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Gap theorems for minimal submanifolds of Euclidean space

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0. Introduction.

The purpose of the present paper is to prove the following

THEOREM A. *Let M be a connected, complete minimal submanifold properly immersed into Euclidean space $\boldsymbol{R}^N$. Suppose that*

$$\text{(0.1)} \qquad \textit{the scalar curvature of } M \textit{ at } x \geqq -\frac{A}{1+|x|^{2+\varepsilon}}$$

for some positive constants A and ε, where $|x|$ stands for the Euclidean norm of $x \in M \subset \boldsymbol{R}^N$. Then:

(I) *M is an m-plane if $m=\dim M \geqq 3$ and M has one end, or if $m=2$, $\varepsilon \geqq 2$ and M has one end.*

(II) *M is a hyperplane if $m=N-1$, $2+\varepsilon>2m$ and M is embedded into $\boldsymbol{R}^N$.*

(III) *M is a catenoid if $m \geqq 3$, $m=N-1$ and M has two ends, or if $m=2$, $N=3$ and M has two embedded ends.*

Since an area-minimizing hypersurface properly embedded into $\boldsymbol{R}^N$ has one end (cf. [**1**]), we have the following

COROLLARY 1. *Let M be an area-minimizing hypersurface properly embedded into $\boldsymbol{R}^N$ satisfying condition* (0.1). *Then M is a hyperplane of $\boldsymbol{R}^N$.*

In case M is a complex submanifold properly embedded into $\boldsymbol{C}^N$, condition (1.0) will imply that the volume of the exterior metric ball $M \cap B_e(r)$ with radius r grows like r^{2m} ($m=\dim_{\boldsymbol{C}} M$) (cf. Lemma 2 (1)), and hence by a theorem of Stoll [**16**], M turns out to be algebraic. In particular, M has one end if $m \geqq 2$ (cf. Lefschetz hyperplane theorem). Thus we have proven

COROLLARY 2. *Let M be a complex submanifold properly embedded into $\boldsymbol{C}^N$.*

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Then M is a complex m-plane, provided that $m=\dim_{C}M$ is greater than or equal to two and the scalar curvature of M satisfies condition (0.1).

We shall give here some remarks and examples to illustrate the roles of several hypotheses in Theorem A.

(1) (Vitter [17]). Let M be an algebraic hypersurface in $\boldsymbol{C}^{m+1}$ ($m=\dim_{C}M$), i. e., $M=\{z\in\boldsymbol{C}^{m+1}: f(z)=0\}$ for $f=f_{(k)}+f_{(k-1)}+\cdots+f_{(0)}$ a polynomial of degree k and $f_{(j)}$ the term of f of degree j. Suppose M is nonsingular at infinity, i. e., the projective hypersurface in $\boldsymbol{CP}^{m+1}$ defined as the zero set of $f_{(k)}$ is nonsingular. Then the sectional curvature of M at z is uniformly bounded in absolute value by $A/|z|^2$ for some positive constant A.

(2) Let M_k be a complex curve of $\boldsymbol{C}^2$ defined by $M_k=\{z=(z_1, z_2): z_1^k+z_2^k=1\}$ ($k\in\{3, 4, \cdots\}$). Then the Gaussian curvature $K(z)$ of M_k at z is given by

$$K(z)=-\frac{(k-1)^2|z_1z_2|^{2(k-2)}}{(|z_1|^{2(k-1)}+|z_2|^{2(k-1)})^3}.$$

(3) Let $X:\boldsymbol{R}^2\to\boldsymbol{R}^3$ be Enneper surface:

$$X(x_1, x_2)=\left(x_1+x_1x_2^2-\frac{x_1^3}{3},\ -x_2-x_2x_1^2+\frac{x_2^3}{3},\ x_1^2-x_2^2\right).$$

Then Gaussian curvature $K(x)$ at x is given by

$$K(x)=-\frac{4}{(1+x_1^2+x_2^2)^4}.$$

Thus Enneper surface satisfies condition (0.1) with $\varepsilon=2/3$. Note that it is not embedded.

(4) Let M be a catenoid of $\boldsymbol{R}^3$, i. e., $M=\{x=(x_1, x_2, x_3): \cosh x_3=\sqrt{x_1^2+x_2^2}\}$. Then the Gaussian curvature $K(x)$ is equal to $-1/(x_1^2+x_2^2)^2$. Thus catenoids satisfy condition (0.1) with $\varepsilon=2$. More generally, m-dimensional catenoids satisfy condition (0.1) with $\varepsilon=2m-2$, and complete minimal surfaces in $\boldsymbol{R}^3$ of finite total curvature and whose ends are embedded satisfy (0.1) with $\varepsilon=2$ (cf. [**15**: p. 801]).

In connection with the above results, we should mention first the two works by Mok, Siu and Yau [**12**] and Greene and Wu [**4**]. In [**12**], they investigated complete noncompact Kähler manifolds and proved that, if the sectional curvature of a complete Kähler manifold M with a pole $o\in M$ is uniformly bounded in absolute value by $A(\varepsilon)/\mathrm{dis}_M(o, *)^{2+\varepsilon}$, then M is biholomorphic to $\boldsymbol{C}^m$, where ε is a positive constant and $A(\varepsilon)$ is a positive number depending on ε. Here we call a point o of M a *pole* if the exponential map at o induces a diffeomorphism between the tangent space at o and M. In addition, they showed that if the curvature of M does not change its sign and $m\geqq 2$, then M is biholomorphic and isometric to $\boldsymbol{C}^m$. Recently, this result was extended by

Greene and Wu [4] to Riemannian manifolds with a pole. The first two assertions of Theorem A was inspired by these works.

During the submission of this paper to the journal, a paper of Schoen [15] appeared. In the latter half of [15], he considers a class of complete minimal hypersurfaces of $\boldsymbol{R}^N$ which are said to be regular at infinity (cf. Definition below) and shows that any complete minimal hypersurface which is regular at infinity and has two ends is a catenoid or a pair of hyperplanes. Our result is closely related to his theorem. In fact, the last assertion of Theorem A has been obtained after his result just mentioned above.

We shall now outline the proof of Theorem A. At the first step, it will be shown that each end of a minimal submanifold M properly immersed into $\boldsymbol{R}^N$ satisfying (0.1) is quasi-isometric to the exterior of a ball of $\boldsymbol{R}^N$. At the second step, we shall consider the Gauss map of M and prove that each end of M behaves reasonably well. Finally we shall show the following

THEOREM B. *Let M be a connected, complete minimal submanifold properly immersed into $\boldsymbol{R}^N$. Suppose that M satisfies condition* (0.1) *and that if* $m=\dim M=2$, *ε is greater than or equal to two and each end of M is embedded. Then M is regular at infinity.*

Here we shall give the following

DEFINITION. A complete minimal immersion $M\to\boldsymbol{R}^N$ is said to be *regular at infinity* if there is a compact subset $K\subset M$ such that $M-K$ consists of k connected components $M_1, \cdots, M_k$ such that each M_j is the graph of functions $\{h_{j;\alpha}\}_{\alpha=1,\cdots,N-m}$ ($m=\dim M$) with bounded slope over the exterior of a bounded region in some m-plane P_j. Moreover if $v_1, \cdots, v_m$ are coordinates of P_j, we require the $h_{j;\alpha}$ have the following asymptotic behaviour for large $|v|$ and $m=2$:

$$h_{j;\alpha}(v)=a_{j;\alpha}\log|v|+b_{j;\alpha}+(c_{j;\alpha,1}v_1+c_{j;\alpha,2}v_2)|v|^{-2}+O|v|^{-1-\delta} \qquad (0<\delta\leqq 1),$$

while for $m\geqq 3$, we require

$$h_{j;\alpha}(v)=b_{j;\alpha}+a_{j;\alpha}|v|^{2-m}+\sum_{\beta=1}^{m}c_{j;\alpha,\beta}v_\beta|v|^{-m}+O|v|^{-m}$$

for some constants $a_{j;\alpha}$, $b_{j;\alpha}$, $c_{j;\alpha,\beta}$.

This definition is an adaptation of Schoen's one in [15] where minimal hypersurfaces are treated.

