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Nonstationary free boundary problem for perfect fluid with surface tension

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§1. Introduction.

We consider a free boundary problem for a nonstationary motion of perfect fluid, which is a model for a flow around a celestial body. We consider only the flow in the plane through the equator. Hence the flow is regarded as a two-dimensional one. For simplicity we assume that the fluid is incompressible, inviscid and irrotational. We also assume that the equator Γ is a unit circle in $\boldsymbol{R}^2$. Self-gravitation of the fluid is neglected and only the gravitation due to the inside of Γ is taken into account. We then look for a time-dependent closed Jordan curve $\gamma(t)$ outside Γ, which, together with Γ, encloses the fluid (see Fig. I) and at the same time look for a stream function V and the pressure P of the fluid. The curve $\gamma(t)$ is assumed to be represented as

$$\gamma(t)=\{(r, \theta)\in\boldsymbol{R}^2 ; r=\gamma(t, \theta), 0\leqq\theta<2\pi\},$$

where $\gamma(\cdot, \cdot)$ is a positive function satisfying $\gamma(t, \theta)>1$. Then the problem to be considered here is formulated as follows.

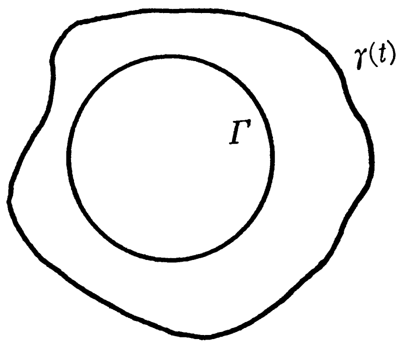

Figure I.

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PROBLEM (NS). Find functions $V(t, r, \theta)$, $P(t, r, \theta)$ and a time-dependent closed Jordan curve $\gamma(t)$ $(0\leqq t\leqq T)$ such that

$$\Delta V(t, r, \theta)=0 \quad \text{in} \quad Q_{T,\gamma}=\bigcup_{0<t<T}\Omega_{\gamma(t)}, \tag{1.1}$$

$$V(t, 1, \theta)=0 \quad \text{for} \quad 0<t<T,\ 0<\theta<2\pi, \tag{1.2}$$

$$\frac{\partial}{\partial\theta}V(t, \gamma(t, \theta), \theta)=\gamma(t, \theta)\frac{\partial}{\partial t}\gamma(t, \theta) \quad (0<t<T), \tag{1.3}$$

$$\frac{1}{r}\frac{\partial^2 V}{\partial t\partial\theta}+\frac{\partial}{\partial r}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \quad \text{in} \quad Q_{T,\gamma}, \tag{1.4}$$

$$-\frac{\partial^2 V}{\partial t\partial r}+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \quad \text{in} \quad Q_{T,\gamma}, \tag{1.5}$$

$$P=\sigma K_{\gamma(t)} \quad \text{on} \quad \gamma(t), \tag{1.6}$$

$$V(0, r, \theta)=V_0(r, \theta), \qquad \gamma(0, \theta)=\gamma_0(\theta), \tag{1.7}$$

$$|\Omega_{\gamma(t)}|=\omega_0. \tag{1.8}$$

Here σ, ω_0 and g are prescribed positive constants. Δ is the Laplacian with respect to r and θ. Physical meanings of the symbols are as follows. The function V is a stream function for the flow, i.e., the velocity vector is given by $(\partial V/\partial y, -\partial V/\partial x)$, P is the pressure. For a fixed time t, the flow region is denoted by $\Omega_{\gamma(t)}$, which is bounded by Γ and $\gamma(t)=\{(r, \theta); r=\gamma(t, \theta)\ (0\leqq\theta<2\pi)\}$. Hence $\Omega_{\gamma(t)}=\{(r, \theta); 1<r<\gamma(t, \theta)\}$. $K_{\gamma(t)}$ is the curvature of $\gamma(t)$, whose sign is chosen to be positive if $\gamma(t)$ is convex. The constant σ is the surface tension coefficient. We make the following hypotheses: the function γ_0, which determines the initial position of the free boundary, belongs to $C^{5+\alpha}(S^1)$. It also satisfies that $\gamma_0>1$ and that

$$\frac{1}{2}\int_0^{2\pi}\gamma_0(\theta)^2 d\theta-\pi=\omega_0. \tag{1.9}$$

The function V_0 is determined by

$$\Delta V_0=0 \ \text{in}\ \Omega_{\gamma(0)}, \qquad V_0(1, \theta)\equiv 0, \quad V_0(\gamma_0(\theta), \theta)\equiv a, \tag{1.10}$$

where $a>0$ is a given constant which determines the magnitude of the circulation of the flow. It is easy to observe that (1.3) is satisfied if (1.3)* below holds for some function $f(t)$ of t.

$$V(t, \gamma(t, \theta), \theta)=\int_0^\theta\gamma(t, \phi)\frac{\partial\gamma}{\partial t}(t, \phi)d\phi+f(t). \tag{1.3*}$$

The condition (1.3) implies that fluid particles on the free boundary remains on the boundary throughout the motion. The equations (1.4) and (1.5) are the Euler equations written in terms of the stream function V.

Our goal is to discuss on the stability of a stationary solution of the problem above. So we consider the stationary version of the problem above:

PROBLEM (S). Find a closed Jordan curve γ and a function V such that the following conditions (1.11)-(1.15) are satisfied.

$$\Delta V = 0 \qquad \text{in } \Omega_\gamma, \tag{1.11}$$

$$V = 0 \qquad \text{on } \Gamma, \tag{1.12}$$

$$V = a \qquad \text{on } \gamma, \tag{1.13}$$

$$\frac{1}{2}|\nabla V|^2 - \frac{g}{r} + \sigma K_\gamma = \text{constant} \qquad \text{on } \gamma, \tag{1.14}$$

$$|\Omega_\gamma| = \omega_0 . \tag{1.15}$$

REMARK. The boundary condition (1.13) corresponds to (1.3) (see also (1.10)): The conditions (1.2, 1.3) and the Euler equations (1.4, 1.5) yield the invariance of the circulation. In this sense the stream function V of Problem (NS) is connected to the parameter a.

This stationary problem is analysed in Okamoto [8] (see also [**9**]). We briefly recall the result there. One easily sees that there is a radially symmetric solution. Namely, if we determine $r_0>1$ by the equality $\pi r_0^2-\pi=\omega_0$, then the circle of radius r_0 with the origin as its center is a solution to Problem (S). Indeed the corresponding stream function V is given by

$$V = V_0(r) = \frac{a}{\log r_0}\log r \qquad (1<r<r_0) . \tag{1.16}$$

We denote this curve (circle) by γ_0 and call it a trivial solution. This radially symmetric solution is a natural one, since all the data are radially symmetric. However, there is a solution without $O(2)$-symmetry, which bifurcates from the trivial solution. This is a main subject in [8]. The result is stated as follows. We define a_n $(n=1, 2, \cdots)$ by

$$a_n = \left(\frac{\sigma(n^2-1)/r_0^2+g/r_0^2}{r_0^{-1}+nR_n/r_0}\right)^{1/2} r_0 \log r_0 ,$$

where $R_n=(r_0^n+r_0^{-n})/(r_0^n-r_0^{-n})$. Then a_n is a bifurcation point provided $a_n \notin \{a_m\}_{m\neq n}$. Namely we have a nontrivial solution in any neighborhood of a_n. As for the geometric properties of the bifurcating solutions, see Fujita et al.

[4] where numerical solutions are given.

We put $a^*=\min_{n\geq 1}a_n$. Then one might think that the trivial solution is stable for $0<a<a^*$ and that it loses stability at $a=a^*$. In this paper we will show that the trivial solution is unstable (in the sense described in § 6) if $a>a^{**}$, where $a^{**}=\min_{n\geq 1}(1+nR_n)^{1/2}a_n$. This is stated in [**9**]. Here we give a complete proof. We also present another method called a small disturbance approximation. Then we show that two stability criterions derived from these two methods coincide.

We remark that in the present paper we are interested in seeing how the free boundary changes its shape. So our main purpose is to give a mathematical tool to see the asymptotic behavior of the free boundary. The existence and uniqueness of the solution is not discussed here. We think that the method employed in Yosihara [**13, 14**] can be applicable to our problem and that it may ensure the local existence of the solution. But our main purpose is to see long time behavior of the free boundary. Hence we use a method different from those in [**13, 14**].

