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Additivity of Jordan $*$-maps between operator algebras

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The addition and Jordan product in operator algebras seem to be closely related. Our aim in this paper is to present a positive answer to the following problem.

Let M be a unital C^*-algebra and N be an associative $*$-algebra. A map ϕ is said to be a Jordan $*$-map from M to N, if ϕ satisfies the following conditions (i)~(iii) [2].

(i) $\phi(x\circ y)=\phi(x)\circ\phi(y)$ for all x and y in M, where $x\circ y=(1/2)(xy+yx)$.

(ii) $\phi(x^*)=\phi(x)^*$ for all $x\in M$.

(iii) ϕ is bijective.

Can we conclude that ϕ is additive?

Unfortunately, the answer to this problem is negative in the one dimensional case, even if ϕ is uniformly continuous, as the following example shows. Let $\phi(\alpha)=\alpha|\alpha|$ for $\alpha\in\boldsymbol{C}$ (the complex number field). Then ϕ is a uniformly continuous Jordan $*$-map from $\boldsymbol{C}$ to $\boldsymbol{C}$ and it is not additive. If, however, M has a system of $n\times n$ matrix units for some $n\geqq 2$, we obtain the following:

THEOREM. *Let M be a C^*-algebra, N be an associative $*$-algebra and ϕ be a Jordan $*$-map from M to N. Suppose that M has a system of $n\times n$ matrix units for some $n\geqq 2$. Then ϕ is additive.*

In [2], additivity of a Jordan $*$-map on an AW^*-algebra with no abelian direct summand was established under the hypothesis of continuity. S. Sakai conjectured that the hypothesis of continuity is redundant (see [2]). This follows from our theorem:

COROLLARY. *Let M be a von Neumann algebra (or more generally an AW^*-algebra) which has no abelian direct summand, let N be a C^*-algebra and let ϕ be a Jordan $*$-map from M to N. Then ϕ is additive. Moreover, there exist central projections e_1, e_2, e_3, e_4 in M such that ϕ is a linear $*$-ring isomorphism on Me_1, ϕ is a linear $*$-ring antiisomorphism on Me_2, ϕ is a conjugate linear $*$-ring isomorphism on Me_3 and ϕ is a conjugate linear $*$-ring antiisomorphism on Me_4.*

Throughout this paper, we always assume that M is a unital C^*-algebra, N

is an associative $*$-algebra, ϕ satisfies the conditions (i)$\sim$(iii) and M has a system of $n\times n$ matrix units for some $n\geqq 2$.

1. Preliminaries.

An element e is called a projection if it is idempotent ($e^2=e$) and selfadjoint ($e^*=e$). The relation $e=ef$ defines a partial ordering of projections, denoted $e\leqq f$. Projections e and f will be said to be orthogonal if $ef=0$. We shall break up the proof of the theorem into a sequence of lemmas.

LEMMA 1 ([**2**, Lemma 1.2]). *Let e and f be projections in M. Then*
(i) *$ef=0$ if and only if $e\circ f=0$,*
(ii) *$e\leqq f$ if and only if $e=e\circ f$.*

Thus ϕ is an order isomorphism from the partially ordered set M_p of the projections in M to N_p in N which preserves orthogonality. So $\phi(1)=1$ and $\phi(0)=0$ follow.

LEMMA 2 ([**3**]). *Let e and f be projections of M. If $ef=0$, then $\phi(\alpha e+\beta f)=\phi(\alpha e)+\phi(\beta f)$ for all α, $\beta\in\boldsymbol{C}$; in particular, $\phi(e+f)=\phi(e)+\phi(f)$.*

