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Some results on ordered fields

Dedicated to Professor Hirosi Nagao on his 60th birthday

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1. Introduction.

One of the purposes of this paper is to study the conjugacy of involutions (elements of order 2) in the automorphism group Aut(F) of an algebraically closed field F of characteristic 0. How many conjugacy classes of involutions are there in Aut(F)? This problem has its origin in the following question communicated to the author by Prof. H. Yabuki.

"*Let σ be an involutory automorphism of a field E. Let $x_1, \cdots, x_n$ be elements of E such that $x_1\sigma(x_1)+\cdots+x_n\sigma(x_n)=0$. Then can one say $x_1=\cdots=x_n=0$?*"

The answer is not always "yes" unless E is an algebraically closed field of characteristic 0. However in the case E is an algebraically closed field of characteristic 0, the answer is always "yes". This, being immediate from Artin-Schreier theory (see Proposition 2.6), led the author to study involutory automorphisms of an algebraically closed field. Since the characteristic of an algebraically closed field admitting an involutory automorphism is forced to be 0, we may restrict our attention to F. Let τ be an involutory automorphism of F. Then the fixed subfield of τ has codimension 2 and is real closed by virtue of Artin-Schreier theory (see [4] Satz 4). Indeed, this is a one-to-one correspondence between the involutions in Aut(F) and the subfields of F with codimension 2. Furthermore, two involutions are conjugate if and only if corresponding fixed subfields are isomorphic. Thus to study the conjugacy of involutions is equivalent to study the isomorphism problem of real closed fields. For this purpose we shall introduce two invariants of an ordered field K and show several properties concerning those invariants.

In the following, ord(Δ) denotes the order type of a totally ordered set Δ. Then our main results are as follows.

In Section 3, we introduce the term "*order-basis*" of K over its subfield L which is analogous to "transcendency basis", and we prove in Theorem A that ord(X) is independent of the choice of an order-basis X of K over L. Then applying Theorem A, we estimate the cardinality of the conjugacy classes of involutions in Aut(F) (see Theorem B).

In Section 4, we introduce the notion "*unifinitely closed subfield*" and then define an equivalence relation $\approx$ in $I_K(L)$, which is the set of all unifinitely closed subfields of K containing a fixed subfield L. Then $I_K(L)^* = I_K(L)/\approx$ admits a total ordering (see Theorem C).

In Section 5, we show the existence of "*a unifinite closure*" (see Definition 5.1) of an arbitrary subfield in a real closed field (see Theorem D). Finally, assuming K to be real closed, we prove in Theorem E that $\mathrm{ord}(I_K(L)^*) = \mathrm{ord}(\tilde{X})$ with the set $\tilde{X}$ of all segments (see Section 2) of an order-basis X.

Throughout this paper, an isomorphism between ordered fields implies order preserving.

The main tool in this paper is Artin-Schreier theory which is originated by E. Artin and O. Schreier [**1**], [**2**], [**3**], [**4**]. There are good textbooks containing the theory [**5**], [**7**], [**8**], [**9**]. Therefore we use the basic results of the theory without reference.

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2. Preliminaries.

In this section we first recall some results on totally ordered sets and then refer to a lexicographic order on a rational function field in several variables over an ordered field. Then we answer the question mentioned in Introduction and finally we prove an interesting fact concerning the generation of an algebraically closed field of characteristic 0.

Let Δ be a totally ordered set. A subset S of Δ is called a *segment* if it satisfies the condition "*whenever* $x \leqq y$ *where* $x \in \Delta$ *and* $y \in S$, *it follows* $x \in S$". In the succeeding, let $\tilde{\Delta}$ denote the set of all segments of Δ. We note that $\tilde{\Delta}$ is a totally ordered set by the inclusion relation and has the initial element $\emptyset$ and the final element Δ. For an element $x \in \Delta$, we set $S(x) = \{y \in \Delta \mid y < x\}$ and $T(x) = \{y \in \Delta \mid y \leqq x\}$. Then $S(x)$ and $T(x)$ are elements of $\tilde{\Delta}$ and $S(x)$ is a predecessor of $T(x)$. Conversely, let T be an element of $\tilde{\Delta}$ with a predecessor S. Then there exists an element x of Δ with $x \in T - S$ and an easy argument shows that $T = T(x)$ and $S = S(x)$. Thus we have the following.