In the case of $m\geqq 3$, the first assertion of Theorem A is an immediate consequence of Theorem B and the last one follows from Theorem B and Theorem 3 in [15] which has been stated above. The remaining parts of Theorem A and Theorem B will be proven in Section 2.

1. Preliminaries.

Let $\iota: M \to \boldsymbol{R}^N$ be an immersion of an m-dimensional smooth manifold M into Euclidean space $\boldsymbol{R}^N$. Throughout this paper, M is assumed to be connected. We consider M as a Riemannian manifold with the induced metric g_M. For any point x of M, we shall denote $\iota(x) \in \boldsymbol{R}^N$ by the same letter x if there is no danger of confusion. Thus the tangent space $T_x M$ is a subspace of the tangent space $T_x \boldsymbol{R}^N$ $(=\boldsymbol{R}^N)$ and it is equipped with the inner product induced from the Euclidean inner product $\langle , \rangle$. We write $T_x M^\perp$ for the normal space to M at $x \in M$ and $X^\perp$ for the normal component of a vector $X \in \boldsymbol{R}^N$. Moreover let us denote by ∇ (resp. $\bar{\nabla}$) the covariant differentiation on M with respect to g_M (resp. the covariant differentiation on $\boldsymbol{R}^N$).

First we have the following

LEMMA 1. *Let f be a smooth function on an open subset U of $\boldsymbol{R}^N$ and denote by $f|_{M \cap U}$ the restriction of f to $M \cap U$. Then*

$$\nabla^2 f|_{M \cap U}(X, Y) = \bar{\nabla}^2 f(X, Y) + \langle \alpha_M(X, Y), (\bar{\nabla} f)^\perp \rangle,$$

where $X, Y \in TM$ and $\alpha_M : TM \times TM \to TM^\perp$ is the second fundamental form of M.

PROOF. This follows immediately from the definitions of the Hessians $\nabla^2 f|_{M \cap U}$, $\bar{\nabla}^2 f$ and the second fundamental form α_M.

Let us now prove

LEMMA 2. *Let $\iota: M \to \boldsymbol{R}^N$ be a proper immersion of an m-dimensional non-compact smooth manifold M into $\boldsymbol{R}^N$. Suppose that there exist positive constants A and ε such that*

$$\text{(1.1)} \qquad \text{the square length } |\alpha_M|^2 \text{ of } \alpha_M \text{ at } x \in M \leqq \frac{A}{1+|x|^{2+\varepsilon}},$$

where $|x|$ stands for the Euclidean distance between $x \in M$ and the origin $0 \in \boldsymbol{R}^N$. Then the following assertions hold:

(1) *There are positive constants β, B and a diffeomorphism $\mu: M \setminus B_e(\beta) \to [\beta, \infty) \times_t \partial B_e(\beta)$ (the warped product of $[\beta, \infty)$ and $\partial B_e(\beta)$ with a warping function t) such that for any vector X tangent to M at $x \in M \setminus B_e(\beta)$,*

$$\frac{1}{B} g_M(X, X) \leqq g_w(\mu_* X, \mu_* X) \leqq B g_w(X, X).$$

Here $B_e(\beta) = \{x \in M : |x| \leqq \beta\}$ and g_w denotes the warped metric on $[\beta, \infty) \times_t \partial B_e(\beta)$. Moreover the $[\beta, \infty)$-component of $\mu(x)$ is equal to $|x|$ for any $x \in M \setminus B_e(\beta)$.

(2) *Suppose that $m \geqq 3$. Then M possesses the Green function $G_M(x, y)$ for the Laplace operator Δ_M. Moreover for a fixed point $x \in M$, there exists a*

positive constant $C(x)$ *such that*

$$\frac{1}{C(x)|x-y|^{m-2}} \leqq G_M(x, y) \leqq \frac{C(x)}{|x-y|^{m-2}}$$

for any $y \in M$.

(3) *Suppose that* $m \geqq 3$. *Then, for any smooth function* Q *on* M *satisfying*

$$|Q(x)| \leqq \frac{D}{1+|x|^{2+\delta}}$$

for some positive constants D *and* δ, *there is a unique solution* U *of equation:* $\Delta_M U + Q = 0$ *such that*

$$|U(x)| \leqq \frac{D'}{1+|x|^{\delta}}$$

for some positive constant D' *which is independent of* Q.

(4) *Suppose that* $m \geqq 3$. *Let* h *be a bounded harmonic function defined on an end* Ω *of* M. *Then there is a constant* h_∞ *such that*

$$|h(x)-h_\infty| \leqq \frac{E \sup|h|}{1+|x|^{m-2}}$$

on Ω, *where* E *is a positive constant independent of* h.

PROOF. For the proof of the first assertion, we put $\bar{r}(v)=|v|$ and $r=\bar{r}|_M$. Then we have by Lemma 1 and assumption (1.1)

$$\begin{aligned} \frac{1}{2}\nabla^2 r^2(X, X) &= \frac{1}{2}\bar{\nabla}^2\bar{r}^2(X, X)+\frac{1}{2}\langle\alpha_M(X, X), (\bar{\nabla}r^2)^{\perp}\rangle \\ &= |X|^2+r\langle\alpha_M(X, X), (\bar{\nabla}r)^{\perp}\rangle \\ &\geqq |X|^2-r|\alpha_M(X, X)| \\ &\geqq \left(1-r\sqrt{\frac{A}{1+r^{2+\varepsilon}}}\right)|X|^2, \end{aligned}$$

for any tangent vector $X \in TM$. Similarly we see that

$$\frac{1}{2}\nabla^2 r^2(X, X) \leqq \left(1+r\sqrt{\frac{A}{1+r^{2+\varepsilon}}}\right)|X|^2.$$

Thus the Hessian $(1/2)\nabla^2 r^2$ satisfies

$$(1-\eta\circ r)g_M \leqq \frac{1}{2}\nabla^2 r^2 \leqq (1+\eta\circ r)g_M \tag{1.2}$$

on M, where $\eta(t)=t[A/(1+t^{2+\varepsilon})]^{1/2}$. In the sequel, we follow the argument in [7] in order to construct a quasi-isometry $\mu: M \setminus B_e(\beta) \to [\beta, \infty) \times_t \partial B_e(\beta)$. At first, note that r^2 is a smooth exhaustion function on M and further it is a strictly convex function on $M \setminus B_e(\beta)$ for some $\beta > 0$, because of (1.2). In particular, we see that the gradient ∇r^2 never vanishes on $M \setminus B_e(\beta)$. Let us now define a vector field X_r on $M \setminus B_e(\beta)$ by $X_r = \nabla r / |\nabla r|^2$. We write $\lambda_p : [\beta, \infty) \to M \setminus B_e(\beta)$ for the maximal integral curve of X_r such that $\lambda_p(\beta) = p \in \partial B_e(\beta)$. Then $r(\lambda_p(t)) = t$ for all $t \geqq \beta$, since $dr(\lambda_p(t))/dt = 1$. Define a smooth map $\nu : [\beta, \infty) \times_t \partial B_e(\beta) \to M \setminus B_e(\beta)$ by $\nu(t, p) = \lambda_p(t)$. Then ν gives a diffeomorphism between $[\beta, \infty) \times_t \partial B_e(\beta)$ and $M \setminus B_e(\beta)$. We shall prove that $\mu = \nu^{-1}$ is a required quasi-isometry. In fact, for any smooth regular curve $\gamma : [0, \delta] \to \partial B_e(\beta)$, set a smooth map $\nu_\gamma : [\beta, \infty) \times [0, \delta] \to M \setminus B_e(\beta)$ by $\nu_\gamma(t, s) = \nu(t, \gamma(s))$. Put $X_\gamma = \nu_{\gamma *}(\partial/\partial t)$ and $Y_\gamma = \nu_{\gamma *}(\partial/\partial s)$. Then we have

$$\frac{\partial}{\partial t} |Y_\gamma(t, s)| = \frac{\nabla^2 r(Y_\gamma(t, s), Y_\gamma(t, s))}{|Y_\gamma(t, s)|\, |\nabla r|^2(\nu_\gamma(t, s))}. \tag{1.3}$$