In fact, if $ef=0$, then, there exists the least upper bound $e\vee f$ in M_p, and $e\vee f=e+f$. Since $\phi|M_p$ is an order isomorphism and preserves orthogonality, there exists $\phi(e)\vee\phi(f)$ in N_p and $\phi(e\vee f)=\phi(e)\vee\phi(f)$. So $\phi(e+f)=\phi(e)+\phi(f)$. Put $a=\alpha e+\beta f$ for arbitrary α, $\beta\in\boldsymbol{C}$. Then

$$\begin{aligned}\phi(a) &= \phi(a\circ(e+f))=\phi(a)\circ\phi(e+f) = \phi(a)\circ(\phi(e)+\phi(f)) \\ &= \phi(a)\circ\phi(e)+\phi(a)\circ\phi(f) = \phi(\alpha e)+\phi(\beta f).\end{aligned}$$

LEMMA 3 ([**2**, Lemma 2.1]). *$\phi|\boldsymbol{C}\cdot 1$ is additive.*

Let $\{e_{ij}\}$ be a system of $n\times n$ matrix units in M with $n\geqq 2$. Put $e=e_{ii}$, $v=e_{ij}(i\neq j)$, $p=(1/2)(e+v^*)(e+v)$ and $q=(1/2)(e-v^*)(e-v)$. Then p and q are orthogonal projections in M. Since $\phi(e)\phi(x)\phi(e)=\phi(exe)$ (note that $exe=((2e-1)\circ x)\circ e$; see [**2**, Lemma 1.6]) and by Lemma 2,

$$\begin{aligned}\phi((\alpha+\beta)\cdot 1)\circ\phi(e_{ii}) &= \phi((\alpha+\beta)\cdot 1)\circ\phi(e) = \phi(e(2\alpha p+2\beta q)e) \\ &= \phi(e)\phi(2\alpha p+2\beta q)\phi(e) = \phi(e)(\phi(2\alpha p)+\phi(2\beta q))\phi(e) \\ &= (\phi(\alpha\cdot 1)+\phi(\beta\cdot 1))\circ\phi(e) = (\phi(\alpha\cdot 1)+\phi(\beta\cdot 1))\circ\phi(e_{ii})\end{aligned}$$

for each i. So our Lemma 3 follows.

COROLLARY 4. (i) *$\phi(-x)=-\phi(x)$ for all $x\in M$.* (ii) *$\phi(\rho x)=\rho\phi(x)$ for all $x\in M$ and all rational number ρ.*

Since $0=\phi(0)=\phi(1+(-1))=\phi(1)+\phi(-1)=1+\phi(-1)$, by Lemma 3, $\phi(-x)=\phi(-1)\circ\phi(x)=-\phi(x)$. For arbitrary integers m $(m\neq 0)$ and n, $m\phi((n/m)x)=\phi(nx)=n\phi(x)$. So $\phi((n/m)x)=(n/m)\phi(x)$.

LEMMA 5 ([**2**]). *Let* $\{e_i : i=1, 2, \cdots, n\}$ *be an orthogonal family of projections in* M *such that* $\sum_i e_i=1$. *Then*

$$\phi(x)=\sum_i \phi(e_i)\phi(x)\phi(e_i)+2\sum_{i<j}\{\phi(e_i), \phi(x), \phi(e_j)\}$$

where $\{x, y, z\}=(1/2)(xyz+zyx)$.

Since $\{\phi(e_i) : i=1, 2, \cdots, n\}$ is an orthogonal family of projections in N such that $\sum_i \phi(e_i)=1$,

$$\begin{aligned}\phi(x)&=\sum_{i,j}\phi(e_i)\phi(x)\phi(e_j)\\&=\sum_i \phi(e_i)\phi(x)\phi(e_i)+2\sum_{i<j}\{\phi(e_i), \phi(x), \phi(e_j)\}.\end{aligned}$$

2. Additivity of Jordan ∗-maps.

LEMMA 6. *Let* e *and* f *be projections in* M. *Then*

$$\phi(\alpha\cdot 1+\beta e+\gamma f)=\phi(\alpha\cdot 1)+\phi(\beta e)+\phi(\gamma f)$$

for all $\alpha, \beta, \gamma\in \boldsymbol{C}$.