PROPOSITION 2.1. *Let Δ and $\tilde{\Delta}$ be as above. Then an element T of $\tilde{\Delta}$ has a predecessor if and only if $T = T(x)$ for some $x \in \Delta$.*

PROPOSITION 2.2. *Let $P(\tilde{\Delta})$ be the set of elements of $\tilde{\Delta}$ which have predecessors. Then we have* $\mathrm{ord}(\Delta)=\mathrm{ord}(P(\tilde{\Delta}))$.

PROPOSITION 2.3. *Let Δ_1 and Δ_2 be totally ordered sets. Then the following two conditions are equivalent.*

1) $\mathrm{ord}(\Delta_1)=\mathrm{ord}(\Delta_2)$.
2) $\mathrm{ord}(\tilde{\Delta}_1)=\mathrm{ord}(\tilde{\Delta}_2)$.

Next we shall define a lexicographic order on $K[U]$, a polynomial ring in a totally ordered set U with coefficients in an ordered field K. Let f be an element of $K[U]$. Then there exist elements $u_1, u_2, \cdots, u_n$ of U such that f is a polynomial in $u_1, \cdots, u_n$ and $u_1<u_2<\cdots<u_n$. Regarding f as a polynomial in u_n with coefficients in $K[u_1, \cdots, u_{n-1}]$, we define inductively that f is positive if the coefficient of the term of the highest degree is a positive element of $K[u_1, \cdots, u_{n-1}]$. In this way, $K[U]$ admits an ordering, which we call a lexicographic order. Since an order defined on a ring has a unique extension to its field of quotients, $K(U)$ becomes an ordered field.

In the remainder, we consider exclusively a lexicographic order on $K(U)$ and we say that $K(U)$ is *a lexicographically ordered field* (*with respect to* U) for convenience' sake.

It should be noted here that $K(U)$ is not archimedean in the case U is not empty, because $u>x$ holds for any $u\in U$ and $x\in K$.

The following statement is an easy consequence of the above.

PROPOSITION 2.4. *Let K_1 and K_2 be ordered fields and let U and V be totally ordered sets with* $\mathrm{ord}(U)=\mathrm{ord}(V)$. *Suppose there exists a homomorphism ϕ from K_1 to K_2. Then there exists such a homomorphism from $K_1(U)$ to $K_2(V)$ that is an extension of ϕ. Consequently, if K_1 is isomorphic to K_2, then $K_1(U)$ is isomorphic to $K_2(V)$.*

PROPOSITION 2.5. *Let $K(U)$ be a lexicographically ordered field. Suppose U is not empty. Then for each non zero element x of $K(U)$, there exists an element u of U such that $u^n<|x|<u^m$ for suitable integers n and m.*

PROOF. For a non zero polynomial f in $u_1, u_2, \cdots, u_r$, where $u_i\in U$ and $u_1<u_2<\cdots<u_r$, it is easy to get $u_r^s<|f|<u_r^t$ for suitable integers s and t. Thus the conclusion is immediate.

Now we answer the question mentioned in Introduction. If the characteristic of E is $p>0$, then considering the case $x_1=\cdots=x_p=1$, we conclude that the answer is "no". Even in the case that the characteristic of E is 0, we have a counterexample (set $E=\boldsymbol{Q}(\sqrt{2})$ and $x_1=x_2=1$, $x_3=\sqrt{2}$, where $\boldsymbol{Q}$ is a prime subfield of E). However we can prove the following.

PROPOSITION 2.6. *Let σ be an involutory automorphism of an algebraically closed field E of characteristic* 0. *Suppose $x_1, x_2, \cdots, x_n \in E$ satisfy $x_1\sigma(x_1)+x_2\sigma(x_2)+\cdots+x_n\sigma(x_n)=0$. Then we have $x_1=x_2=\cdots=x_n=0$.*

PROOF. Let K be the fixed subfield of σ. Then by Artin-Schreier theory, K is real closed and $E=K(\sqrt{-1})$. Let $x_i=a_i+b_i\sqrt{-1}$, where a_i and b_i are elements of K. From the assumption, it follows that $a_1^2+b_1^2+\cdots+a_n^2+b_n^2=0$. But then, since K is formally real, this occurs only in the case $a_1=b_1=\cdots=a_n=b_n=0$, which implies $x_1=x_2=\cdots=x_n=0$. Thus Proposition 2.6 is proved.