This paper is composed of seven sections. In § 2 we formulate Problem (NS) by the perturbation method using some function spaces. The problem is transformed to seeking a zero point of a certain mapping defined in a Banach space. In sections 3, 4 and 5 we are concerned with a linearization of the equation, i. e., we derive a Fréchet derivative of the mapping above. It will be very important to note that we linearize the equation at the trivial solution with real parameter a. Therefore the structure of the linearized equation varies with a. In § 6 we prove a theorem which implies the instability of the trivial solution for $a>a^{**}$. In § 7 we give a small disturbance approximation for our problem and show that the stability condition given by it is the same as that given in § 6.

§ 2. Formulation by the perturbation method.

We first define function spaces. Let $T>0$ and $0<\beta<\alpha<1$ be fixed.

FUNCTION SPACES.

$$X=\Big\{u\in\bigcap_{j=0}^{3}C^j([0,\,T]\,;C^{5-j+\alpha}(S^1))\ ;$$

$$u(0,\,\theta)\equiv 0,\ \frac{\partial u}{\partial t}(0,\,\theta)\equiv 0,\ \int_0^{2\pi}u(t,\,\theta)d\theta\equiv 0\Big\},$$

$$Y=\Big\{u\in C^0([0,\,T]\,;C^\beta(S^1))\ ;\ \int_0^{2\pi}u(t,\,\theta)d\theta\equiv 0\Big\}.$$

X and Y are Banach spaces with canonical norms. Our plan to catch a solution is as follows. We first give a $u \in X$. By this function u we construct a time-dependent closed Jordan curve $\{\gamma_u(t)\}_{0<t<T}$ satisfying (1.8). Then we solve a Dirichlet problem (1.1, 1.2), (1.3)* and (1.7) for V in a domain bounded by Γ and $\gamma_u(t)$, regarding t as a parameter. Denoting by V_u the solution thus obtained, we solve (1.4) and (1.5) which is a Cauchy-Riemann equation. Then we define a mapping F by

$$F(\gamma_0, u) = \frac{\partial}{\partial \theta}(P|_{\gamma_u} - \sigma K_{u(t)}),$$

where $K_{u(t)}$ is the curvature of $\gamma_{u(t)}$. Observe that γ_u is a solution for $\gamma(0)=\gamma_0$ if and only if $F(\gamma_0, u)=0$. Hence our task is to investigate a zero-point of F.

We begin with the definition of γ_u. For a sufficiently small $u \in X$ we define a function γ_u on $[0, T] \times S^1$ by

$$\gamma_u(t, \theta) = \gamma_0(\theta) + u(t, \theta) + g_u(t), \tag{2.1}$$

where the function g_u is defined by

$$g_u(t) = \frac{1}{2\pi}\left(-\int_0^{2\pi} \bar{\gamma}_u(t, \theta)d\theta + \sqrt{\left\{\int_0^{2\pi} \bar{\gamma}_u(t, \theta)d\theta\right\}^2 - 2\pi\int_0^{2\pi}(\bar{\gamma}_u^2 - \gamma_0^2)d\theta}\right) \tag{2.2}$$

with $\bar{\gamma}_u = \gamma_0 + u$. The function g_u is defined so that

$$\int_0^{2\pi} \gamma_u(t, \theta)\frac{\partial}{\partial t}\gamma_u(t, \theta)d\theta \equiv 0.$$

From this equality and (1.9) we easily see that $|\Omega_{u(t)}| \equiv \omega_0$. Hereafter we write $\Omega_{u(t)}$ instead of $\Omega_{\gamma_u(t)}$. Observe that g_u is three times continuously differentiable and that $g_u(0) = g_u'(0) = 0$. We next prove the following

PROPOSITION 2.1. *There exist unique f and V_u satisfying*

$$\Delta V_u = 0 \quad in \quad Q_{T,u} \equiv Q_{T,\gamma_u}, \tag{2.3}$$

$$V_u = 0 \quad on \quad \Gamma, \quad 0<t<T, \tag{2.4}$$

$$V_u = \int_0^\theta \gamma_u(t, \phi)\frac{\partial}{\partial t}\gamma_u(t, \phi)d\phi + f_u(t) \quad on \quad \gamma(t), \tag{2.5}$$

$$V_u(0, r, \theta) = V_0(r, \theta) \quad in \quad \Omega_0, \tag{2.6}$$

$$\frac{d}{dt}\int_0^{2\pi} \frac{\partial V_u}{\partial r}(t, 1, \theta)d\theta \equiv 0. \tag{2.7}$$

Furthermore we have $f_u \in C^2([0, T])$, $(\partial/\partial t)^j V_u(t, \cdot) \in C^{5-j+\alpha}(\overline{\Omega_{u(t)}})$ $(j=0, 1, 2)$. *Using a canonical pull-back from* $\overline{\Omega_{u(t)}}$ *to* $\overline{\Omega_0}$, *we can regard* $(\partial^j V_u/\partial t^j)|_{\gamma_u} \in C([0, T]; C^{5-j+\alpha}(S^1))$.

PROOF. We begin with a heuristic argument. We first define W_u and Z_u by the equations below.

$$(2.8)\qquad \begin{cases} \Delta W_u = 0 & \text{in } Q_{T,u}, \\ W_u = 0 \quad \text{on } \Gamma, & W_u = \int_0^\theta \gamma_u \frac{\partial}{\partial t}\gamma_u \quad \text{on } \gamma_u(t). \end{cases}$$

$$(2.9)\qquad \begin{cases} \Delta Z_u = 0 & \text{in } Q_{T,u}, \\ Z_u = 0 \quad \text{on } \Gamma, & Z_u = 1 \quad \text{on } \gamma_u(t). \end{cases}$$

Then, for each t, the functions $W_u(t, \cdot)$ and $Z_u(t, \cdot)$ belong to $C^{5+\alpha}(\overline{\Omega_{u(t)}})$. If the solution exists, then it must satisfy $V_u = W_u + f_u(t)Z_u$. Putting this into (2.7), we have

$$(2.10)\qquad f_u'(t)\int_0^{2\pi}\frac{\partial}{\partial r}Z_u(t, 1, \theta)d\theta + f_u(t)\int_0^{2\pi}\frac{\partial^2}{\partial t\partial r}Z_u(t, 1, \theta)d\theta + \int_0^{2\pi}\frac{\partial^2}{\partial t\partial r}W_u(t, 1, \theta)d\theta = 0.$$

On the other hand, putting $t=0$ in (2.5), we obtain $f_u(0)=a$ (if the solution exists).

To prove the proposition we solve the ordinary differential equation (2.10) with the initial condition $f_u(0)=a$. Since u is small, it holds that

$$Z_u \sim \frac{\log r}{\log r_0},$$

hence

$$\int_0^{2\pi}\frac{\partial}{\partial r}Z_u(t, 1, \theta)d\theta \sim \frac{2\pi}{\log r_0} > 0.$$

Therefore (2.10) is uniquely solvable. Then we define V_u by the equality $V_u = W_u + f_u(t)Z_u$. The functions f_u and V_u thus defined give the solution. It remains to examine smoothness of them. It is easy to see that $W_u(t, \cdot)$, $Z_u(t, \cdot) \in C^{5+\alpha}(\overline{\Omega_{u(t)}})$. The function $(\partial/\partial t)W_u$ satisfies

$$\Delta\frac{\partial}{\partial t}W_u = 0 \qquad \text{in } Q_{T,u},$$

$$\frac{\partial}{\partial t}W_u = 0 \qquad \text{on } \Gamma,$$

$$\frac{\partial}{\partial t}W_u = \int_0^\theta \gamma_u \frac{\partial^2}{\partial t^2}\gamma_u d\theta + \int_0^\theta \left(\frac{\partial}{\partial t}\gamma_u\right)^2 d\theta - \frac{\partial}{\partial r}W_u \frac{\partial}{\partial t}\gamma_u \qquad \text{on } \gamma_u(t).$$

The boundary condition on $\gamma_u(t)$ comes from the differentiation of the condition on (2.8). Hence, for a fixed t, the function $(\partial/\partial t)W_u(t, \cdot)$ belongs to $C^{4+\alpha}$. Other smoothness properties for W_u and Z_u can be verified similarly. By virtue of (2.10) we can conclude that $f_u \in C^2$. Q. E. D.