Put

$$x=\alpha\cdot 1+\beta e+\gamma f,\quad y=\phi(\alpha\cdot 1)+\phi(\beta e)+\phi(\gamma f)\quad \text{and}\quad e'=1-e.$$

Since $\{\phi(e), \phi(x), \phi(e')\}=\phi(\{e, x, e'\})$ ([**2**, Corollary 2.2]; note that $2(e\circ x)\circ e'=\{e, x, e'\}$), it follows that

$$\begin{aligned}\phi(e)\phi(x)\phi(e)&=\phi(exe)=\phi(e((\alpha+\beta)\cdot 1+\gamma f)e)\\&=\phi(e)\phi((\alpha+\beta+\gamma)f+(\alpha+\beta)(1-f))\phi(e)\\&=\phi(e)(\phi((\alpha+\beta+\gamma)f)+\phi((\alpha+\beta)(1-f)))\phi(e)\\&=\phi(e)(\phi(\alpha f)+\phi(\beta f)+\phi(\gamma f)+\phi(\alpha(1-f))+\phi(\beta(1-f)))\phi(e)\\&=\phi(e)(\phi(\alpha\cdot 1)+\phi(\beta\cdot 1)+\phi(\gamma f))\phi(e)=\phi(e)y\phi(e),\end{aligned}$$

$$\begin{aligned}\phi(e')\phi(x)\phi(e')&=\phi(e'xe')=\phi(e'(\alpha\cdot 1+\gamma f)e')\\&=\phi(e')(\phi((\alpha+\gamma)f)+\phi(\alpha(1-f)))\phi(e')\\&=\phi(e')(\phi(\alpha f)+\phi(\gamma f)+\phi(\alpha(1-f)))\phi(e')\\&=\phi(e')(\phi(\alpha\cdot 1)+\phi(\gamma f))\phi(e')=\phi(e')y\phi(e')\quad \text{and}\end{aligned}$$

$$\begin{aligned}\{\phi(e), \phi(x), \phi(e')\}&=\phi(\{e, x, e'\})=\phi(\{e, \gamma f, e'\})\\&=\{\phi(e), \phi(\gamma f), \phi(e')\}=\{\phi(e), y, \phi(e')\}.\end{aligned}$$

Therefore

$$\phi(x) = \phi(e)\phi(x)\phi(e)+\phi(e')\phi(x)\phi(e')+2\{\phi(e), \phi(x), \phi(e')\}$$
$$= \phi(e)y\phi(e)+\phi(e')y\phi(e')+2\{\phi(e), y, \phi(e')\} = y$$

by Lemma 5.

LEMMA 7. *Let u and v be symmetries (selfadjoint unitaries) in M. Then $\phi(\alpha u+\beta v)=\phi(\alpha u)+\phi(\beta v)$ for all $\alpha, \beta\in \boldsymbol{C}$.*

Put $e=(1/2)(1+u)$ (resp. $f=(1/2)(1+v)$). Then e (resp. f) is a projection in M. Hence

$$\begin{aligned}\phi(\alpha u+\beta v) &= \phi(2\alpha e+2\beta f-(\alpha+\beta)\cdot 1)\\ &= \phi(2\alpha e)+\phi(2\beta f)-\phi((\alpha+\beta)\cdot 1)\\ &= \phi(2\alpha e)+\phi(2\beta f)-(\phi(\alpha\cdot 1)+\phi(\beta\cdot 1))\\ &= \phi(\alpha\cdot 1)\circ(2\phi(e)-1)+\phi(\beta\cdot 1)\circ(2\phi(f)-1)\end{aligned}$$

by Lemma 6, Corollary 4 and Lemma 3. On the other hand,

$$\begin{aligned}2\phi(e)-1 &= \phi(e)-(1-\phi(e)) = \phi(e)-\phi(1-e)\\ &= \phi(e-(1-e)) = \phi(u)\end{aligned}$$

and similarly

$$2\phi(f)-1 = \phi(v).$$

Therefore

$$\phi(\alpha u+\beta v) = \phi(\alpha u)+\phi(\beta v).$$

LEMMA 8. *Let h and k be selfadjoint elements in M. Then*

$$\phi(\alpha h+\beta k) = \phi(\alpha h)+\phi(\beta k)$$

for all $\alpha, \beta\in \boldsymbol{C}$.