Finally, let F be an algebraically closed field of characteristic 0. Then by virtue of Artin-Schreier theory, there exists a subfield K such that F is quadratic over K. Let N_K denote the norm mapping in F over K and set $S_K=\{x\in F \mid N_K(x)=1\}$.

LEMMA 2.7. *Let ψ be an automorphism of F which fixes each element of S_K. Then ψ is the identity.*

PROOF. Let x be a positive element of K with $x<1$. Then, since K is real closed and $1-x^2>0$, there exists an element y of K with $y^2=1-x^2$. Setting $z_1=x+y\sqrt{-1}$, and $z_2=x-y\sqrt{-1}$, we obtain $N_K(z_1)=N_K(z_2)=1$. Hence by the assumption, we get $z_1=\psi(z_1)$ and $z_2=\psi(z_2)$. Therefore $x=(z_1+z_2)/2$ is fixed by ψ. If w is a positive element of K with $w>1$, then $1/w<1$, hence it follows $\psi(1/w)=1/w$. Thus we have $\psi(w)=w$. Therefore each positive element of K is fixed by ψ. Thus ψ fixes each element of K. Considering $\sqrt{-1}\in S_K$ and $F=K(\sqrt{-1})$, we conclude that ψ is the identity.

In the sequel, $\boldsymbol{Q}$ denotes a prime subfield of a field of characteristic 0.

PROPOSITION 2.8. $F=\boldsymbol{Q}(S_K)$.

PROOF. Put $E=\boldsymbol{Q}(S_K)$. Assume $F\neq E$. Then F admits a nontrivial E-automorphism, which contradicts Lemma 2.7. Thus $F=E$, which proves the assertion.

Applying Proposition 2.8 in the case F is the algebraic closure of $\boldsymbol{Q}$ and K is the field of real algebraic numbers, we see the following.

COROLLARY 2.9. *An algebraic number is a $\boldsymbol{Q}$-linear combination of algebraic numbers with absolute value* 1.

3. Order-bases.

In this section K denotes an ordered field and L denotes its subfield. According to Artin-Schreier [2], we define several basic terminologies concerning K.

DEFINITION 3.1. Let x be an element of K.

If $|x|>y$ holds for all $y \in L$, then x is said to be *infinitely large with respect to L.*

If $|y|>|x|>0$ holds for every non zero element y of L, then x is said to be *infinitely small with respect to L.*

If x is neither infinitely large nor infinitely small with respect to L, then x is said to be *finite with respect to L.*

If each element of K is finite with respect to L, then K is called *archimedean over L.*

We note that K is archimedean over $\boldsymbol{Q}$ if and only if K is an archimedean ordered field in the usual sense.

The following proposition is an easy consequence of the above definition. Hence the proof will be omitted.

PROPOSITION 3.2. *The following assertions hold.*

1) *For a non zero element x of K, x is infinitely large with respect to L if and only if $1/x$ is infinitely small with respect to L.*

2) *If $x-y$ is infinitely large with respect to L for $x \in K$ and $y \in L$, then x is infinitely large with respect to L.*

3) *If an element x of K is infinitely small with respect to L, then $x+y<z$ holds whenever y and z are elements of L with $y<z$.*

4) *If x and y are elements of K which are infinitely small with respect to L, then $|x|+|y|$ is also infinitely small with respect to L.*

5) *If K is not archimedean over L, then there exists such an element of K that is infinitely large with respect to L.*

6) *K is archimedean over K itself.*

7) *Let M be an intermediate subfield between K and L. If K is archimedean over L, then K is archimedean over M.*

8) *If K is archimedean over L and L is archimedean over its subfield N, then K is archimedean over N.*

In the following, an intermediate subfield M between K and L is said to be *maximal archimedean over L* if it is archimedean over L and no other intermediate subfield containing M is archimedean over L. By Proposition 3.2,8) and Zorn's lemma, the following is immediate.

PROPOSITION 3.3. *There exists a subfield of K which is maximal archimedean over L.*

DEFINITION 3.4. A subset X of positive elements of K is said to be *order-*

independent over L provided that each element x of X is infinitely large with respect to $L(S(X))$, where $S(X)=\{y\in X \mid y<x\}$. If X is order-independent over L and no other subset of K containing X is order-independent over L, then X is called an *order-basis* of K over L.

By Zorn's lemma, the following is easy to prove.