PROPOSITION 2.2. *Let V_u be the function given in the preceding proposition. Then for each t, we have a unique q_u such that*

$$\frac{1}{r}\frac{\partial^2}{\partial t \partial \theta} V_u + \frac{\partial}{\partial r} q_u = 0 \qquad in \quad \Omega_{u(t)}, \tag{2.11}$$

$$-\frac{\partial^2}{\partial t \partial r} V_u + \frac{1}{r}\frac{\partial}{\partial \theta} q_u = 0 \qquad in \quad \Omega_{u(t)}, \tag{2.12}$$

$$\int_{\Omega_{u(t)}} q_u = 0 . \tag{2.13}$$

Moreover it satisfies that $q_u(t) \in C^{4+\alpha}$.

PROOF. The function

$$\bar{q}_u(t, r, \theta) = -\int_1^r \frac{1}{\rho}\frac{\partial^2}{\partial t \partial \theta} V_u(t, \rho, \theta) d\rho + \int_0^\theta \frac{\partial^2}{\partial t \partial r} V_u(t, 1, \phi) d\phi$$

is single-valued because of (2.7) and satisfies (2.11) and (2.12) because of the harmonicity of V. Choose a constant c so that $\bar{q}_u + c$ satisfies (2.13). Then we will have the desired function. Uniqueness is obvious. Q. E. D.

NOTATION.

$$M = \left\{\gamma_0 \in C^{5+\alpha}(S^1) \ ; \ \frac{1}{2}\int_0^{2\pi} \gamma_0(\theta)^2 d\theta - \pi = \omega_0, \ \|\gamma_0 - r_0\|_{5+\alpha} < \frac{r_0 - 1}{2}\right\}.$$

Of course M is a Banach manifold.

We finally define a mapping $F: M \times X \to Y$ by the following equality:

$$F(\gamma_0, u) = \frac{\partial}{\partial \theta}\left(\left\{q_u - \frac{1}{2}|\nabla V_u|^2 + \frac{g}{r}\right\}\Big|_{\gamma_{u(t)}} - \sigma K_{u(t)}\right). \tag{2.14}$$

Then γ_u solves our problem if and only if $F(\gamma_0, u) = 0$. In the following sections we investigate a zero point of the mapping F when γ_0 is sufficiently smooth and close to r_0. Here and in what follows we regard a function defined on $\gamma_u(t)$ as a function on S^1 by means of a canonical correspondence $(1, \theta) \to (r_0 + u(\theta), \theta)$.

§ 3. Derivative of F.

In this section we linearize the mapping F. Namely we derive a concrete expression of the Fréchet derivative of F. To do this we prepare several symbols: the Fréchet derivative with respect to u is denoted by D_u. We put

$\zeta = D_u \gamma_u(w)$ for an arbitrary $w \in X$. Hence we have

(3.1) $$\zeta = w - \int_0^{2\pi} \gamma_0(\theta) w(t, \theta) d\theta \Big(\Big(\int_0^{2\pi} \bar{\gamma}_u \Big)^2 - 2\pi \int_0^{2\pi} (\bar{\gamma}_u{}^2 - \gamma_0^2) \Big)^{-1/2}.$$

For $w \in X$ we denote by $U_u(w)$ the solution U of the following Dirichlet problem.

(3.2) $$\Delta U = 0 \quad \text{in } \Omega_u, \qquad U = 0 \quad \text{on } \Gamma,$$

(3.3) $$U = \int_0^\theta \Big(\frac{\partial^2}{\partial t^2} (\zeta \gamma_u) \Big) d\theta + D_u(f'_u) w - \frac{\partial V_u}{\partial r} \frac{\partial \zeta}{\partial t} - D_u \Big(\frac{\partial V_u}{\partial r} \Big|_{\gamma_u} \Big) w \cdot \frac{\partial}{\partial t} \gamma_u - \frac{\partial^2 V_u}{\partial t \partial r} \zeta \quad \text{on } \gamma_u(t).$$

We also consider the following Dirichlet problem

(3.4) $$\Delta U = 0 \quad \text{in } \Omega_{u(t)}, \qquad U = 0 \quad \text{on } \Gamma,$$

(3.5) $$U = \int_0^\theta \frac{\partial}{\partial t} (\zeta \gamma_u) d\theta + D_u(f_u)(w) - \frac{\partial V_u}{\partial r} \zeta.$$

The solution of this equation is denoted by $U_u^*(w)$. Lastly we define a mapping Φ_u: for $z \in C^1(\Omega_{u(t)})$ with $\int_0^{2\pi} (\partial/\partial r) z(1, \theta) d\theta = 0$ and $\Delta z = 0$, we define $\Phi_u z$ by the equation below.

(3.6) $$\begin{cases} \dfrac{1}{r} \dfrac{\partial}{\partial \theta} z + \dfrac{\partial}{\partial r} \Phi_u z = 0 & \text{in } \Omega_u, \\ -\dfrac{\partial}{\partial r} z + \dfrac{1}{r} \dfrac{\partial}{\partial \theta} \Phi_u z = 0 & \text{in } \Omega_u, \\ \displaystyle\int_{\Omega_u} \Phi_u z = 0. \end{cases}$$

Now we give a theorem concerning the linearization of F, which plays a fundamental role in the stability analysis.

THEOREM 3.1. *For a fixed $\gamma_0 \in M$, the mapping $F(\gamma_0, \cdot): X \to Y$ is Fréchet differentiable. The derivative $D_u F(\gamma_0, u)$ with respect to u is given by*

(3.7) $$\begin{aligned} D_u F(\gamma_0, u) w = & \frac{\partial}{\partial \theta} \Phi_u \big(U_u(w) \big) + \frac{\partial}{\partial \theta} \Big(\frac{\partial}{\partial r} q_u \Big|_{\gamma_u} \cdot \zeta \Big) \\ & - \frac{\partial}{\partial \theta} \Big(\frac{\partial V_u}{\partial r} \Big(\frac{\partial}{\partial r} U_u^*(w) + \frac{\partial^2 V_u}{\partial r^2} \zeta \Big) + \frac{1}{\gamma_u^2} \frac{\partial V_u}{\partial \theta} \Big(\frac{\partial}{\partial \theta} U_u^*(w) + \frac{\partial^2 V_u}{\partial r \partial \theta} \zeta \Big) \\ & - \frac{2}{\gamma_u^3} \Big(\frac{\partial}{\partial \theta} V_u \Big)^2 \zeta - \frac{g \zeta}{\gamma_u^2} \Big) - \sigma \frac{\partial}{\partial \theta} \big(f_0(u) w + f_1(u) w' + f_2(u) w'' \big), \end{aligned}$$

where

$$f_0(u) = \frac{\partial}{\partial u} K_{u(t)}, \qquad f_1(u) = \frac{\partial}{\partial u'} K_{u(t)} \qquad \text{and} \qquad f_2(u) = \frac{\partial}{\partial u''} K_{u(t)}.$$

COROLLARY 3.1.

$$D_uF(r_0, 0)w = \frac{\partial}{\partial\theta}\Phi_0\big(U_0(w)\big) - \frac{\partial}{\partial\theta}\Big(\frac{a}{r_0\log r_0}\Big(\frac{\partial}{\partial r}U_0^*(w) - \frac{aw}{r_0^2\log r_0}\Big)\Big) - \frac{g}{r_0^2}\frac{\partial w}{\partial\theta} + \frac{\sigma}{r_0^2}\Big(\frac{\partial^3 w}{\partial\theta^3} + \frac{\partial w}{\partial\theta}\Big), \tag{3.8}$$

where $U_0(w)$ *and* $U_0^*(w)$ *are harmonic in* $1<r<r_0$ *and satisfy* $U_0(w)=U_0^*(w)=0$ *on* $[0, T]\times\Gamma$,

$$U_0(w) = \int_0^\theta r_0\frac{\partial^2 w}{\partial t^2}d\theta - \frac{a}{r_0\log r_0}\frac{\partial w}{\partial t} \qquad on \quad r=r_0,\ 0<t<T,$$

$$U_0^*(w) = \int_0^\theta r_0\frac{\partial w}{\partial t}d\theta - \frac{aw}{r_0\log r_0} \qquad on \quad r=r_0,\ 0<t<T.$$

In particular, we have

$$U_0(w) = \frac{\partial}{\partial t}U_0^*(w).$$

PROOF OF THE COROLLARY. If one note that $\zeta=w$ and $\partial V_u/\partial t\equiv 0$ in the case of $u=0$ and $\gamma_0=r_0$, then (3.8) follows from (3.7).