In fact, let $\{e_i\}$ be the diagonal projections of the given system of matrix units $\{e_{ij}\}$ of M. Put $\gamma=\|h\|+\|k\|$, $h_1=\gamma^{-1}h$ and $k_1=\gamma^{-1}k$. Then there exist symmetries u_i, u_{ij} (resp. v_i, v_{ij}) such that $e_ih_1e_i=e_iu_ie_i$ and $\{e_i, h_1, e_j\}=\{e_i, u_{ij}, e_j\}$ $(i\neq j)$ (resp. $e_ik_1e_i=e_iv_ie_i$ and $\{e_i, k_1, e_j\}=\{e_i, v_{ij}, e_j\}$ $(i\neq j)$) (see the proof of Lemma 1 in [1] and Lemma 3.5 in [2]; in fact, let

$$u_i = e_ih_1e_i+(e_i-e_ih_1e_ih_1e_i)^{1/2}e_{ij}+e_{ji}(e_i-e_ih_1e_ih_1e_i)^{1/2}$$
$$-e_{ji}h_1e_{ij}+1-e_i-e_j \quad (i\neq j)$$

and let

$$u_{ij} = e_ih_1e_j+e_jh_1e_i+(e_i-e_ih_1e_jh_1e_i)^{1/2}$$
$$-(e_j-e_jh_1e_ih_1e_j)^{1/2}+1-e_i-e_j \quad (i\neq j),$$

then u_i and u_{ij} enjoy all the requirements). Put

$$x=\alpha h+\beta k,\quad y=\phi(\alpha h)+\phi(\beta k),\quad w_i=\alpha u_i+\beta v_i,\quad w_{ij}=\alpha u_{ij}+\beta v_{ij},$$
$$z_i=\phi(\alpha u_i)+\phi(\beta v_i) \text{ and } z_{ij}=\phi(\alpha u_{ij})+\phi(\beta v_{ij}).$$

Then $\phi(w_i)=z_i$ and $\phi(w_{ij})=z_{ij}$ by Lemma 7. Hence

$$\begin{aligned}
\phi(x) &= \sum_i \phi(e_i)\phi(x)\phi(e_i)+2\sum_{i<j}\{\phi(e_i),\ \phi(x),\ \phi(e_j)\} \\
&= \sum_i \phi(e_i x e_i)+2\sum_{i<j}\phi(\{e_i,\ x,\ e_j\}) \\
&= \phi(\gamma\cdot 1)\circ(\sum_i \phi(e_i w_i e_i)+2\sum_{i<j}\phi(\{e_i,\ w_{ij},\ e_j\})) \\
&= \phi(\gamma\cdot 1)\circ(\sum_i \phi(e_i)z_i\phi(e_i)+2\sum_{i<j}\{\phi(e_i),\ z_{ij},\ \phi(e_j)\}) \\
&= \phi(\gamma\cdot 1)\circ(\sum_i \phi(e_i)(\phi(\alpha u_i)+\phi(\beta v_i))\phi(e_i) \\
&\qquad +2\sum_{i<j}\{\phi(e_i),\ \phi(\alpha u_{ij})+\phi(\beta v_{ij}),\ \phi(e_j)\}) \\
&= \phi(\gamma\cdot 1)\circ(\sum_i \phi(e_i(\alpha h_1)e_i)+\sum_i \phi(e_i(\beta k_1)e_i) \\
&\qquad +2\sum_{i<j}\phi(\{e_i,\ \alpha h_1,\ e_j\})+2\sum_{i<j}\phi(\{e_i,\ \beta k_1,\ e_j\})) \\
&= \phi(\gamma\cdot 1)\circ(\sum_i \phi(e_i)(\phi(\alpha h_1)+\phi(\beta k_1))\phi(e_i) \\
&\qquad +2\sum_{i<j}\{\phi(e_i),\ \phi(\alpha h_1)+\phi(\beta k_1),\ \phi(e_j)\}) \\
&= \sum_i \phi(e_i)y\phi(e_i)+2\sum_{i<j}\{\phi(e_i),\ y,\ \phi(e_j)\} \\
&= y.
\end{aligned}$$