PROPOSITION 3.5. *Let X be order-independent over L. Then there exists an order-basis of K over L that contains X.*

LEMMA 3.6. *If K is algebraic over L, then K is archimedean over L.*

PROOF. Suppose false. Then there exists an element x of K that is infinitely large with respect to L by Proposition 3.2, 5). Let f be the minimal polynomial of x over L, and let n be the degree of f. Then $1=1-f(x)/x^n$ must be infinitely small with respect to L because f is monic, a contradiction. Hence the lemma is proved.

PROPOSITION 3.7. *If a subset X of K is order-independent over L, then X is algebraically independent over L.*

PROOF. By Definition 3.4 and Lemma 3.6, each element x of X is transcendental over $L(S(x))$. Thus the conclusion is obvious.

Let $\{x\}$ be order-independent over L, where $x\in K$. Then by Proposition 3.7, $L[x]$ is identified with a polynomial ring in x with coefficients in L. Since x is positive and infinitely large with respect to L, we conclude that for each polynomial $f\in L[x]$, f is positive if and only if the leading coefficient of f is positive. Thus the restriction of the order of K to $L[x]$ coincides with a lexicographic order on $L[x]$. Hence a subfield $L(x)$ of K can be identified with a lexicographically ordered field $L(x)$ with respect to x. Inductively for a finite subset X which is order-independent over L, $L(X)$ is identified with a lexicographically ordered field with respect to X. Thus, for any subset which is order-independent over L, we have the same conclusion.

We summarize the above as the following.

PROPOSITION 3.8. *Let X be a subset of K which is order-independent over L. If a totally ordered set U has the same order type as X, then $L(X)$ is isomorphic to a lexicographically ordered field $L(U)$ with respect to U.*

COROLLARY 3.9. *Let $X\subseteq K$ be order-independent over L. Suppose K is algebraic over $L(X)$. Then X is an order-basis of K over L.*

COROLLARY 3.10. *Let X be an order-basis of K over L. Then K is archimedean over $L(X)$.*

The converse of Corollary 3.10 is false. For example, let $K=\boldsymbol{Q}(u_1, u_2)$ be a lexicographically ordered field with respect to $\{u_1, u_2\}$, where $u_1<u_2$. Then it follows from Proposition 2.5 that each element of K is finite with respect to $\boldsymbol{Q}(u_2)$. Therefore K is archimedean over $\boldsymbol{Q}(u_2)$. However $\{u_2\}$ is not an order-basis of K over $\boldsymbol{Q}$.

It should be noted here that K is archimedean over $\boldsymbol{Q}$ if and only if the empty set $\emptyset$ is an order-basis of K over $\boldsymbol{Q}$.

LEMMA 3.11. *Let X be an order-basis of K over L. If $X\neq\emptyset$, then for any non zero element y of K, there exist an element x of X and suitable integers n, m such that $x^n<|y|<x^m$.*

PROOF. By Corollary 3.10, K is archimedean over $L(X)$. It follows that there exist elements z and w of $L(X)$ such that $0<z<|y|<w$. But then by virtue of Proposition 2.5 and Proposition 3.8, there exist elements x_1 and x_2 of X such that $x_1^r<z<|y|<w<x_2^s$, where r and s are suitable integers. Put $x=\max\{x_1, x_2\}$. Then we can find integers n and m such that $x^n<x_1^r$ and $x_2^s<x^m$, which implies Lemma 3.11.

LEMMA 3.12. *Let X be an order-basis of K over L. Let y be a positive element of K which is infinitely large with respect to L. Then there exists a unique element x of X such that $x<y^m<x^n$ for suitable positive integers n and m.*

PROOF. Set $S=\{z\in X|z<y\}$. First assume that y is infinitely large with respect to $L(S)$. Then it follows that $S\cup\{y\}$ is order-independent over L. On the other hand, since $X\cup\{y\}$ cannot be order-independent over L, $X-S$ must contain the minimal element x. Furthermore x is not infinitely large with respect to $L(S\cup\{y\})$. Hence we conclude that $x<y^n<x^n$ for a suitable positive integer n by Lemma 3.11.

Next assume that y is not infinitely large with respect to $L(S)$. Then there exists some element z of S such that $y<z^r$ for a suitable positive integer r by Lemma 3.11. Hence we have $z<y<z^r$. Thus the existence part of the assertion is verified.