PROOF OF THEOREM 3.1. First we show that

$$D_u(q_u|_{\gamma_u})w = \Phi_u(U_u(w)). \tag{3.9}$$

To this end we extend $\partial V_u/\partial t$ to a function on some neighborhood of $Q_{T,u}$ in such a way that the extended function has the same order of smoothness as $\partial V_u/\partial t$. Put $Y_u=\partial V_u/\partial t$. Recall that $Y\in C^{4+\alpha}$ (Proposition 2.1). For small u and w we have

$$\begin{aligned} q_{u+w}|_{\gamma_{u+w}} - q_u|_{\gamma_u} &= \Phi_{u+w}[Y_{u+w}]|_{\gamma_{u+w}} - \Phi_{u+w}[Y_u]|_{\gamma_{u+w}} \\ &\quad + \Phi_{u+w}[Y_u]|_{\gamma_{u+w}} - \Phi_u[Y_u]|_{\gamma_u} \\ &\equiv Q_1 + Q_2. \end{aligned}$$

Observe that

$$\Phi_{u+w}[Y_u]|_{\gamma_{u+w}} = -\int_1^{\gamma_{u+w}}\frac{1}{\rho}\frac{\partial Y_u}{\partial\theta}d\rho + \int_0^\theta\frac{\partial Y_u}{\partial r}(t, 1, \theta)d\theta + \text{constant},$$

$$\Phi_u[Y_u]|_{\gamma_u} = -\int_1^{\gamma_u}\frac{1}{\rho}\frac{\partial Y_u}{\partial\theta}d\rho + \int_0^\theta\frac{\partial Y_u}{\partial r}(t, 1, \theta)d\theta + \text{constant}.$$

Therefore we have

$$Q_2 = -\int_{\gamma_u}^{\gamma_{u+w}}\frac{1}{\rho}\frac{\partial Y_u}{\partial\theta}d\rho + \text{constant}.$$

By the definition of F, constant terms give no contribution because of the presence of $\partial/\partial\theta$. Hence, in what follows, we omit constant terms. Using (2.11), we obtain

$$Q_2-\frac{\partial q_u}{\partial r}(t, \gamma_u(t, \theta), \theta)\zeta$$
$$=-\int_0^1(\gamma_{u+w}-\gamma_u)\frac{1}{\gamma_u+\eta(\gamma_{u+w}-\gamma_u)}\frac{\partial Y_u}{\partial\theta}(t, \gamma_u+\eta(\gamma_{u+w}-\gamma_u), \theta)d\eta$$
$$+\frac{1}{\gamma_u}\frac{\partial Y_u}{\partial\theta}(t, \gamma_u, \theta)\zeta$$
$$=-\int_0^1(\gamma_{u+w}-\gamma_u-\zeta)\frac{1}{\gamma_u+\eta(\gamma_{u+w}-\gamma_u)}\frac{\partial Y_u}{\partial\theta}(t, \gamma_u+\eta(\gamma_{u+w}-\gamma_u), \theta)d\eta$$
$$-\int_0^1\left(\frac{1}{\gamma_u+\eta(\gamma_{u+w}-\gamma_u)}\frac{\partial Y_u}{\partial\theta}(t, \gamma_u+\eta(\gamma_{u+w}-\gamma_u), \theta)-\frac{1}{\gamma_u}\frac{\partial Y_u}{\partial\theta}(t, \gamma_u, \theta)\right)\zeta d\eta$$
$$\equiv Q_2'+Q_2''.$$

By the definition we obtain $\|\gamma_{u+w}-\gamma_u\|_{4+\alpha}\leqq c\|w\|_{4+\alpha}$ and $\|\gamma_{u+w}-\gamma_u-\zeta\|_{4+\alpha}\leqq c\|w\|_{4+\alpha}^2$ uniformly in t. (Here and hereafter c means a generic constant which is different in different context.) Then we easily obtain $\|Q_2'\|_{1+\beta}\leqq c\|w\|_{4+\alpha}^2$ and $\|Q_2''\|_{1+\beta}\leqq c\|w\|_{4+\alpha}^2$ uniformly in t, hence $\|Q_2\|_Y\leqq c\|w\|_X^2$.

Next we consider Q_1. Note first that Y_u is characterized by the following equation

$$\Delta Y_u=0 \quad \text{in} \quad \Omega_{u(t)},$$
$$Y_u=0 \quad \text{on} \quad \Gamma,$$
$$Y_u=\frac{1}{2}\frac{\partial^2}{\partial t^2}\int_0^\theta(\gamma_u(t, \phi))^2 d\phi+f_u'(t)-\frac{\partial V_u}{\partial r}\frac{\partial\gamma_u}{\partial t} \quad \text{on} \quad \gamma_u.$$

Therefore we have

$$\Delta(Y_{u+w}-Y_u)=-\Delta Y_u \quad \text{in} \quad \Omega_{u+w},$$
$$Y_{u+w}-Y_u=0 \quad \text{on} \quad \Gamma,$$
$$Y_{u+w}-Y_u=Y_{u+w}|_{\gamma_{u+w}}-Y_u|_{\gamma_u}-(Y_u|_{\gamma_{u+w}}-Y_u|_{\gamma_u}) \quad \text{on} \quad \gamma_{u+w}.$$

Using this equation and the defining equation of $U_u(w)$ we will prove that

$$\|(Y_{u+w}-Y_u)\circ\phi-U_u(w)\|_{1+\beta}\leqq c\|w\|_{4+\alpha}^2 \quad \text{uniformly in } t, \tag{3.10}$$

where $\phi:\Omega_u\to\Omega_{u+w}$ is defined by

$$\phi(r, \theta)=(\rho, \theta), \qquad \rho=\frac{(r_0-1+u+w)r-w}{r_0-1+u}.$$

To this end, we compute and obtain

$$\Delta((Y_{u+w}-Y_u)\circ\phi-U_u(w))=O(|w|^2) \quad \text{in} \quad \Omega_u,$$
$$(Y_{u+w}-Y_u)\circ\phi-U_u(w)=0 \quad \text{on} \quad \Gamma,$$
$$(Y_{u+w}-Y_u)\circ\phi-U_u(w)=O(|w|^2) \quad \text{on} \quad \gamma_u.$$

The precise meaning of the right hand side will be explained below. Put $Z(u, w)\equiv(Y_{u+w}-Y_u)\circ\phi-U_u(w)$ and observe that

$$\begin{aligned}\Delta Z(u, w) &= \Delta\{(Y_{u+w}-Y_u)\circ\phi\}\\ &= -(\tilde{\Delta}Y_u)\circ\phi+\frac{w(r-1)}{r\{(r_0-1+u+w)r-w\}}\frac{\partial Y}{\partial\rho}\frac{\partial\rho}{\partial r}+\frac{2w(r-1)+w^2(r-1)^2}{r^2\{(r_0-1+u+w)r-w\}^2}\frac{\partial^2 Y}{\partial\theta^2}\\ &\quad+\frac{1}{r^2}\Big(\frac{\partial^2 Y}{\partial\rho^2}\Big(\frac{\partial\rho}{\partial\theta}\Big)^2+\frac{\partial^2 Y}{\partial\rho\partial\theta}\frac{\partial\rho}{\partial r}+\frac{\partial Y}{\partial\rho}\frac{\partial^2\rho}{\partial\theta^2}\Big),\end{aligned}$$

where $Y=Y_{u+w}-Y_u$ and $\tilde{\Delta}$ is the Laplacian with respect to ρ and θ. Therefore we have $\|\Delta Z(u, w)\|_\beta\leqq c\|w\|_{4+\alpha}^2$, where c depends only on $\|u(t)\|_{4+\alpha}$ and $\|\partial V_u/\partial t\|_{3+\alpha}$. As for the boundary conditions, $Z(u, w)$ evidently satisfies $Z(u, w)=0$ on Γ. Furthermore we have