On the other hand, we see by (1.2) that

$$\frac{1}{r}[(1 - \eta \circ r) g_M - (dr)^2] \leqq \nabla^2 r \leqq \frac{1}{r}[(1 + \eta \circ r) g_M - (dr)^2], \tag{1.4}$$

and hence by (1.3) and (1.4) we have

$$\frac{1 - \eta(t)}{t} |Y_\gamma(t, s)|^2 \leqq \nabla^2 r(Y_\gamma(t, s), Y_\gamma(t, s)) \leqq \frac{1 + \eta(t)}{t} |Y_\gamma(t, s)|^2. \tag{1.5}$$

Moreover it follows from (1.4) that

$$1 - \xi \circ r \leqq |\nabla r|^2 \leqq 1 \tag{1.6}$$

on $M \setminus B_e(\beta)$, [where $\xi(t) = 2\int_\beta^t u\eta(u) du / t^2 - \beta^2(a-1)/t^2$ and $a = \min\{|\nabla r|^2(p) : p \in \partial B_e(\beta)\}$ (cf. [7: Lemma 2]). Therefore by (1.3), (1.5) and (1.6), we obtain

$$\frac{1 - \eta(t)}{t} \leqq \frac{\partial}{\partial t} \log |Y_\gamma(t, s)| \leqq \frac{1 + \eta(t)}{t(1 - \xi(t))}$$

so that, integrating the both sides from β to t, we have

$$\frac{t}{\beta} \exp \int_\beta^t - \frac{\eta(u)}{u} du \leqq \frac{|Y_\gamma(t, s)|}{|Y_\gamma(\beta, s)|} \leqq \frac{t}{\beta} \exp \int_\beta^t \frac{\phi(u)}{u} du,$$

where $\phi(t) = (\eta(t) + \xi(t))/(1 - \xi(t))$. This implies that

$$\frac{t}{\beta} \exp \int_\beta^\infty - \frac{\eta(u)}{u} du \leqq \frac{|Y_\gamma(t, s)|}{|Y_\gamma(\beta, s)|} \leqq \frac{t}{\beta} \exp \int_\beta^\infty \frac{\phi(u)}{u} du. \tag{1.7}$$

Thus it turns out from (1.6) and (1.7) that μ is a required map. This completes the proof of the first assertion.

Let us next show the second assertion. We put

$$\Phi(t)=\int_t^\infty s^{1-m}\exp\left[-\int_\beta^s \frac{\phi(u)}{u}du\right]ds,$$

$$\Psi(t)=\int_t^\infty s^{1-m}\exp\left[\int_\beta^s \frac{m\eta(u)}{u}du\right]ds,$$

where $\phi(t)=\{(m-1)\xi(t)+m\eta(t)\}/(1-\xi(t))$. Then by (1.4) and (1.6), we see that $\Delta_M\Phi\circ r\geqq 0$ and $\Delta_M\Psi\circ r\leqq 0$ on $M\setminus B_e(\beta)$. The maximum principle implies that for some positive constants $C_1(x)$ and $C_2(x)$,

$$C_1(x)\Phi\circ r(y)\leqq G_M(x, y)\leqq C_2(x)\Psi\circ r(y)$$

outside a compact set. This shows the second assertion, since $\Phi(t)\geqq C_3t^{2-m}$ and $\Psi(t)\leqq C_4t^{2-m}$ for some positive constants C_3 and C_4.

To prove the third assertion, we set

$$\Sigma(t)=\int_t^\infty\left[\frac{1}{\tau(s)}\int_0^s\sigma(u)\tau(u)du\right]ds,$$

where $\sigma(t)=D/(1+t^{2+\delta})$, $\tau(t)=t^{m-1}\exp\left[-\int_1^t m\eta(s)/s\, ds\right]$, and η is as in (1.2). Then by (1.4) and (1.6), we have

$$\Delta_M\Sigma\circ r+\sigma\circ r\leqq 0$$

on $M\setminus B_e(R)$, for some positive constant R. Therefore it follows from the same argument as in the proof of Theorem 5.4 in [**9**] that

$$\begin{aligned}\int_M Q(y)G_M(x, y)dy&\leqq\int_M\sigma\circ r(y)G_M(x, y)dy\\&\leqq D'\Sigma\circ r(x),\end{aligned}$$

where D' is a positive constant independent of Q. Thus $U(x)=\int_M Q(y)G_M(x, y)dy$ is the required solution.

It remains to prove the last assertion. For any $t>\beta$, we denote by $M_j(t)$ $(j=1, \cdots, k)$ the connected components of $M\setminus B_e(t)$. Let us define immersions $\iota_j(t)$ from $M_j(t)$ into the unit sphere $S^{N-1}(1)$ of $\boldsymbol{R}^N$ by $\iota_j(t)(x)=t^{-1}\iota(x)$. Then by condition (1.1), we see that the second fundamental forms $\alpha_j(t)$ of the immersions tend to 0 as $t\to+\infty$. Therefore each $M_j(t)$ turns out to be diffeomorphic to the unit sphere $S^{m-1}(1)$ of $\boldsymbol{R}^m$. Thus taking account of the first assertion, we may assume that each $M_j(t)$ (t is fixed) is a domain of $\boldsymbol{R}^m$ equipped with a Riemannian metric g_j satisfying

$$\frac{1}{\lambda}|X|^2\leqq g_j(X, X)\leqq\lambda|X|^2$$

for some $\lambda \geqq 1$ and every $X \in \boldsymbol{TR}^m$. Let h be a bounded harmonic function defined on $M_j(t)$. Then it follows from Theorem 5 of Moser [**13**] that $h(x)$ tends to a constant h_∞ as $x \to \infty$. Moreover by the second assertion, we get

$$|h(x)-h_\infty| \leqq \frac{E \sup|h|}{|x|^{m-2}},$$

where E is a positive constant independent of h. This completes the proof of Lemma 2.

REMARK. Let M and β be as in Lemma 2. If the dimension of M is not less than 3 and $\partial B_e(\beta)$ is disconnected, that is, M has at least two ends, then M possesses nonconstant bounded harmonic functions with finite Dirichlet norm (cf. the proof of Lemma 2 (2) and [**8**: Corollary (5.8)]).

COROLLARY 3. *Let M be as in Lemma 2. Suppose that $m \geqq 3$. Then M has the only one end if and only if there are no bounded harmonic functions on M except constants.*

REMARK (cf. [**10**]). Let $\iota: M \to \boldsymbol{R}^N$ be a minimal immersion of an m-dimensional smooth manifold M into $\boldsymbol{R}^N$. Suppose that $m \geqq 3$. Then:

(1) For any $x \in M$ and $R \geqq 0$, the Green function $G_R(x, y)$ of $B_e(x; R)$ $(=\{y \in M : |y-x| \leqq R\})$ satisfies

$$G_R(x, y) \leqq \frac{1}{(m-1)\omega_{m-1}}\left(\frac{1}{|x-y|^{m-2}}-\frac{1}{R^{m-2}}\right),$$

where ω_{m-1} stands for the volume of the Euclidean unit sphere of dimension $m-1$.

(2) For any smooth function Q on M such that

$$|Q(x)| \leqq \frac{D}{1+|x|^{2+\delta}},$$

where D and δ are positive constants, there is one and only one solution U of equation: $\Delta_M U + Q = 0$ satisfying

$$|U(x)| \leqq \frac{D'}{1+|x|^{\delta}},$$

where D' is a positive constant independent of Q.

We shall now consider the Gauss map of an immersion $\iota: M \to \boldsymbol{R}^N$ from a smooth manifold M of dimension m into Euclidean space $\boldsymbol{R}^N$ and give a crucial lemma (cf. Lemma 5) for the proof of the main theorem.

