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A theorem on $P_\kappa(\lambda)$

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1. Introduction.

Let κ be a regular uncountable cardinal. Consider the set

$$S(\kappa, \kappa^+)=\{x\in P_\kappa(\kappa^+):|x|=|x\cap\kappa|^+\}.$$

A question has been raised, notably in [1], [2] and [5], whether $S(\kappa, \kappa^+)$ can be stationary.

By a result of Baumgartner, cf. [1], if $S(\kappa, \kappa^+)$ is stationary then κ is (weakly) inaccessible and $0^\#$ exists. We show (in Corollary 4.3) that if $S(\kappa, \kappa^+)$ is stationary then the function $f(\xi)=\xi^+$ on κ has the Galvin-Hajnal norm κ^+.

2. Some facts about $P_\kappa(\lambda)$.

Throughout this paper, κ is a fixed regular uncountable cardinal; all other greek letters denote ordinal numbers. If x is a set of ordinals, then

$$\bar{x}=\text{the order type of } x;$$

as usual, $|x|$ is the cardinality of x. For $\lambda\geq\kappa$,

$$P_\kappa(\lambda)=\{x\subset\lambda:|x|<\kappa\}.$$

2.1. DEFINITION ([4]). A set $C\subseteq P_\kappa(\lambda)$ is *closed* if whenever $D\subseteq C$ is a chain under inclusion with $|D|<\kappa$, then $\bigcup D\in C$. C is *unbounded* if for every $x\in P_\kappa(\lambda)$ there is a $y\in C$ with $x\subseteq y$. C is a *club* if it is closed and unbounded. A set $S\subseteq P_\kappa(\lambda)$ is *stationary* if $S\cap C\neq\emptyset$ for all clubs C.

2.2. PROPOSITION. *A subset C of κ is a club iff C is a club in $P_\kappa(\kappa)$; also, κ is a club in $P_\kappa(\kappa)$.*

2.3. PROPOSITION. *Let $\kappa\leq\alpha\leq\beta$. If C is a club in $P_\kappa(\alpha)$ then the set*

$$\{x\in P_\kappa(\beta): x\cap\alpha\in C\}$$

is a club in $P_\kappa(\beta)$.

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2.4. LEMMA (Abraham). *Let* $\kappa \leq \alpha \leq \beta$. *If C is a club in* $P_\kappa(\beta)$ *then the set*

$$\{x \cap \alpha : x \in C\}$$

contains a club in $P_\kappa(\alpha)$.

2.5. COROLLARY. *Let* $\kappa \leq \alpha \leq \beta$. *A set* $S \subseteq P_\kappa(\alpha)$ *is stationary in* $P_\kappa(\alpha)$ *if and only if the set*

$$\{x \in P_\kappa(\beta) : x \cap \alpha \in S\}$$

is stationary in $P_\kappa(\beta)$.

3. Ordinal functions on $P_\kappa(\lambda)$.

3.1. DEFINITION. Let f and g be ordinal functions on $P_\kappa(\lambda)$.

$$f < g \quad \text{iff} \quad \{x : f(x) < g(x)\} \quad \text{contains a club.}$$

The λ-*norm* of f is the rank of f in the well-founded relation $<$:

$$\|f\|_\lambda = \sup\{\|g\|_\lambda + 1 : g < f\}.$$

(If $\lambda = \kappa$, $\|f\| = \|f\|_\kappa$ is the Galvin-Hajnal norm, cf. [3].)

3.2. For $\gamma < \kappa$, let c_γ be the constant function with value γ.
We have

$$\|c_\gamma\|_\lambda = \gamma.$$

For almost all x (i. e. for all x in a club $C \subseteq P_\kappa(\lambda)$),

$$\kappa_x = x \cap \kappa$$

is an ordinal, and

$$\|\kappa_x\|_\lambda = \kappa.$$

3.3. PROPOSITION. *For all* α *such that* $\kappa \leq \alpha \leq \lambda$, *and all* $x \in P_\kappa(\lambda)$, *let*

$$\alpha_x = \overline{x \cap \alpha}.$$

Then

$$\|\alpha_x\|_\lambda = \alpha.$$

(The proof is by induction.)

3.4. PROPOSITION ([2]). *An ordinal function f is canonical if* $\|f\|_\lambda \leq \|g\|_\lambda$ *implies* $f \leq g$. *For each* $\eta < \lambda^+$ *there is a canonical function* f_η *on* $P_\kappa(\lambda)$ *such that* $\|f_\eta\|_\lambda = \eta$.
(The functions in 3.2 and 3.3 are canonical, for $\eta \leq \lambda$.)

3.5. For every $\eta < \lambda^+$, if f_η is the η^{th} canonical function on $P_\kappa(\lambda)$, then

$$f_\eta(x) < \lambda_x^+.$$

It follows that

$$\|\lambda_x^+\|_\lambda \geq \lambda^+.$$

In particular,

(3.6) $$\|\xi \mapsto \xi^+\| \geq \kappa^+.$$

4. A theorem on $P_\kappa(\lambda)$.

4.1. THEOREM. *Let $\kappa \leq \alpha \leq \lambda$, and let f be an ordinal function on $P_\kappa(\alpha)$ with $\|f\|_\alpha \leq \lambda$. The following holds for almost all $x \in P_\kappa(\lambda)$:*

$$\overline{x \cap \|f\|_\alpha} \leq f(x \cap \alpha).$$

If $\alpha = \kappa$ then Theorem 1 takes the following form:

4.2. THEOREM. *If f is an ordinal function on κ, $\|f\| = \beta \leq \lambda$, then*

$$\beta_x \leq f(\kappa_x)$$

for almost all $x \in P_\kappa(\lambda)$.