$$\begin{aligned}Z(u, w)|_{\gamma_{u(t)}} &= Y_{u+w}|_{\gamma_{u+w}}-Y_u|_{\gamma_u}-\{Y_u|_{\gamma_{u+w}}-Y_u|_{\gamma_u}\}\\ &\quad-\int_0^\theta\Big(\frac{\partial^2}{\partial t^2}(\zeta\gamma_u)\Big)d\theta-D_u(f'_u)w+\frac{\partial V_u}{\partial r}\frac{\partial\zeta}{\partial t}+\frac{\partial^2 V_u}{\partial t\partial r}\zeta+D_u\Big(\frac{\partial V_u}{\partial r}\Big|_{\gamma_u}\Big)w\cdot\frac{\partial}{\partial t}\gamma_u\\ &= \frac{1}{2}\frac{d^2}{dt^2}\int_0^\theta\gamma_{u+w}^2-\frac{1}{2}\frac{d^2}{dt^2}\int_0^\theta\gamma_u^2-\frac{d^2}{dt^2}\int_0^\theta\zeta\gamma_u+f'_{u+w}(t)-f'_u(t)-D_u(f'_u)(w)\\ &\quad-\frac{\partial V_{u+w}}{\partial r}\frac{\partial\gamma_{u+w}}{\partial t}+\frac{\partial V_u}{\partial r}\frac{\partial\gamma_u}{\partial t}+\frac{\partial V_u}{\partial r}\frac{\partial\zeta}{\partial t}+D_u\Big(\frac{\partial V_u}{\partial r}\Big|_{\gamma_u}\Big)w\cdot\frac{\partial}{\partial t}\gamma_u\\ &\quad-\frac{\partial V_u}{\partial t}\Big|_{\gamma_{u+w}}+\frac{\partial V_u}{\partial t}\Big|_{\gamma_u}+\frac{\partial^2 V_u}{\partial t\partial\gamma}\zeta.\end{aligned}$$

Hence we have $\|Z(u, w)|_{\gamma_{u(t)}}\|_{1+\beta}\leqq c\|w\|_{4+\alpha}^2$. By the Schauder estimate we obtain (3.10). We have thus proved (3.9).

Next, observing that

$$|\nabla V_u|^2=\Big(\frac{\partial V_u}{\partial r}\Big)^2+\frac{1}{\gamma_u^2}\Big(\frac{\partial V_u}{\partial\theta}\Big)^2,$$

we have

$$D_u\Big(\frac{1}{2}|\nabla V_u|^2\Big)(w)=\frac{\partial V_u}{\partial r}D_u\Big(\frac{\partial V_u}{\partial r}\Big)(w)+\frac{1}{\gamma_u^2}\frac{\partial V_u}{\partial\theta}D_u\Big(\frac{\partial V_u}{\partial\theta}\Big)(w)-\frac{1}{\gamma_u^3}\Big(\frac{\partial V_u}{\partial\theta}\Big)^2\zeta.$$

Using the method above, by which we have dealt with Q_1, we see that

$$D_u\Big(\frac{\partial V_u}{\partial r}\Big)(w)=\frac{\partial}{\partial r}U_u^*(w)+\frac{\partial^2 V_u}{\partial r^2}\zeta,$$
$$D_u\Big(\frac{\partial V_u}{\partial\theta}\Big)(w)=\frac{\partial}{\partial\theta}U_u^*(w)+\frac{\partial^2 V_u}{\partial r\partial\theta}\zeta.$$

The principal idea and the computation is the same as those in [8]. From these formulas we obtain the expression for the derivative of $D_u(|\nabla V_u|^2/2)(w)$.

Other parts of the proof is straightforward. Hence we omit it. Q. E. D.

REMARK 3.1. From the proof we see that there is a constant c independent of T such that

$$\|F(r_0, u)-F(r_0, 0)-D_uF(r_0, 0)u\|_Y \leqq c\|u\|_X^2 .$$

§ 4. Spectral analysis of $D_uF(\gamma_0, 0)$.

In this section we represent $D_uF(r_0, 0)$ more concretely. Express $w \in X$ by the Fourier series:

$$w(t, \theta) = \sum_{n\neq 0} w_n(t)e^{in\theta}.$$

Then the function $Z\equiv(\partial/\partial\theta)U_0^*(w)$ is characterized by

$$\begin{cases} \Delta Z = 0 \quad \text{in } 1<r<r_0, \qquad Z = 0 \quad \text{on } \Gamma, \\ Z = r_0\dfrac{\partial w}{\partial t} - \dfrac{a}{r_0 \log r_0}\dfrac{\partial w}{\partial \theta} \\ \quad = r_0 \sum\limits_{n\neq 0} w_n'(t)e^{in\theta} - \dfrac{a}{r_0\log r_0}\sum\limits_{n\neq 0} inw_n(t)e^{in\theta} \qquad \text{on } r=r_0 . \end{cases}$$

Therefore we have

$$\frac{\partial}{\partial\theta}U_0^*(w) = r_0 \sum_{n\neq 0}\frac{r^n-r^{-n}}{r_0^n-r_0^{-n}}w_n'(t)e^{in\theta} - \frac{a}{r_0\log r_0}\sum_{n\neq 0}\frac{r^n-r^{-n}}{r_0^n-r_0^{-n}}w_n(t)ine^{in\theta}.$$

Observe next that $\Phi_0(r^ne^{in\theta})=-ir^ne^{in\theta}$, $\Phi_0(r^{-n}e^{in\theta})=ir^{-n}e^{in\theta}$. From these equalities we have

$$\begin{aligned} &\frac{\partial}{\partial\theta}\Phi_0(U_0(w)) = \Phi_0\Big(\frac{\partial^2}{\partial\theta\partial t}U_0^*(w)\Big) \\ &= r_0 \sum_{n\neq 0}\frac{-r^n-r^{-n}}{r_0^n-r_0^{-n}}iw_n''(t)e^{in\theta} - \frac{a}{r_0\log r_0}\sum_{n\neq 0}\frac{r^n+r^{-n}}{r_0^n-r_0^{-n}}w_n'(t)ne^{in\theta} \\ &= -ir_0\sum_{n\neq 0}R_nw_n''(t)e^{in\theta} - \frac{a}{r_0\log r_0}\sum_{n\neq 0}nR_nw_n'(t)e^{in\theta} \qquad \text{on } r=r_0, \end{aligned}$$

$$\text{where } R_n = \frac{r_0^n+r_0^{-n}}{r_0^n-r_0^{-n}} \quad (n\in \boldsymbol{Z}\setminus\{0\}).$$

On the other hand, $(\partial^2/\partial\theta\partial r)U_0^*(w)$ is computed as follows.

$$\frac{\partial^2}{\partial\theta\partial r}U_0^*(w)|_{r=r_0} = \sum_{n\neq 0} nR_nw_n'(t)e^{in\theta} - \frac{a}{r_0\log r_0}\sum_{n\neq 0}\frac{in^2}{r_0}R_nw_ne^{in\theta}.$$

We now introduce a linear operator H which is defined by

$$Hv=\sum_{n\neq 0}(-iR_n)v_n e^{in\theta} \qquad \text{for} \quad v=\sum_{n\neq 0} v_n e^{in\theta}. \tag{4.1}$$

Then, putting $b=a/(r_0^2\log r_0)$, we have

$$\frac{\partial}{\partial\theta}\Phi_0(U_0(w))|_{r=r_0}=r_0H\frac{\partial^2 w}{\partial t^2}-r_0bH\frac{\partial^2 w}{\partial t\partial\theta},$$

$$\frac{\partial^2}{\partial r\partial\theta}U_0^*(w)|_{r=r_0}=H\frac{\partial^2 w}{\partial t\partial\theta}-bH\frac{\partial^2 w}{\partial\theta^2}.$$

Therefore $D_uF(r_0, 0)$ is represented as follows.

$$D_uF(r_0, 0)w=r_0H\frac{\partial^2 w}{\partial t^2}-2r_0bH\frac{\partial^2 w}{\partial t\partial\theta}+r_0b^2H\frac{\partial^2 w}{\partial\theta^2}+\Big(r_0b^2+\frac{\sigma-g}{r_0^2}\Big)\frac{\partial w}{\partial\theta}+\frac{\sigma}{r_0^2}\frac{\partial^3 w}{\partial\theta^3}.$$

REMARK. The quantity R_n tends to ± 1 exponentially as $n\to\pm\infty$. Hence the operator H is the Hilbert transform on S^1 plus a "smoothing operator".