Let us begin by giving some notations. Let P be an m-dimensional subspace (called an m-plane) of $\boldsymbol{R}^N$. We write π_P (resp. $P^\perp$) for the orthogonal projection from $\boldsymbol{R}^N$ to P (resp. the orthogonal complement of P). For two

m-planes P and Q, define an 'inner product' $\langle\!\langle P, Q\rangle\!\rangle$ by

$$\langle\!\langle P, Q\rangle\!\rangle=\min\{|\pi_Q(X)| : X\in P,\ |X|=1\}.$$

When P and Q are oriented m-planes, we set

$$\langle P, Q\rangle=\det(\langle e_i, f_j\rangle),$$

where $\{e_i\}_{i=1,\cdots,m}$ and $\{f_i\}_{i=1,\cdots,m}$ are, respectively, oriented orthonormal bases of P and Q. Then we have the following

LEMMA 3.

(1) $\langle\!\langle P, Q\rangle\!\rangle=\langle\!\langle Q, P\rangle\!\rangle=\langle\!\langle P^{\perp}, Q^{\perp}\rangle\!\rangle$.

(2) $\langle\!\langle P, Q\rangle\!\rangle=0$ *if and only if* $P\cap Q^{\perp}\neq\{0\}$.

(3) $\langle\!\langle P, Q\rangle\!\rangle^k\leqq|\langle P, Q\rangle|\leqq\langle\!\langle P, Q\rangle\!\rangle \quad (k=\min\{m, N-m\})$.

PROOF. The first two assertions are clear from the definition of $\langle\!\langle P, Q\rangle\!\rangle$. For the last one, it may be assumed that $N-m\geqq m$ and moreover that suitable orthonormal bases $\{e_i\}_{i=1,\cdots,m}$, $\{f_i\}_{i=1,\cdots,m}$ and $\{\hat{f}_i\}_{i=1,\cdots,N-m}$, respectively, of P, Q and $Q^{\perp}$ satisfy

$$e_i=\frac{f_i+\lambda_i\hat{f}_i}{\sqrt{1+\lambda_i^2}}$$

for some $\lambda_i\in R$ $(i=1, \cdots, m)$. Then we have

$$|\langle P, Q\rangle|=|\det(\langle e_i, f_j\rangle)|=\prod_{i=1}^{m}\mu_i \qquad (\mu_i=1/\sqrt{1+\lambda_i^2}).$$

On the other hand, it follows from the definition of $\langle\!\langle P, Q\rangle\!\rangle$ that

$$\langle\!\langle P, Q\rangle\!\rangle=\min\left\{\sqrt{\sum_{i=1}^{m}t_i\mu_i^2} : \sum_{i=1}^{m}t_i=1,\ t_i\geqq 0\right\}$$
$$\leqq\mu_i \qquad (i=1, \cdots, m),$$

and hence, we have

$$\langle\!\langle P, Q\rangle\!\rangle^m\leqq\prod_{i=1}^{m}\mu_i=|\langle P, Q\rangle|.$$

Furthermore we see that $|\langle P, Q\rangle|\leqq\langle\!\langle P, Q\rangle\!\rangle$, because

$$\sum_{i=1}^{m}t_i\mu_i^2\geqq\prod_{i=1}^{m}\mu_i^{2t_i}\geqq\prod_{i=1}^{m}\mu_i^2.$$

Thus the last assertion has been proven. This completes the proof of Lemma 3.

LEMMA 4. *Let* $\iota: M\to\boldsymbol{R}^N$ *be an immersion of an* m*-dimensional smooth manifold* M *into Euclidean space* $\boldsymbol{R}^N$ *such that the induced metric on* M *is complete. Suppose that there are an* m*-plane* P, *a compact subset* K *of* M (K

may be empty) and a positive constant ε *such that*

$$\langle\!\langle T_x M, P\rangle\!\rangle \geqq \varepsilon > 0 \tag{1.8}$$

on a noncompact connected component Ω *of* $M\setminus K$. *Denote by* $\bar{r}_{P^\perp}$ *the Euclidean distance to* $P^\perp$ *and by* $r_{P^\perp}$ *the restriction of* $\bar{r}_{P^\perp}$ *to* M. *Set* $\Omega_t=\{x\in\Omega : r_{P^\perp}(x)\geqq t\}$, $P_t=\{v\in P: |v|\geqq t\}$, *and* $T=\max\{r_{P^\perp}(x): x\in\partial\Omega\}$. *Then the following assertions hold:*

(1) *The restriction* $\pi_P|_{\Omega_T}$ *of the orthogonal projection* $\pi_P: \boldsymbol{R}^N\to P$ *to* Ω_T *defines a finite covering map from* Ω_T *onto* P_T.

(2) *In the case when* m *is greater than or equal to* 3, Ω_T *can be realized as a graph of some smooth functions* $\{h_1, \cdots, h_{N-m}\}$ *defined on* P_T (*i.e.*, $\Omega_T=\{(v, h_1(v), \cdots, h_{N-m}(v)): v\in P_T\}$). *Moreover these functions satisfy*

$$1+\sum_{i=1}^{N-m}\left|\frac{\partial h_i}{\partial v_\alpha}\right|^2 \leqq \varepsilon^2 \qquad (\alpha=1, \cdots, m), \tag{1.9}$$

where $(v_1, \cdots, v_m)$ *is a canonical coordinate system on* P.

(3) *If* K *is empty, the second assertion holds on* P *without any restriction on* m.

PROOF. Without loss of generality, we may assume that K is a compact domain with smooth boundary ∂K. Since the vector field $\bar{\nabla}\bar{r}_{P^\perp}$ is parallel to P, we see by (1.8) that

$$|\nabla r_{P^\perp}| \geqq \varepsilon \tag{1.10}$$

on Ω. This inequality enables us to define a smooth vector field X on a neighborhood of $\bar{\Omega}$ by $X=\nabla r_{P^\perp}/|\nabla r_{P^\perp}|^2$. Let $\gamma:(a, b)\to\Omega$ be an integral curve of X. Observe that $|r_P(\gamma(t))-r_P(\gamma(s))|=|t-s|$ for any $t, s\in(a, b)$. This implies that $|t-s|\leqq \mathrm{dis}_M(\gamma(t), \gamma(s))$. On the other hand, it follows from (1.10) that $\mathrm{dis}_M(\gamma(t), \gamma(s))\leqq\varepsilon^{-1}|t-s|$. Thus we obtain

$$|t-s| \leqq \mathrm{dis}_M(\gamma(t), \gamma(s)) \leqq \frac{1}{\varepsilon}|t-s| \tag{1.11}$$

for any $t, s\in(a, b)$. Let us now fix a number $t_0>T$ and let $\gamma:(a, b)\to\Omega$ be a maximal integral curve such that $t_0\in(a, b)$ and $\gamma(t_0)\in\partial\Omega_{t_0}$. Then inequality (1.11) tells us that $\gamma((a, t_0])$ is contained in the closure of the (intrinsic) metric ball around $\gamma(t_0)$ with radius $\varepsilon^{-1}(t_0-a)$. Therefore we see by (1.11) that γ can be defined on $[a, t_0]$ and that $\gamma(a)$ belongs to $\partial\Omega$. Hence the correspondence: $\gamma(t_0)\to\gamma(a)$ defines an injective map from $\partial\Omega_{t_0}$ into $\partial\Omega$ such that $\mathrm{dis}_M(\gamma(t_0), \gamma(a))\leqq\varepsilon^{-1}(t_0-a)\leqq\varepsilon^{-1}t_0$. This shows that $\partial\Omega_{t_0}$ is a compact hypersurface of M for every $t_0\in(T, \infty)$, so that $\pi_P|_{\Omega_T}: \Omega_T\to P_T$ is a proper immersion. Thus the assertion (1) turns out to be true. As for the second assertion, note that P_T is simply connected if $m\geqq 3$ and hence, $\pi_P|_{\Omega_T}: \Omega_T\to P_T$ is actually a diffeomorphism between Ω_T and P_T. Consequently Ω_T can be realized as a graph of some smooth functions $\{h_i\}_{i=1,\cdots,N-m}$ defined on P_T. It is clear from condition (1.8) that these functions satisfy inequality (1.9). The last statement (3) follows from the above proof of the first assertion. This completes the proof of Lemma 4.