PROOF OF THEOREM. Now we come to prove our theorem. Let h_j, k_j ($j=1, 2$) be selfadjoint elements in M such that $x=h_1+ih_2$, $y=k_1+ik_2$ ($i^2=-1$). By Lemma 8,

$$\begin{aligned}
\phi(x+y) &= \phi((h_1+k_1)+i(h_2+k_2)) \\
&= \phi(h_1+k_1)+\phi(i\cdot 1)\circ\phi(h_2+k_2) \\
&= (\phi(h_1)+\phi(k_1))+\phi(i\cdot 1)\circ(\phi(h_2)+\phi(k_2)) \\
&= (\phi(h_1)+\phi(ih_2))+(\phi(k_1)+\phi(ik_2)) \\
&= \phi(x)+\phi(y).
\end{aligned}$$

This completes the proof.

PROOF OF COROLLARY. We need the following lemma which is well known to specialists. But for the sake of completeness we give here a proof.

LEMMA 9. *Keep the notations and assumptions in Corollary in mind, let a and b be any mutually commuting elements in M, then* $\phi(a)\phi(b)=\phi(b)\phi(a)$. *In particular,* ϕ*(the center of M) = the center of N.*

In fact, if $ab=ba$, then $a\circ(b\circ x)=b\circ(a\circ x)$ for all $x\in M$, and so $\phi(a)\circ(\phi(b)\circ\phi(x))=\phi(b)\circ(\phi(a)\circ\phi(x))$ for all $x\in M$. So $\phi(a)\phi(b)-\phi(b)\phi(a)$ is a central element in N, which implies that $\phi(a)\phi(b)-\phi(b)\phi(a)$ commutes with $\phi(a)$. Hence, by a theorem of Kleinecke ([**4**]), $z=\phi(a)\phi(b)-\phi(b)\phi(a)$ is a normal quasi-nilpotent element in N, and so, $z=0$, that is, $\phi(a)\phi(b)=\phi(b)\phi(a)$.

Let $\{p_i\}$ be a family of central orthogonal projections in M such that $\vee p_i=1$ where Mp_1 has no finite type I direct summand and $Mp_i\ (i\geqq 2)$ is homogeneous of type $\mathrm{I}_{n_i}\ (n_i\geqq 2)$. Then $\phi|Mp_i$ is a Jordan $*$-map from Mp_i to $N\phi(p_i)$, because $\phi(p_i)$ is a central projection in N for each i by Lemma 9. By our theorem, it follows, for each i, that

$$\begin{aligned}\phi(x+y)\phi(p_i) &= \phi(xp_i+yp_i) = \phi(x)\phi(p_i)+\phi(y)\phi(p_i)\\ &= (\phi(x)+\phi(y))\phi(p_i)\end{aligned}$$

for every pair x and y in M, because each Mp_i has a system of $n_i\times n_i$ matrix units for some integer n_i with $n_i\geqq 2$. Let $a=\phi(x+y)-\phi(x)-\phi(y)\ (\in N)$ and let b be the inverse image of a under ϕ in M. Then $\phi(bp_i)=\phi(b)\phi(p_i)=a\phi(p_i)=0$ for each i. The injectivity of ϕ tells us that $bp_i=0$ for each i. Since M is an AW^*-algebra, this implies that $b=0$ and so $a=0$, that is, $\phi(x+y)=\phi(x)+\phi(y)$ for all x and y in M. The rest of the proof is the same as in [**2**, Theorem 3.10].

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