Now we see that $D_uF(r_0, 0)w=f$ $(\in Y)$ is equivalent to the Cauchy problem for the hyperbolic equation below:

$$\frac{\partial^2 w}{\partial t^2}-2b\frac{\partial^2 w}{\partial t\partial\theta}+b^2\frac{\partial^2 w}{\partial\theta^2}+H^{-1}\Big(b^2+\frac{\sigma-g}{r_0^3}\Big)\frac{\partial w}{\partial\theta}+H^{-1}\frac{\sigma}{r_0^3}\frac{\partial^3 w}{\partial\theta^3} \tag{4.2}$$
$$=\frac{1}{r_0}H^{-1}f(t, \theta)\equiv g\in Y,$$
$$w(0, \theta)\equiv 0, \qquad \frac{\partial w}{\partial t}(0, \theta)\equiv 0.$$

We rewrite this hyperbolic equation in the framework of the Hille-Yosida theory for semi-groups of operators. To this end, put $v=\partial w/\partial t$. Then the equation (4.2) is rewritten as

$$\frac{d}{dt}\binom{w}{v}=\begin{pmatrix}0 & I\\ L & 2b(\partial/\partial\theta)\end{pmatrix}\binom{w}{v}+\binom{0}{g} \tag{4.3}$$
$$\equiv A\binom{w}{v}+\binom{0}{g},$$

where the operator L is defined by

$$L=-\frac{\sigma}{r_0^3}H^{-1}\frac{\partial^3}{\partial\theta^3}-b^2\frac{\partial^2}{\partial\theta^2}-\Big(b^2+\frac{\sigma-g}{r_0^3}\Big)H^{-1}\frac{\partial}{\partial\theta}.$$

We consider (4.3) in a function space $E\equiv\dot{H}^{3/2}(S^1)\times\dot{L}^2(S^1)$. Here and in what follows $H^t(S^1)$ means a Sobolev space in the L^2-category and $\dot{H}^t(S^1)=H^t(S^1)/\boldsymbol{R}$. The following lemma is important in the stability analysis of the trivial solution.

LEMMA 4.1. *The spectrum of A is composed only of eigenvalues which lie on the imaginary axis except for finite number of them. More concretely,*

$$\sigma(A)=\{\lambda_+(n),\ \lambda_-(n)\,;\ n=\pm 1,\ \pm 2,\ \pm 3,\ \cdots\},$$

with

$$\lambda_\pm(n)=\frac{1}{r_0^2\log r_0}\left(ian\pm i\sqrt{\frac{n\{(1+nR_n)a_{|n|}^2-a^2\}}{R_n}}\right).$$

PROOF. First step. We show that $\lambda_\pm(n)$'s are eigenvalues and that there is no other eigenvalue. Let λ be an eigenvalue and $(w, v)\in E$ be an eigenvector associated with λ. Then it holds that

$$\lambda w=v\,,$$

$$\lambda v=-\frac{\sigma}{r_0^3}H^{-1}\frac{\partial^3 w}{\partial\theta^3}-b^2\frac{\partial^2 w}{\partial\theta^2}-\left(b^2+\frac{\sigma-g}{r_0^3}\right)H^{-1}\frac{\partial w}{\partial\theta}+2b\frac{\partial v}{\partial\theta}.$$

Therefore we have

$$\lambda^2 w=-\frac{\sigma}{r_0^3}H^{-1}\frac{\partial^3 w}{\partial\theta^3}-b^2\frac{\partial^2 w}{\partial\theta^2}-\left(b^2+\frac{\sigma-g}{r_0^3}\right)H^{-1}\frac{\partial w}{\partial\theta}+2b\lambda\frac{\partial w}{\partial\theta}.$$

Using the Fourier expansion $w=\sum_{n\neq 0}w_n e^{in\theta}$, we obtain

$$\lambda^2 w_n=-\frac{\sigma}{r_0^3}\,\frac{-in^3}{-iR_n}w_n-b^2(-n^2)w_n-\left(b^2+\frac{\sigma-g}{r_0^3}\right)\frac{in}{-iR_n}w_n+2b\lambda inw_n$$

for all $n\in\boldsymbol{Z}\setminus\{0\}$. Since $w\neq 0$, it should hold that

$$\lambda^2=-\frac{\sigma}{r_0^3}\,\frac{n^3}{R_n}+b^2n^2+\left(b^2+\frac{\sigma-g}{r_0^3}\right)\frac{n}{R_n}+2b\lambda in$$

for some $n\in\boldsymbol{Z}\setminus\{0\}$. Hence

$$\lambda=\frac{i}{r_0^2\log r_0}\left(an\pm\sqrt{\frac{n\{(1+nR_n)a_{|n|}^2-a^2\}}{R_n}}\right),$$

therefore we have proved the claim.

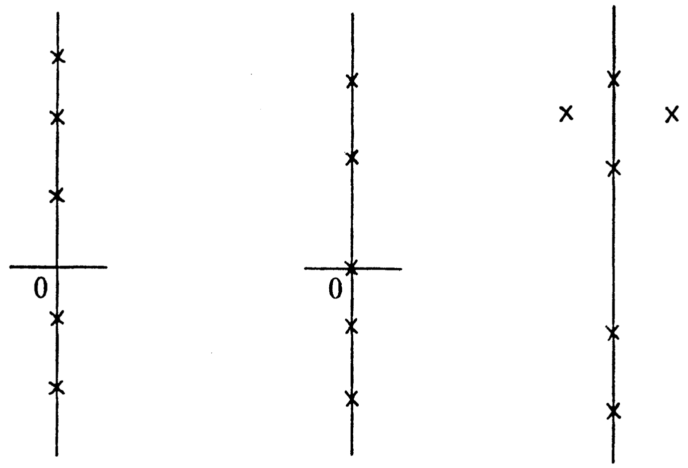
Second step. To complete the proof it is sufficient to show that $\lambda\notin\{\lambda_l\}_{l\neq 0}$ belongs to the resolvent set. For this purpose we consider the following operator:

$$A_0=\begin{pmatrix}0 & I\\ -\sigma r_0^{-3}H^{-1}\partial^3/\partial\theta^3 & 0\end{pmatrix}.$$

Then it is easily verified that A_0 is a bounded operator from $\dot{H}^3(S^1)\times\dot{H}^{3/2}(S^1)$ onto E and that A_0^{-1} and $(A-A_0)A_0^{-1}$ are compact operators in E. If $\lambda\notin\{\lambda_l\}_{l\neq 0}$, then $\lambda-A$ is injective. We can rewrite it as $\lambda-A=\{\lambda A_0^{-1}-(A-A_0)A_0^{-1}-I\}A_0$. The operator inside $\{\ \}$ is a sum of a compact operator and the identity in E.

Further, it is injective. Hence, by the Riesz-Schauder theory, it must be an isomorphism from E onto itself. Consequently $(\lambda-A)^{-1}$ is a bounded operator.
Q. E. D.

The position of the eigenvalues is illustrated in Fig. II.



$0<a<a^*$ $a=a^*$ $a^{**}<a$

Figure II.

§ 5. Linearization with respect to γ_0.

In this section we derive a linearized operator with respect to the initial value γ_0. To describe it we need the following function space:

$$\dot{C}^{5+\alpha}(S^1)=\left\{\eta\in C^{5+\alpha}(S^1) : \int_0^{2\pi}\eta(\theta)d\theta=0\right\}.$$

Note that $\dot{C}^{5+\alpha}(S^1)$ can be regarded as the tangent space of M at $\gamma_0=r_0$.