To show the uniqueness of x, suppose x_1 and x_2 of X satisfy $x_1<y^m<x_1^n$ and $x_2<y^r<x_2^s$. Then it follows $x_1^r<x_2^{sm}$ and $x_2^m<x_1^{nr}$, which implies $x_1=x_2$. This completes the proof of Lemma 3.12.

THEOREM A. *Let X and Y be order-bases of K over L. Then* $\mathrm{ord}(X)=\mathrm{ord}(Y)$.

PROOF. If $X=\emptyset$, then by Corollary 3.10, K is archimedean over L. It follows that Y is also empty. Similarly $Y=\emptyset$ implies $X=\emptyset$. Therefore in the following, we may assume that neither X nor Y is empty. By Lemma 3.12, for each element of Y we can assign an element of X. Denote this mapping from Y to X by τ. Similarly we can define a mapping ρ from X to Y. Let y_1 and y_2 be elements of Y with $y_1<y_2$. Then by the definition of τ and Lemma 3.12, we have $\tau(y_1)<y_1^m<\tau(y_1)^n$ and $\tau(y_2)<y_2^r<\tau(y_2)^s$ for suitable positive integers m, n, r, s. This implies $\tau(y_1)<\tau(y_2)$. Therefore τ preserves the order.

Next considering $y_1^m<\tau(y_1)^n<y_1^{mn}$, we get $\rho\tau(y_1)=y_1$. Hence $\rho\tau$ is the identity of Y. Similarly $\tau\rho$ is the identity of X. Thus we conclude that τ is an order isomorphism from Y onto X. This proves Theorem A.

REMARK 3.13. Let K_1 and K_2 be ordered fields, and let X_1 (resp. X_2) be an order-basis of K_1 (resp. K_2) over its prime subfield $\boldsymbol{Q}$. If K_1 is isomorphic to K_2, then by Theorem A we have $\mathrm{ord}(X_1)=\mathrm{ord}(X_2)$. We should remark that the converse of this fact is not true even for real closed fields. For example, let E_i be a real closure of $\boldsymbol{Q}(u_i)$, $i=1, 2$, where u_1 and u_2 are real numbers which are algebraically independent over $\boldsymbol{Q}$. Then E_i's are archimedean over $\boldsymbol{Q}$. Hence $\emptyset$ is an order-basis of E_i over $\boldsymbol{Q}$, $i=1, 2$. However, E_1 and E_2 are not isomorphic (see for example, [**6**], 546).

Now we are ready to prove Theorem B which is a generalization of Satz 11 in [**2**].

THEOREM B. *Let F be an algebraically closed field of characteristic* 0 *and let α, β, γ and 2^β be as follows,*

α = *the cardinality of the conjugacy classes of involutions in* Aut(F),

β = *the transcendency degree of F over $\boldsymbol{Q}$,*

γ = *the cardinality immediately after β,*

2^β = *the cardinality of the power set of a set with cardinality β.*

Then we have the following.

1) *If $\beta=0$, then $\alpha=1$.*
2) *If β is finite and $\beta\neq 0$, then α is continuum.*
3) *If β is infinite, then $\gamma\leqq\alpha\leqq 2^\beta$.*

PROOF. If $\beta=0$, then Artin had shown $\alpha=1$ (see Satz in [**1**], 323).

If β is finite and $\beta\neq 0$, then since F is countable, it follows that α cannot exceed the cardinality of continuum (= the cardinality of the power set of F). On the other hand, let $x_1, \cdots, x_\beta$ be a transcendency basis of F over $\boldsymbol{Q}$ and let Y be a transcendency basis of $\boldsymbol{R}$ (= the field of real numbers) over $\boldsymbol{Q}$. Then for elements $y_1, \cdots, y_\beta$ of Y, we have a purely algebraic isomorphism from $\boldsymbol{Q}(y_1, \cdots, y_\beta)$ onto $\boldsymbol{Q}(x_1, \cdots, x_\beta)$. Hence $\boldsymbol{Q}(x_1, \cdots, x_\beta)$ admits an ordering induced by this isomorphism. Let K_1 be a real closure of $\boldsymbol{Q}(x_1, \cdots, x_\beta)$ in F and let K_2 be a real closure of $\boldsymbol{Q}(y_1, \cdots, y_\beta)$ in $\boldsymbol{R}$. We note that F is a quadratic extension of K_1. By Theorem 8 in Chap. VI of [8], K_1 is isomorphic to K_2. Since distinct choices of β elements as a subset from Y yield non isomorphic real closed fields, we conclude that α is at least the cardinality of continuum. This proves 2).