LEMMA 5. *Let $\iota: M\to \boldsymbol{R}^N$ be a proper immersion from an m-dimensional smooth manifold M into Euclidean space $\boldsymbol{R}^N$. Suppose that*

$$m \geqq 3, \tag{1.12}$$

$$|\alpha_M|^2(x) \leqq \frac{A}{1+|x|^{2+\varepsilon}}, \tag{1.13}$$

$$|\nabla H_M|(x) \leqq \frac{A'}{1+|x|^{2+\varepsilon'}} \tag{1.14}$$

for some positive constants A, A', ε, and ε', where H_M stands for the mean curvature normal of the immersion $\iota: M\to \boldsymbol{R}^N$. Let $\{M_j\}_{j=1,\cdots,k}$ be the connected components of $M\setminus B_e(\beta)$, where $B_e(\beta)$ is as in Lemma 2(1). Then for each M_j, the assertions below are true:

(1) *There exist an m-plane P and a positive constant B such that*

$$\langle\!\langle T_xM, P\rangle\!\rangle \geqq \frac{B}{1+|x|^{\varepsilon}} \tag{1.15}$$

for every $x\in M_j$.

(2) *For some nonnegative constant T, $M_{j,T}=\{x\in M_j: r_{P^\perp}(x)\geqq T\}$ can be realized as a graph of some smooth functions $\{h_1, \cdots, h_{N-m}\}$ defined on $P_T=\{v\in P: |v|\geqq T\}$, where $r_{P^\perp}=\mathrm{dis}_{\boldsymbol{R}^N}(P^\perp, *)$. Moreover each h_i satisfies*

$$\left|\frac{\partial h_i}{\partial v_\alpha}\right| \leqq \frac{C}{1+|v|^{\varepsilon}} \qquad (\alpha=1, \cdots, m) \tag{1.16}$$

for some positive constant C, where $(v_1, \cdots, v_m)$ is a canonical coordinate system on P.

(3) *Let h be a harmonic function on M_j such that $|h(x)|\leqq D(1+r_P(x)^{1-\eta})$ for some $D>0$ and $\eta>0$. Then there are constants h_∞, D' and E such that*

$$|h(x)-h_\infty| \leqq \frac{D'}{1+|x|^{m-2}}, \qquad |\nabla h(x)| \leqq \frac{E}{1+|x|^{m-1}}.$$

PROOF. It is enough to prove Lemma 5 in case M is oriented. Let P be any oriented m-plane of $\boldsymbol{R}^N$. Then the computation by Fischer-Corbrie [**2**: Lemma 1.1] and the above assumptions (1.13) and (1.14) show that

$$|\Delta_M\langle T_xM^\perp, P^\perp\rangle| \leqq 2|\alpha_M|^2(x)+|\nabla H_M|$$
$$\leqq \frac{2A}{1+|x|^{2+\varepsilon}}+\frac{A'}{1+|x|^{2+\varepsilon'}}$$

on M. Hence it follows from Lemma 2(3) that there is a unique smooth function U_P on M such that

$$\Delta_M\langle T_xM^\perp, P^\perp\rangle = \Delta_M U_P$$
$$|U_P(x)| \leqq \frac{B}{1+|x|^\delta} \qquad (\delta=\min\{\varepsilon, \varepsilon'\}),$$

where B is a positive constant independent of P. This implies that $\langle T_xM^\perp, P^\perp\rangle - U_P(x)$ is a bounded harmonic function on M, and hence it follows from Lemma 2(4) that for some positive constants $C(P)$ and D,

$$|\langle T_xM^\perp, P^\perp\rangle - U_P(x) - C(P)| \leqq \frac{D}{1+|x|^{m-2}}$$

on M_j. Here D is independent of P. Thus we have

$$|\langle T_xM^\perp, P^\perp\rangle - C(P)| \leqq \frac{B}{1+|x|^\delta}+\frac{D}{1+|x|^{m-2}} \tag{1.17}$$

for any $x\in M_j$ and every oriented m-plane P of $\boldsymbol{R}^N$. It is clear from (1.17) that for some oriented plane P, $C(P)$ is equal to 1, that is,

$$\langle T_xM^\perp, P^\perp\rangle \geqq 1-\frac{B}{1+|x|^\delta}-\frac{D}{1+|x|^{m-2}}$$

on M_j. Because of Lemma 3 and (1.13), this implies the first assertion of the lemma, from which the second one can be derived (cf. the proof of Lemma 4).

Now we shall show the last assertion, making use of the second one. Let us identify $M_{j,T}$ with P_T through the orthogonal projection from $M_{j,T}$ onto P_T. Then the Riemannian metric g_M on M can be written on P_T as follows:

$$g_M = \sum_{\alpha,\beta=1}^{m} g_{\alpha\beta}dv_\alpha dv_\beta$$
$$g_{\alpha\beta} = \delta_{\alpha\beta}+\sum_{i=1}^{N-m}\frac{\partial h_i}{\partial v_\alpha}\frac{\partial h_i}{\partial v_\beta},$$

where $(v_1, \cdots, v_m)$ is a canonical coordinate system on P. Let us extend $g_{\alpha\beta}$ to a smooth function $\tilde{g}_{\alpha\beta}$ defined on P so that $\tilde{g}=\sum \tilde{g}_{\alpha\beta}dv_\alpha dv_\beta$ becomes a Riemannian metric on P. Observe that for some positive constant C_1,

$$\left|\frac{\partial g_{\alpha\beta}}{\partial v_\gamma}\right| \leqq \frac{C_1}{1+|v|^{1+\varepsilon/2}} \qquad (\alpha, \beta, \gamma=1, \cdots, m)$$

on P, because of (1.13) and (1.16). This shows that for some positive constant C_2,

$$|a^{\alpha\beta}(u)-a^{\alpha\beta}(v)| \leqq C_2|u-v| \tag{1.18}$$

for any $u, v\in P$, where $a^{\alpha\beta}=\sqrt{G}\,\tilde{g}^{\alpha\beta}$, $G=\det(\tilde{g}_{\alpha\beta})$, and $(\tilde{g}^{\alpha\beta})=(\tilde{g}_{\alpha\beta})^{-1}$. Let h be a harmonic function on M_j satisfying $|h(x)|\leqq D(1+r_{P^\perp}(x)^{1-\eta})$ for some $D>0$ and $\eta>0$. Take a smooth function $\tilde{h}$ on P which coincides with h on P_T. Then the Laplacian $\tilde{\Delta}\tilde{h}$ of $\tilde{h}$ with respect to $\tilde{g}$ is a smooth function on P whose support is compact. We set

$$\tilde{U}(v) = -\int_P \tilde{G}(v, w)\tilde{\Delta}\tilde{h}(w)dw\,,$$

where $\tilde{G}(v, w)$ denotes the Green function of $\tilde{\Delta}$. Then $\tilde{\Delta}\tilde{h}=\tilde{\Delta}\tilde{U}$ on P, so that $\tilde{h}-\tilde{U}$ is a harmonic function with respect to $\tilde{g}$ such that $|(\tilde{h}-\tilde{U})(v)|\leqq D'(1+|v|^{1-\eta})$ for some $D'\geqq 0$. It follows from (1.16), (1.18) and Lemma 6 below that $\tilde{h}-\tilde{U}$ must be constant, and hence h is a bounded harmonic function on M_j. Consequently we have $|h(v)-h_\infty|\leqq D''|v|^{2-m}$ for some constants h_∞ and $D''\geqq 0$. Moreover since $|\nabla_v\tilde{G}(v, w)|\leqq E|v-w|^{1-m}$ (cf. Lemma 6), we obtain $|\nabla h|(v)=|\tilde{\nabla}\tilde{U}|(v)\leqq E'|v|^{1-m}$ for large $|v|$, where E and E' are positive constants. This completes the proof of Lemma 5.