THEOREM 5.1. *F is a C^1-mapping with respect to (γ_0, u). We have*

$$D_{\gamma_0}F(r_0, 0)\eta=-r_0HL\eta=L(-r_0H\eta) \qquad (\eta\in\dot{C}^{5+\alpha}(S^1)). \tag{5.1}$$

PROOF. We only prove the equality (5.1). For $u=0$ we have $\gamma_u(t, \theta)\equiv\gamma_0(\theta)$, $V_u\equiv V_0$, $q_u\equiv 0$. Hence it holds that

$$F(\gamma_0, 0)=\frac{\partial}{\partial\theta}\left(\left(-\frac{1}{2}|\nabla V_0|^2+\frac{g}{r}\right)\Big|_{\gamma_0}-\sigma K_{\gamma_0}\right). \tag{5.2}$$

We put the right hand side $(\partial/\partial\theta)F^*(\gamma_0)$. Then, by the formula in Okamoto [8] or [9], we obtain

$$D_{r_0}F^*(r_0)\eta = -\frac{a}{r_0 \log r_0}\Big(\frac{\partial U}{\partial r}\Big|_{r=r_0} - \frac{a\eta}{r_0^2 \log r_0}\Big) - \frac{g}{r_0^2}\eta + \frac{\sigma}{r_0^2}(\eta+\eta''),$$

where U is the solution of

$$\begin{cases} \Delta U = 0 & \text{in } 1<r<r_0, \\ U = 0 \quad \text{on } r=1, & U = -\dfrac{a\eta}{r_0 \log r_0} \quad \text{on } r=r_0. \end{cases}$$

From this we easily see that

$$\begin{aligned} D_{r_0}F(r_0, 0)\eta &= \frac{a^2}{(r_0 \log r_0)^2}\sum_{n\neq 0}\frac{in^2}{r_0}R_n\eta_n e^{in\theta} - \frac{g}{r_0^2}\sum_{n\neq 0} in\eta_n e^{in\theta} \\ &\quad + \frac{a^2}{r_0^3(\log r_0)^2}\sum_{n\neq 0} in\eta_n e^{in\theta} + \frac{\sigma}{r_0^2}\sum_{n\neq 0}(in - in^3)\eta_n e^{in\theta} \\ &= r_0 b^2 H\frac{\partial^2\eta}{\partial\theta^2} + \frac{\sigma - g}{r_0^2}\frac{\partial\eta}{\partial\theta} + \frac{\sigma}{r_0^2}\frac{\partial^3\eta}{\partial\theta^3} + r_0 b^2\frac{\partial\eta}{\partial\theta} \\ &= -r_0 HL\eta. \end{aligned}$$

Therefore the proof is completed. Q. E. D.

§6. Instability of the trivial solution.

In this section we will show that the trivial solution is unstable when $a > a^{**} \equiv \min_{n\geq 1}(1+nR_n)^{1/2}a_n$. To be more precise, we make the following

DEFINITION 6.1. The trivial solution is called δ-stable if the condition below is satisfied. There is a constant $\delta \in (0, 1]$ such that the solution exists globally and uniquely and satisfies

$$\sup_{0<t<\infty}\sum_{j=0}^{3}\Big\|\frac{\partial^j}{\partial t^j}(\gamma_u - r_0)(t)\Big\|_{5-j+\alpha} < \varepsilon$$

provided $\|\gamma_0 - r_0\|_{5+\alpha} < \delta\varepsilon$, $\gamma_0 \in M$ for sufficiently small ε.

Our goal is to show

THEOREM 6.1. *Suppose that $a > a^{**}$. Then the trivial solution is not δ-stable for any δ.*

PROOF. The proof is carried out by showing a contradiction. So, assume that $a > a^{**}$ and that the trivial solution is δ-stable. Let δ and ε be as in Definition 6.1. By the definition of the Fréchet derivative, we have

$$\|F(\gamma_0, w) - F(r_0, 0) - D_u F(r_0, 0)w - D_{r_0}F(r_0, 0)\eta\|_Y \leq c\|w\|_X^2 + c\|\eta\|_{5+\alpha}^2 \tag{6.1}$$

for sufficiently small $w \in X$ and $\eta = \gamma_0 - \frac{1}{2\pi}\int_0^{2\pi} \gamma_0 \in \dot{C}^{5+\alpha}(S^1)$. Here the constant c is independent of w, η and T (see Remark 3.1). Since $F(r_0, 0)=0$ and $F(\gamma_0, u)=0$, we obtain

$$\|D_u F(r_0, 0)u + D_{\gamma_0} F(r_0, 0)\eta\|_Y \leqq c\|u\|_X^2 + c\|\eta\|_{5+\alpha}^2.$$

Putting $v = u - r_0 H\eta$, this inequality is rewritten as

$$\left\|\frac{\partial^2 v}{\partial t^2} - 2b\frac{\partial^2 v}{\partial t \partial \theta} - Lv\right\|_Y \leqq c\|u\|_X^2 + c\|\eta\|_{5+\alpha}^2. \tag{6.2}$$

Since $a > a^{**}$, we have for some n and $(w_n, y_n) \in E$

$$\lambda_n w_n = y_n, \qquad \lambda_n y_n = Lw_n + 2b\frac{\partial}{\partial \theta} y_n, \qquad \mathrm{Re}(\lambda_n) > 0.$$

Putting $\eta = \delta\varepsilon y_n$, we have

$$\begin{aligned}\left(\frac{\partial^2 v}{\partial t^2} - 2b\frac{\partial^2 v}{\partial t\partial\theta} - Lv,\ y_n\right)_{L^2(S^1)} &= \frac{d}{dt}\left(\begin{pmatrix} v \\ v_t\end{pmatrix}, \begin{pmatrix} w_n \\ y_n\end{pmatrix}\right) - \left(A\begin{pmatrix} v \\ v_t\end{pmatrix}, \begin{pmatrix} w_n \\ y_n\end{pmatrix}\right) \\ &= \frac{d}{dt}\left(\begin{pmatrix} v \\ v_t\end{pmatrix}, \begin{pmatrix} w_n \\ y_n\end{pmatrix}\right) - \lambda_n\left(\begin{pmatrix} v \\ v_t\end{pmatrix}, \begin{pmatrix} w_n \\ y_n\end{pmatrix}\right).\end{aligned}$$

Let $\phi(t) = e^{-\lambda_n t}\left(\begin{pmatrix} v \\ v_t\end{pmatrix}, \begin{pmatrix} w_n \\ y_n\end{pmatrix}\right)$. Then we obtain

$$\left|e^{\lambda_n t}\frac{d\phi}{dt}\right| \leqq c\|u\|_X^2 + c\delta^2\varepsilon^2,$$

hence

$$|\phi(t) - \phi(0)| \leqq \frac{\{c\|u\|_X^2 + c\delta^2\varepsilon^2\}}{\mathrm{Re}\,\lambda_n} \leqq c'\|u\|_X^2 + c'\delta^2\varepsilon^2.$$

Since

$$\mathrm{Re}\,\phi(0) = \left(\begin{pmatrix} -r_0 H\delta\varepsilon y_n \\ 0\end{pmatrix}, \begin{pmatrix} w_n \\ y_n\end{pmatrix}\right) \equiv d\delta\varepsilon$$

is different from zero, we may assume that $\mathrm{Re}\,\phi(0) > 0$. Then

$$\mathrm{Re}\,\phi(t) \geqq \mathrm{Re}\,\phi(0) - c'\|u\|_X^2 - c'\delta^2\varepsilon^2 \geqq (d - c'\delta\varepsilon)\delta\varepsilon - c'\varepsilon^2.$$

If $\varepsilon > 0$ is sufficiently small so that $(d - c'\delta\varepsilon)\delta\varepsilon - c'\varepsilon^2 > d\delta\varepsilon/2$, then we have

$$\mathrm{Re}\left(\begin{pmatrix} v \\ v_t\end{pmatrix}(t), \begin{pmatrix} w_n \\ y_n\end{pmatrix}\right) \geqq \frac{1}{2} d\delta\varepsilon e^{\mathrm{Re}\,\lambda_n t}.$$

Since $\mathrm{Re}\,\lambda_n > 0$, this inequality contradicts the stability assumption. Therefore we have completed the proof. Q. E. D.

§ 7. Stability analysis via the small disturbance approximation.