4.3. COROLLARY. *If the set $S(\kappa, \kappa^+)$ is stationary, then the function $\xi \mapsto \xi^+$ on κ has norm κ^+.*

PROOF of the Corollary. Assume that $S(\kappa, \kappa^+)$ is a stationary subset of $P_\kappa(\kappa^+)$. If $\|\xi^+\| > \kappa^+$ then there is a function f on κ such that $f(\xi) < \xi^+$ for almost all ξ, and $\|f\| = \kappa^+$. Thus it suffices to prove that for any function f on κ such that $f(\xi) \leq \xi^+$, if $\|f\| = \kappa^+$ then $\{\xi < \kappa : f(\xi) = \xi^+\}$ is stationary. Let f be such a function.

By Theorem 4.2 (with $\beta = \lambda = \kappa^+$) the set

$$\{x \in P_\kappa(\kappa^+) : \bar{x} \leq f(\kappa_x)\}$$

contains a club. Since $S(\kappa, \kappa^+)$ is stationary, so is the set

$$\{x \in P_\kappa(\kappa^+) : f(\kappa_x) = \kappa_x^+\}.$$

By Corollary 2.5, $\{\xi < \kappa : f(\xi) = \xi^+\}$ is a stationary subset of κ. □

PROOF of Theorem 4.1. Let $\kappa \leq \alpha \leq \lambda$. We prove the theorem by induction on $\|f\|_\alpha$.

Assume that f is an ordinal function on $P_\kappa(\alpha)$ such that the theorem fails; let $\beta = \|f\|_\alpha \leq \lambda$. Hence the set

$$X = \{x \in P_\kappa(\lambda) : \overline{x \cap \beta} > f(x \cap \alpha)\}$$

is stationary. By normality, there is a stationary set $Y \subseteq X$ and some $\gamma < \beta$ such that for every $x \in Y$,

$$\gamma = \text{the } f(x\cap\alpha)^{\text{th}} \text{ element of } x\cap\beta\,.$$

Let $g<f$ be a function on $P_\kappa(\alpha)$ such that $\|g\|_\alpha=\gamma$.

By the induction hypothesis,

$$\overline{\overline{x\cap\gamma}} \le g(x\cap\alpha)$$

holds for almost all $x\in P_\kappa(\lambda)$.

On the other hand, the set

$$\{x\in P_\kappa(\lambda) : \overline{\overline{x\cap\gamma}}=f(x\cap\alpha)\}$$

is stationary. On the other hand, the set

$$\{x\in P_\kappa(\alpha) : g(x)<f(x)\}$$

contains a club in $P_\kappa(\alpha)$, and so

$$\{x\in P_\kappa(\lambda) : g(x\cap\alpha)<f(x\cap\alpha)\}$$

contains a club in $P_\kappa(\lambda)$; hence for almost all $x\in P_\kappa(\lambda)$,

$$\overline{\overline{x\cap\gamma}} < f(x\cap\alpha)\,.$$

A contradiction. □

5. Final remarks.

It is well known that stationary subsets of κ remain stationary in generic extensions obtained by $<\kappa$-closed forcing. This is not the case for stationary subsets of $P_\kappa(\lambda)$: In [1], Baldwin proves that if κ is weakly inaccessible and $\lambda>\kappa$ is Ramsey, then the set

$$S=\{x\in P_\kappa(\lambda) : |x|\ge\kappa_x^{++}\}$$

is stationary. If λ is changed to κ^+ by the Lévy collapse, then S is no longer stationary, even though the forcing is $<\kappa$-closed. By Theorem 4.2, $|x|\le\kappa_x^+$ for almost all $x\in P_\kappa(\kappa^+)$.

We conclude by stating two open problems: What is the consistency strength of the following?

(5.1) $S(\kappa, \kappa^+)$ is stationary

and

(5.2) $\|\xi\mapsto\xi^+\|=\kappa^+$.

(5.1) holds if κ is κ^+-supercompact (cf. [1]), and implies (5.2) by 4.3.

6. Added in proof (May 1986).

Hans-Dieter Donder, Aki Kanamori and Jean-Pierre Levinski have kindly brought to my attention the following partial answers to the open problems (5.1) and (5.2):

6.1 (Baumgartner [6]). If κ is ineffable then the function $\xi\mapsto\xi^+$ has norm κ^+. If $V=L$ then $\|\xi^+\|=\kappa^+$ if and only if κ is ineffable.

6.2 (Donder). If κ is inaccessible but not Mahlo and if $\|\xi^+\|=\kappa^+$ then $L[U]$ exists.

6.3 (Levinski [7]). If $S(\kappa, \kappa^+)$ is stationary then 0^+ exists.

References

[1] S. Baldwin, The consistency strength of certain stationary subsets of $P_\kappa(\lambda)$, Proc. Amer. Math. Soc., **92** (1984), 90-92.

[2] S. Baldwin, Generalizing the Mahlo hierarchy, with applications to the Mitchell models, Ann. Pure Appl. Logic, to appear.

[3] F. Galvin and A. Hajnal, Inequalities for cardinal powers, Ann. of Math., **101** (1975), 491-498.

[4] T. Jech, Some combinatorial problems concerning uncountable cardinals, Ann. Math. Logic, **5** (1973), 165-198.

[5] W. Zwicker, Partial results on splitting stationary subsets of $P_\kappa(\lambda)$, to appear.

[6] J. Baumgartner, Ineffability properties of cardinals II, Logic, Found. of Math. and Comp. Theory, D. Reidel, Dordrecht and Boston, 1977, pp. 87-106.

[7] J.-P. Levinski, Instances of the conjecture of Chang, Israel J. Math., **48** (1984), 225-243.

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