LEMMA 6 (Widman [18]). *Consider an equation of divergence form*

$$Lf = \sum_{\alpha,\beta=1}^{m}\frac{\partial}{\partial v_\alpha}\Big(a^{\alpha\beta}\frac{\partial f}{\partial v_\beta}\Big) = 0$$

on $\boldsymbol{R}^N$ with the properties:

$$\lambda|\xi|^2 \leqq \sum a^{\alpha\beta}(v)\xi_\alpha\xi_\beta \leqq \Lambda|\xi|^2 \qquad (v, \xi\in\boldsymbol{R}^m)$$

$$|a^{\alpha\beta}(u)-a^{\alpha\beta}(v)| \leqq \omega(|u-v|)\,,$$

where $\omega(t)$ is a nondecreasing function such that $\int_0^\infty \omega(t)/t\, dt<+\infty$ and $\omega(2t)\leqq K\omega(t)$ for some constant K. Then a solution f of $Lf=0$ on a domain Ω of $\boldsymbol{R}^m$ satisfies

$$|df(v)| \leqq \rho^{-1}(v)\sup|f|K(m, \lambda, \Lambda, \omega)\,,$$

where $\rho(v)=\mathrm{dis}_{\boldsymbol{R}^m}(v, \partial\Omega)$ and $K(m, \lambda, \Lambda, \omega)$ is a positive constant depending only on m, λ, Λ and ω. In particular, if f is a solution of $Lf=0$ on $\boldsymbol{R}^m$ with $|f(v)|\leqq D(1+|v|^{1-\eta})$ for some positive constants D and η, then f must be a constant.

Before concluding this section, we shall show a result similar to Lemma 5 for the case of $m=2$.

LEMMA 7. *Let $\iota: M\to\boldsymbol{R}^N$ be a proper immersion from a surface M into $\boldsymbol{R}^N$*

satisfying (1.13) *and* (1.14). *Let* $\{M_j\}_{j=1,\cdots,k}$ *be as in Lemma* 5. *Then for each* M_j, *the following assertions hold:*

(1) *There exists a plane* P *and a positive constant* B *such that*

$$\langle\!\langle T_x M, P\rangle\!\rangle \geqq 1-\frac{B}{1+|x|^{\varepsilon/2}}$$

on M_j.

(2) *Let* $(v_1, \cdots, v_N)$ *be coordinates of* $\boldsymbol{R}^N=P\oplus P^{\perp}$ *and set* $h_\alpha=v_\alpha\circ\iota: M\to\boldsymbol{R}^N$ $(\alpha=3, \cdots, N)$. *Then*

$$|\nabla h_\alpha|(x)\leqq\frac{C}{1+|x|^{\varepsilon/2}}$$

on M_j, *for some positive constant* C.

(3) *Let* h *be a harmonic function on* M_j *such that* $|h(x)|\leqq D\log|x|+E$ *for some positive constants* D *and* E. *Then* $h(x)$ *has the form:*

$$h(x)=D'\log|x|+f(x),$$

where D' *is a constant which vanishes if so does* D, *and* $f(x)$ *has a finite limit as* $|x|\to+\infty$.

PROOF. Using the same diffeomorphism $\mu: M_j\to[\beta, \infty)\times S^1$ as in Lemma 2, we take coordinates (r, θ) on M_j $(r(x)=|x|)$. Then the metric g_M on M_j can be expressed as follows: $g_M=a(r, \theta)dr^2+b(r, \theta)r^2d\theta^2$. Observe that

$$\begin{aligned} &|1-a(r, \theta)|\leqq C_1/r^\varepsilon \\ &\exp(-C_2/r^\varepsilon)\leqq b(r, \theta)\leqq\exp(C_2/r^\varepsilon) \end{aligned}\tag{1.19}$$

for some positive constants C_1 and C_2 (cf. the proof of Lemma 2). Set $\tilde{r}=1/r$ and take coordinates $(\tilde{r}, \theta)$ on M_j. Then g_M has the form: $g_M=\tilde{a}(\tilde{r}, \theta)\tilde{r}^{-4}\tilde{g}$, where $\tilde{a}(\tilde{r}, \theta)=a(r^{-1}, \theta)$, $\tilde{g}=d\tilde{r}^2+\tilde{c}(\tilde{r}, \theta)\tilde{r}^2d\theta^2$ and $\tilde{c}(\tilde{r}, \theta)=b(r^{-1}, \theta)/a(r^{-1}, \theta)$. Set $v=\tilde{r}\cos\theta$ and $w=\tilde{r}\sin\theta$. Then $\tilde{g}$ has the form:

$$\begin{aligned}\tilde{g}=&\left(1+[\tilde{c}(\tilde{r}, \theta)-1]\frac{w^2}{v^2+w^2}\right)dv^2-2[\tilde{c}(\tilde{r}, \theta)-1]\frac{vw}{v^2+w^2}dvdw\\ &+(1+[\tilde{c}(\tilde{r}, \theta)-1])\frac{w^2}{v^2+w^2}dw^2.\end{aligned}$$

Since $|\tilde{c}(\tilde{r}, \theta)-1|\leqq C_3\tilde{r}^\varepsilon$ for some $C_3\geqq0$ by (1.19), $\tilde{g}$ defines a metric on $\Omega=\{(v, w)\in\boldsymbol{R}^2: v^2+w^2<\beta^{-2}\}$ whose coefficients are smooth on $\Omega^*=\{(v, w)\in\Omega: (v, w)\neq(0, 0)\}$ and Hölder continuous at $(0, 0)$. Hence we can apply the results in [3] to the Laplacian $\tilde{\Delta}$ of $\tilde{g}$. Then it is not hard to see that for any smooth function $\tilde{Q}(\tilde{r}, \theta)$ on Ω^* with $|\tilde{Q}(\tilde{r}, \theta)|\leqq C_4\tilde{r}^{\delta-2}$ for constants $C_4>0$ and $\delta>0$, there exists a unique solution $\tilde{U}$ on Ω^* of equation: $\tilde{\Delta}\tilde{U}+\tilde{Q}=0$ on Ω^*, $\tilde{U}=0$ on

$\partial\Omega$ and $\lim_{\tilde{r}\to 0}\tilde{U}(\tilde{r}, \theta)=0$. This implies that for any smooth function $Q(r, \theta)$ on M_j with $|Q(r, \theta)|\leqq C_5 r^{-2-\delta}$ for constants $C_5>0$ and $\delta>0$, there is a unique solution U on M_j of equation: $\Delta_M U+Q=0$ on M_j, $U=0$ on ∂M_j and $\lim_{r\to\infty}U(r, \theta)=0$. Thus it turns out from the same argument as in Lemma 5 that there is a plane P of $\boldsymbol{R}^N$ such that

$$(1.20) \qquad \langle\!\langle T_x M, P\rangle\!\rangle \to 1 \quad \text{as} \quad x\in M_j \to \infty .$$

This shows that for large T, $\Pi_P|_{M_{j,T}}: M_{j,T}\to P_T$ defines a finite covering map from $M_{j,T}$ onto P_T (cf. Lemma 4). Moreover if we take coordinates $(v_1, \cdots, v_N)$ in $\boldsymbol{R}^N=P\oplus P^{\perp}$ and set $h_\alpha=v_\alpha|_{M_j}$ $(\alpha=3, \cdots, N)$, we see from (1.20) that $|\nabla h_\alpha|(x)$ converges to 0 as $x\in M_j\to\infty$. Hence by assumption (1.13), we obtain $|\nabla h_\alpha|\leqq C_6|x|^{-\varepsilon}$ $(\alpha=3, \cdots, N)$ and further $\langle\!\langle T_x M, P\rangle\!\rangle\geqq 1-C_7|x|^{\varepsilon}$ for some positive constants C_6 and C_7. This shows the first two assertions of the lemma. It remains to prove the last one. Let h be a harmonic function on M_j such that $|h(x)|\leqq D\log|x|+E$. Note here that M_j possesses a positive harmonic function G_∞ such that $G_\infty(x)\sim\log|x|$ at infinity. Then suitable choice of constants D' and E' makes $D'G_\infty(x)+E'+h(x)$ a positive harmonic function on M_j. Hence it follows from Theorem 5 in [3] that $h(x)$ has the form: $h(x)=D''G(x)+F(x)$, where $F(x)$ is a bounded harmonic function on M_j which has a finite limit as x goes to ∞. This proves the last assertion of the lemma.

2. Proofs of Theorem A and Theorem B.

We keep the notations in the preceding sections. We shall first prove Theorem B and then give a proof of Theorem A.