In this section we consider the stability of the trivial solution using a "small disturbance approximation". This hydrodynamical technique is a most convenient method used in the water wave theory (see, e. g., [**1**, **3**, **5**, **6**, **11**, **12**]). The important feature of this approximation is to transform the original free boundary problem to a problem in a fixed domain. Our approximate equation is written as follows. We first assume that the difference $\gamma(t, \theta)-r_0$ is small, whence the domain $\Omega_{\gamma(t)}$ is replaced by $\Omega_0=\{1<r<r_0\}$. V and P are functions defined in $[0, \infty)\times\Omega_0$, which satisfy

$$\Delta V=0 \quad \text{in} \quad 0\leqq t<\infty,\ (r, \theta)\in\Omega_0, \tag{7.1}$$

$$V=0 \quad \text{on} \quad 0\leqq t<\infty,\ r=1, \tag{7.2}$$

$$\frac{1}{r}\frac{\partial^2 V}{\partial t\partial\theta}+\frac{\partial}{\partial r}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \quad \text{in} \quad [0, \infty)\times\Omega_0, \tag{7.3}$$

$$-\frac{\partial^2 V^2}{\partial t\partial r}+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \quad \text{in} \quad [0, \infty)\times\Omega_0. \tag{7.4}$$

We put $\gamma=r_0+\eta(t, \theta)$. We assume that η is small. Hence η^2, $(\partial\eta/\partial\theta)^2$, etc. are neglected. By the boundary condition (1.3) we have

$$\frac{\partial V}{\partial\theta}+\frac{\partial V}{\partial r}\frac{\partial\eta}{\partial\theta}=r_0\frac{\partial\eta}{\partial t} \quad \text{on} \quad 0\leqq t<\infty,\ r=r_0,\ \theta\in S^1. \tag{7.5}$$

Observe that

$$K_{\gamma(t)}=\frac{(r_0+\eta)^2+2(\eta_\theta)^2-(r_0+\eta)\eta_{\theta\theta}}{\{(r_0+\eta)^2+(\eta_\theta)^2\}^{3/2}}.$$

Therefore we approximate the Laplace equation (1.6) by

$$P=\frac{\sigma}{r_0}\left(1-\frac{\eta}{r_0}-\frac{\eta_{\theta\theta}}{r_0}\right)+\text{constant} \quad \text{on} \quad 0\leqq t<\infty,\ r=r_0. \tag{7.6}$$

Now our approximate problem is to find functions V and P defined on $[0, \infty)\times\Omega_0$, η defined on $[0, \infty)\times S^1$ satisfying (7.1-7.6). If η is determined, then $r_0+\eta$ represents the " free boundary ".

In order to solve the equations above we eliminate η from (7.5) and (7.6). To this end we seek a solution V in the following form:

$$V=\sum_{n\neq 0} f_n(t)g_n(r)e^{in\theta}+f_0(t)g_0(r).$$

By (7.1, 7.2) the functions $g_n(r)$ must be of the form

$$g_n(r) = r^n - r^{-n} \qquad (n \neq 0),$$

$$g_0(r) = \log r$$

or their constant multiples. On the other hand, the invariance of the circulation (the Kelvin theorem) yields that $f_0'(t) \equiv 0$. Indeed, this is also checked by requiring that (7.3) and (7.4) must be satisfied by some single-valued function P. Now we have

$$V = \sum_{n \neq 0} f_n(t)(r^n - r^{-n})e^{in\theta} + \frac{a}{\log r_0} \log r\,. \tag{7.7}$$

The unknowns are $f_n(t)$ $(n = \pm 1, \pm 2, \cdots)$. We denote by W the first term of the right hand side. Since we assume that the disturbance is small, we assume that W is small (i.e., all $f_n(t)$ are small). This assumption enables us to approximate $|\nabla V|^2$ by

$$\begin{aligned}\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \theta}\right)^2 &= \left(\frac{\partial W}{\partial r} + \frac{a}{\log r_0}\frac{1}{r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 \\ &\simeq \frac{2a}{\log r_0}\frac{1}{r}\frac{\partial W}{\partial r} + \left(\frac{a}{\log r_0}\frac{1}{r}\right)^2.\end{aligned} \tag{7.8}$$

The equations (7.3) and (7.4) yield

$$P + \frac{1}{2}|\nabla V|^2 - \frac{g}{r} = -\sum_{n \neq 0} f_n'(t)(r^n + r^{-n})ie^{in\theta} + \text{constant}. \tag{7.9}$$

By (7.8) and (7.9) we obtain

$$\begin{aligned}P = \frac{g}{r} - \sum_{n \neq 0} f_n'(t)(r^n + r^{-n})ie^{in\theta} - \frac{1}{2}\left(\frac{a}{\log r_0}\right)^2 r^{-2} \\ - \frac{a}{\log r_0}\frac{1}{r}\frac{\partial W}{\partial r} + \text{constant} \qquad \text{in } [0, \infty) \times \Omega_0.\end{aligned} \tag{7.10}$$

On the boundary $r = r_0$, we replace $1/r$ by $(1 - \eta/r_0)/r_0$. Then we have

$$\begin{aligned}P(t, r_0, \theta) = \frac{g}{r_0}\left(1 - \frac{\eta}{r_0}\right) - \sum_{n \neq 0} f_n'(t)(r_0^n + r_0^{-n})ie^{in\theta} \\ - \frac{1}{2}\left(\frac{a}{r_0 \log r_0}\right)^2\left(1 - \frac{2\eta}{r_0}\right) - \frac{a}{r_0^2 \log r_0}\sum_{n \neq 0} f_n(t)(r_0^n + r_0^{-n})ne^{in\theta} \\ + \text{constant} \qquad \text{on } [0, \infty) \times S^1.\end{aligned} \tag{7.11}$$

To eliminate η we now rewrite (7.5) as follows:

$$\left(r_0 \frac{\partial}{\partial t} - \frac{a}{r_0 \log r_0}\frac{\partial}{\partial \theta}\right)\eta = \sum_{n \neq 0} f_n(t)(r_0^n - r_0^{-n})ine^{in\theta}. \tag{7.12}$$

Eliminating P from (7.6) and (7.11), applying the operator in the left hand side of (7.12), we obtain the following equation which is represented only by $f_n(t)$'s:

$$(7.13)\qquad -\frac{\sigma}{r_0^2}\Big(1+\frac{\partial^2}{\partial\theta^2}\Big)\sum_{n\neq 0} f_n(t)(r_0^n-r_0^{-n})ine^{in\theta}$$

$$=\Big(-\frac{g}{r_0^2}+\frac{a^2}{r_0^3(\log r_0)^2}\Big)\sum_{n\neq 0} f_n(t)(r_0^n-r_0^{-n})ine^{in\theta}$$

$$-r_0\sum_{n\neq 0} f_n''(t)(r_0^n+r_0^{-n})e^{in\theta}-\frac{2a}{r_0\log r_0}\sum_{n\neq 0} f_n'(t)(r_0^n+r_0^{-n})ne^{in\theta}$$

$$+\frac{a^2}{r_0^3(\log r_0)^2}\sum_{n\neq 0} f_n(t)(r_0^n+r_0^{-n})in^2e^{in\theta}.$$

Therefore for all $n\neq 0$, it holds that

$$(7.14)\qquad f_n''(t)-2bnif_n'(t)+\frac{n}{R_n}\Big(\frac{\sigma(n^2-1)}{r_0^3}+\frac{g}{r_0^2}-b^2(1+nR_n)\Big)f_n(t)=0,$$

where $b=a/(r_0^2\log r_0)$, $R_n=(r_0^n+r_0^{-n})/(r_0^n-r_0^{-n})$. Now we have reached the following theorem.

Theorem 7.1. *In the linearized sense, the trivial solution is exponentially unstable if and only if*

$$a>\min\{(1+nR_n)^{1/2}a_n;\ n\in\boldsymbol{N}\}.$$

Proof. By the definition of a_n, the characteristic equation for (7.14) is written as

$$\lambda^2-\frac{2ani\lambda}{r_0^2\log r_0}+\frac{n}{R_n}\,\frac{a_{|n|}^2-a^2}{r_0^4(\log r_0)^2}(1+nR_n)=0.$$

Hence some of the real parts of the roots of these equations become positive if and only if $a>\min\{(1+nR_n)^{1/2}a_n;\ n\in\boldsymbol{N}\}$. Q. E. D.

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