+1}\}_{i<\omega}$ are independent over A (in T_n) and each $\bar{a}_{2i+1}$ satisfies $\varphi_i(\bar{a}_0, \bar{x})$. But this is a contradiction, since we are assuming T_n is pseudo-superstable.

The following theorem extends Lachlan's theorem.

THEOREM 3.5. *Let T_n $(n<\omega)$ and T be as in the statement of Corollary* 3.4. *Then $I(\omega, T)=1$ or $I(\omega, T)\geqq\omega$.*

PROOF. Suppose that $1<I(\omega, T)<\omega$. Then there is a powerful type $p(\bar{x})$ $\in S(T)$. Let $\bar{a}$ be a realization of p and M a prime (atomic) model over $\bar{a}$. Let $\bar{b}$ and $\bar{d}$ be realizations of p which are independent. Since M is weakly saturated, we can choose $\bar{b}$ and $\bar{d}$ from M. So both $\bar{b}$ and $\bar{d}$ are isolated by $\bar{a}$. Hence we have $w_R(p)\geqq 2$. But this contradicts the statement of Corollary 3.4.

By Theorem 3.5, if a (countable) stable theory T has only finitely many (>1) countable models then there is a finite sublanguage L_0 of $L(T)$ for which $T\restriction L_0$ becomes non-pseudosuperstable. So, by Theorem 2.1 of [8], there are L_0-formulas $\{\varphi_i(x, \bar{y}_i)\}_{i<\omega}$, finite numbers n_i and parameters $\{\bar{a}^j_i\}_{i,j<\omega}$ such that (1) $\{\varphi_i(x, \bar{a}^j_i) : j<\omega\}$ are n_i-inconsistent for every $i<\omega$, and (2) $\{\varphi_i(x, \bar{a}^{\eta(i)}_i) : i<\omega\}$ are consistent for every $\eta\in{}^\omega\omega$.

Many model theoreticians believe that every ω-categorical stable theory is superstable (hence pseudo-superstable). If this is true then from Theorem 3.5 we can easily deduce the following weaker version of Theorem 2.2: Let T_n be ω-categorical stable theories such that $T_n \subset T_{n+1}$. Then $I(\omega, T)=1$ or $I(\omega, T) \geqq \omega$.

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