PROOF OF THEOREM B. Let M be a minimal submanifold properly immersed into $\boldsymbol{R}^N$ satisfying condition (0.1). Let us consider first the case of $m\geqq 3$. Then Lemma 5 can be applied to M. Fix an end M_j of M and realize $M_{j,T}$ as a graph over P_T: $M_{j,T}=\{(v_1, \cdots, v_m, h_{m+1}(v), \cdots, h_N(v)): v=(v_1, \cdots, v_m)\in P_T\}$. Since M is minimal, each h_α $(\alpha=m+1, \cdots, N)$ is harmonic on M. Moreover by (1.16), we have $|\nabla h_\alpha(v)|\leqq C_1|v|^{-\varepsilon}$ for a constant $C_1\geqq 0$. This implies that $|h_\alpha(v)|\leqq C_2|v|^{1-\varepsilon}$ for a constant $C_2\geqq 0$. Therefore by Lemma 5(3), we have

$$(2.1) \qquad |h_\alpha(v)-h_{\alpha,\infty}|\leqq\frac{C_3}{1+|v|^{m-2}}, \qquad |\nabla h_\alpha(v)|\leqq\frac{C_4}{1+|v|^{m-1}}$$

for some constants $h_{\alpha,\infty}$, C_3 and C_4. Now we extend h_α to a smooth function $\tilde{h}_\alpha$ on P. Then we have

$$\sum_{\beta=1}^{m}\frac{\partial^2\tilde{h}_\alpha}{\partial v_\beta^2}=\tilde{\Delta}\tilde{h}_\alpha-\sum_{\substack{\beta,\gamma=1\\ \beta\neq\gamma}}^{m}\tilde{g}^{\beta\gamma}\frac{\partial^2\tilde{h}_\alpha}{\partial v_\beta\partial v_\gamma}-\frac{1}{\sqrt{G}}\sum_{\beta,\gamma=1}^{m}\frac{\partial\sqrt{G}\,\tilde{g}^{\beta\gamma}}{\partial v_\beta}\frac{\partial\tilde{h}_\alpha}{\partial v_\gamma}.$$

Observe that by (0.1) and (2.1), $|g^{\beta\gamma}|=O|v|^{2-2m}$ $(\beta\neq\gamma)$, $|\partial^2 h_\alpha/\partial v_\beta \partial v_\gamma|=O|v|^{-1-\varepsilon/2}$ and $|G^{-1/2}(\partial\sqrt{G}\tilde{g}^{\beta\gamma}/\partial v_\gamma)|=O|v|^{-m-\varepsilon/2}$. This shows that $\sum_{\beta=1}^m \partial^2\tilde{h}_\alpha/\partial v_\beta^2= O|v|^{-2m+1-\varepsilon/2}$. Hence we see that

$$\tilde{h}_\alpha(v)=h_{\alpha,\infty}-\frac{1}{(m-1)\omega_m}\int_P \frac{Q(w)}{|v-w|^{m-1}}dw\,,$$

where ω_m denotes the volume of unit sphere in $\boldsymbol{R}^m$ and $Q(v)=\sum_{\beta=1}^m \partial^2\tilde{h}_\alpha/\partial v_\beta^2$. Noting the decay rate of $Q(v)$ and the following inequality: $|v-w|^{2-m}=|v|^{2-m} -(m-2)v\cdot w|v|^{-m}+O|v|^{-m}|w|^2$ for $|w|\leqq(1/2)|v|$, we obtain

$$h_\alpha(v)=h_{\alpha,\infty}+a_\alpha|v|^{2-m}+\sum_{\beta=1}^m C_{\alpha\beta}v_\beta|v|^{-m}+O|v|^{-m}$$

$$a_\alpha=-\frac{1}{(m-2)\omega_m}\int_P Q(w)dw$$

$$C_{\alpha\beta}=\frac{1}{\omega_m}\int_P w_\beta Q(w)dw\,.$$

This proves Theorem B in case of $m\geqq3$. Let us now suppose that $m=2$, $\varepsilon\geqq2$ and M has embedded ends. Then Lemma 7 can be applied to M. Since M has embedded ends, it is shown that for each end M_j, there exists a plane P of $\boldsymbol{R}^N$ and $M_{j,T}$ is realized as a graph over P_T: $M_{j,T}=\{(v_1, v_2, h_3(v), \cdots, h_N(v)): v=(v_1, v_2)\in P_T\}$. Moreover because of Lemma 7 and $\varepsilon\geqq2$, each h_α $(\alpha=3, \cdots, N)$ is a harmonic function on M_j of the form: $h_\alpha(x)=D\log|x|+f_\alpha(x)$, where $f_\alpha(x)$ converges to a constant as $x\in M_j\to\infty$ ($D=0$ if $\varepsilon>2$). Then it turns out from the same argument as in the case $m\geqq3$ that the h_α has the following asymptotic behaviour:

$$h_\alpha(v)=a_\alpha\log|v|+b_\alpha+(C_{\alpha,1}v_1+C_{\alpha,2}v_2)|v|^{-2}+O|v|^{-1-\delta} \qquad (0<\delta\leqq1)\,.$$

We remark here that if $\varepsilon>2$, we have $a_\alpha=0$ and $\delta=1$; if $N=3$, we have $\delta=1$ (cf. [**15**: Proposition 3] for the case of $N=3$). This completes the proof of Theorem B.

PROOF OF THEOREM A. Let M be a connected, complete minimal submanifold properly immersed into $\boldsymbol{R}^N$ satisfying condition (0.1). The first assertion is an immediate consequence of Theorem B in the case of $m\geqq3$. We consider the case $m=2$ and suppose $\varepsilon\geqq2$ and M has one end. Then by Lemma 7, there is a plane of $\boldsymbol{R}^N$ such that the restriction h_α of each component v_α of coordinates $(v_3, \cdots, v_N)$ in $P^\perp$ has the form: $h_\alpha(x)=D\log|x|+E$ $(\alpha=3, \cdots, N)$. Since M has one end, the maximum principle says that each h_α must be constant, that is, M must be a plane. Now we shall prove the second assertion. Suppose $m=N-1$, $2+\varepsilon>2m$ and M is embedded. Then by Lemma 5 and Lemma 7, we have a hyperplane P such that for each end M_j, $M_{j,T}$ can be realized as a graph over P_T: $M_{j,T}=\{(v, h_j(v)):v\in P_T\}$. Since $2+\varepsilon>2m$, each $h_j(x)$ tends to a constant C_j as $x\in M_j\to\infty$. We take the smallest constant, say C_1, among C_j's. Then by the maximum principle, h_1 must be bounded from below by C_1, that is, h_1-C_1 is positive on M. Then if $m\geqq 3$, $h_1(x)-C_1\geqq D_1(1+|x|^{m-2})^{-1}$; if $m=2$, $h_1(x)-C_1\geqq D_2$, where D_1 and D_2 are positive constants. This leads us to a contradiction, because of Lemma 5 and Lemma 7. The last assertion of Theorem A follows from Theorem B and Theorem 3 in [**15**]. This completes the proof of Theorem A.

3. Other results.

In this section, we shall give three results below, making use of Lemma 4 and Lemma 5 in Section 1.

Initially, we have the following

THEOREM 1. *Let M be a complex submanifold immersed into $\boldsymbol{C}^N$. Suppose that the induced metric on M is complete and that there are a complex m-plane P ($m=\dim_C M$), a compact subset K of M, and a positive constant ε satisfying*

$$\langle\!\langle T_z M, P\rangle\!\rangle \geqq \varepsilon > 0$$

for any $z\in M\setminus K$. Then M is a complex m-plane if $m\geqq 2$, or $m=1$ and K is empty.

PROOF. The same notations will be used as in Lemma 4. Consider first the case $m\geqq 2$. Then Lemma 4 (2) shows that M_T is a graph of some holomorphic functions $\{h_1, \cdots, h_{N-m}\}$ defined on P_T. Therefore the Hartogs extension theorem tells us that each h_i has a unique extension, denoted by the same letter h_i, to P. Moreover, it follows from (1.9) that each h_i is a constant or a polynomial of degree 1. This proves that M is a complex m-plane of $\boldsymbol{C}^N$. The same proof is available for the case $m=1$ and K is empty, by Lemma 4 (3). This concludes the proof of Theorem 1.

The following theorem is an immediate consequence of Theorem 7 in [**6**], Lemma 3 and Lemma 4.

THEOREM 2. *Let M be a complete minimal submanifold of dimension m immersed into $\boldsymbol{R}^N$. Then M is an m-plane of $\boldsymbol{R}^N$, provided that there are an m-plane P of $\boldsymbol{R}^N$ and a positive constant ε satisfying*

$$\langle T_x M, P\rangle \geqq \varepsilon \quad \textit{for every} \quad x\in M,$$

$$\varepsilon \geqq \cos^k\left(\frac{\pi}{2}\sqrt{\delta k}\right), \qquad k=\min\{m, N-m\}, \qquad \delta=\begin{cases} 1 & \textit{if} \quad k=1 \\ 2 & \textit{if} \quad k\geqq 2. \end{cases}$$

Finally, let us consider an m-dimensional, connected noncompact Riemannian submanifold M properly immersed into $\boldsymbol{R}^N$. Assume that M satisfies conditions (1.13) and (1.14) and furthermore the constant ε of (1.13) is greater than 2. We first consider the case of $m \geqq 3$. Then applying Lemma 5 to M, we see that for each end M_j, there is an m-plane P such that for some $T \geqq 0$, $M_{j,T} = \{x \in M_j : \mathrm{dis}_{\boldsymbol{R}^N}(x, P^{\perp}) \geqq T\}$ can be realized as a graph of some smooth functions $\{h_{m+1}, \cdots, h_N\}$ on $P_T = \{v \in P : |v| \geqq T\}$. Moreover the $h_\alpha(v)$ tends to a constant $h_{\alpha,\infty}$ as $|v| \to \infty$. In fact, since h_α satisfies

$$\Delta_M h_\alpha = \langle H_M, (\bar{\nabla} h)^{\perp} \rangle \leqq |H_M| \leqq \frac{A''}{1+|x|^{1+\varepsilon/2}}$$

for a constant A'', there exists a unique solution U_α on M_j of equation: $\Delta_M U_\alpha = \Delta_M h_\alpha$ and $U_\alpha(x)$ goes to 0 as $|x| \to \infty$ (cf. Lemma 2(3)). This implies that $U_\alpha - h_\alpha$ is a bounded harmonic function on M_j and hence it tends to a constant. Suppose now that M has one end. (Note that M has one end if M has nonnegative Ricci curvature, since M has no nonconstant bounded harmonic functions (cf. [**19**] and Corollary 3).) Then if M is not an m-plane, we can find a (sufficiently large) ball B of $\boldsymbol{R}^N$ such that for some point x of M, M is tangent to ∂B from the inside of B. Hence the second fundamental form α_M satisfies

$$\langle \alpha_M(X, X), \nu \rangle \geqq C \langle X, X \rangle \tag{3.1}$$

for any $X \in T_x M$, where ν is the outer unit normal to ∂B and C is a positive constant. In the case of $m=2$, we can apply Lemma 7 to M and obtain (3.1). Thus we have the following

THEOREM 3. *Let M be an m-dimensional, connected noncompact Riemannian submanifold properly immersed into $\boldsymbol{R}^N$ satisfying* (1.13) *with* $\varepsilon > 2$ *and* (1.14). *Then:*

(I) *M is a hyperplane if $m = N-1$ and the sectional curvature is nonnegative.*

(II) *M is an m-plane if M has one end and if, for any point x of M, there is a subspace T of $T_x M$ such that* $\dim T > N-m$ *and the sectional curvature for any plane of T is nonnegative.*

The first assertion follows from the above argument and the results of [**5**] and [**14**]. The second one is a consequence of the above argument and Otsuki's lemma (cf. [**11**: p. 28]).

References

[1] E. Bombieri and E. Giusti, Harnack's inequality for elliptic differential equations on minimal surfaces, Invent. Math., **15** (1972), 24-46.

[2] D. Fischer-Corbrie, Some rigidity theorems for minimal submanifolds of the sphere Acta Math., **145** (1980), 29-46.

[3] D. Gilbarg and J. Serrin, On isolated singularities of second order elliptic differential equations, J. Analyse Math., **4** (1955/56), 309-340.
[4] R.E. Greene and H. Wu, Gap theorems for noncompact Riemannian manifolds, Duke Math. J., **44** (1982), 731-756.
[5] P. Hartman and L. Nirenberg, On spherical image maps whose Jacobian do not change sign, Amer. J. Math., **81** (1959), 901-920.
[6] S. Hildebrandt, J. Jost and K.-O. Widman, Harmonic mappings and minimal submanifolds, Invent. Math., **62** (1980), 269-298.
[7] A. Kasue, On Riemannian manifolds admitting certain strictly convex functions, Osaka J. Math., **18** (1981), 577-582.
[8] A. Kasue, A Laplacian comparison theorem and function theoretic properties of a complete Riemannian manifold, Japan. J. Math., **18** (1982), 309-341.
[9] A. Kasue, Applications of Laplacian and Hessian comparison theorems, Geometry of Geodesics and Related Topics, Adv. Studies in Pure Math., **3**, Kinokuniya, Tokyo, 1984, pp. 333-386.
[10] A. Kasue, Estimates for solutions of Poisson equations and their applications to submanifolds, Geometry of Submanifolds, Lecture Notes in Math., **1090**, Springer, pp. 1-14.
[11] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. II, Interscience, New York, 1969.
[12] N. Mok, Y.-T. Siu and S.-T. Yau, The Poincaré-Lelong equation on complete Kähler manifolds, Compositio Math., **44** (1981), 183-218.
[13] J. Moser, Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math., **14** (1961), 577-591.
[14] S. Sacksteder, On hypersurfaces with nonnegative sectional curvatures, Amer. J. Math., **82** (1960), 609-630.
[15] R. Schoen, Uniqueness, symmetry, and embeddedness of minimal surfaces, J. Differential Geometry, **18** (1983), 791-809.
[16] W. Stoll, The growth of the area of a transcendental analytic set, I, II, Math. Ann., **156** (1964), 47-78 and 144-170.
[17] A. Vitter, On the curvature of complex hypersurfaces, Indiana Univ. Math. J., **23** (1974), 813-826.
[18] K.-O. Widman, Inequalities for Green functions of second order elliptic operators, Report nr 8-1972, Linköping University.
[19] S.-T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math., **28** (1975), 201-228.
[20] M. T. Anderson, Compactification of a minimal submanifold in Euclidean space by the Gauss map, preprint.
[21] A. Kasue and K. Sugahara, Gap theorems for certain submanifolds of Euclidean space and Hyperbolic space form, preprint; ——— II, to appear in the proceeding of the Taniguchi Symposium "Curvature and Topology of Riemannian manifolds".
[22] A. Kasue, On manifolds of asymptotically nonnegative curvature, preprint.

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Added in proof. During the submission of this paper to the journal, the author received a preprint [**20**] in which Anderson investigated complete minimal submanifolds in $\boldsymbol{R}^n$ of finite total scalar curvature. Especially, the proof of his main theorem there tells us that for a complete minimal submanifold M of dimension $m \geqq 3$ immersed into $\boldsymbol{R}^n$, the immersion is proper and the second fundamental form α_M satisfies: $|\alpha_M| \leqq c/|x|^m$ for some positive constant c, if the total scalar curvature $\int_M |\alpha_M|^m$ is finite. It is easy to see that the total scalar curvature is finite if the immersion is proper and $|\alpha_M| \leqq c/|x|^{1+\varepsilon}$ for some constants $c>0$ and $\varepsilon>0$. Moreover some improvements of Theorem A (I) and Theorem 3 have been given in [**21**].

Finally, manifolds as in Theorem A and Theorem 3 belong to a class of Riemannian manifolds of asymptotically nonnegative curvature. In [**22**], several results on such manifolds, including some gap theorems similar to the results in this paper, have been proved.