Finally assume β is infinite. In this case cardinality of F coincides with β. It follows that $\alpha \leqq 2^{\beta}$. To show another inequality let X be a transcendency basis of F over $\boldsymbol{Q}$. If X is totally ordered by some means, then we obtain a lexicographically ordered field $\boldsymbol{Q}(X)$. But then there exists a real closure K of $\boldsymbol{Q}(X)$ in F (note that F is quadratic over K). By Corollary 3.9, X is an order-basis of K over $\boldsymbol{Q}$. Hence it follows from Theorem A that distinct type of orderings of X yield non isomorphic real closures of $\boldsymbol{Q}(X)$ in F. Therefore we get $\gamma \leqq \alpha$, because there exist at least γ types of well orderings of X. This implies 3). Thus the proof of the theorem is completed.

4. Unifinitely closed subfields.

In this section K also denotes an ordered field.

DEFINITION 4.1. Let L be a subfield of K and let x be an element of K.

If $x-y$ is finite with respect to L for all $y \in L$, then x is said to be *unifinite with respect to L*.

If L contains every element of K that is unifinite with respect to L, then L is said to be *unifinitely closed in K*.

The following is an easy consequence of the above definition.

PROPOSITION 4.2. 1) *If an element x of K is unifinite with respect to L, then x is finite with respect to L.*

2) *If K is archimedean over L, then each element of K is unifinite with respect to L.*

REMARK 4.3. 1) The converse statement of Proposition 4.2, 1) is false. To see this, let $\boldsymbol{Q}(u)$ be a lexicographically ordered field with respect to u. Then $2+1/u$ is finite with respect to $\boldsymbol{Q}$, but it is not unifinite with respect to $\boldsymbol{Q}$.

2) Let $K=\boldsymbol{Q}(\sqrt{2})$. Then K admits a usual ordering induced by the order of $\boldsymbol{R}$. Set $K(u)$ be a lexicographically ordered field with respect to u. Put $x=\sqrt{2}+1/u$. Then x is unifinite with respect to $\boldsymbol{Q}$ but x^2 is not unifinite with respect to $\boldsymbol{Q}$, because x^2-2 is infinitely small with respect to $\boldsymbol{Q}$. This implies that the assumption that x is unifinite with respect to a subfield L of K is not sufficient for $L(x)$ to be archimedean over L.

3) Let L be a subfield of K and let M be an intermediate subfield between K and L which is archimedean over L. Suppose M is unifinitely closed in K.

Then by Proposition 3.2, 7) and Proposition 4.2, 2), it is easy to show that M is maximal archimedean over L. Conversely, let N be an intermediate subfield between K and L which is maximal archimedean over L. Then under the assumption that K is real closed, we can show that N is unifinitely closed in K (see Section 5). However, the conclusion is not true without real closedness of K (see Remark 4.9).

LEMMA 4.4. *Let x be an element of K and let y_1 and y_2 be elements of a subfield L of K. Suppose $x-y_i$ is 0 or infinitely small with respect to L, $i=1, 2$. Then $y_1=y_2$.*

PROOF. Consider the inequality,

$$|y_1-y_2| \leqq |y_1-x|+|x-y_2|.$$

Since the right hand side is 0 or infinitely small with respect to L by Proposition 3.2, 4), it follows that y_1-y_2 is 0 or infinitely small with respect to L, whereas y_1-y_2 is contained in L. This occurs only in the case $y_1=y_2$. This proves the lemma.

LEMMA 4.5. *Let L and M be subfields of K and let ϕ be a mapping from L to M. Suppose $x-\phi(x)$ is 0 or infinitely small with respect to M. Then ϕ is an isomorphism from L into M.*

PROOF. Let x and y be elements of L. Then,

$$|xy-\phi(x)\phi(y)| \leqq |x(y-\phi(y))|+|\phi(y)(x-\phi(x))|.$$

We note here that x is not infinitely large with respect to M, because,

$$|x| \leqq |x-\phi(x)|+|\phi(x)| \leqq 2|\phi(x)|.$$

Therefore the right hand side of the first inequality is 0 or infinitely small with respect to M. Hence $xy-\phi(x)\phi(y)$ is 0 or infinitely small with respect to M. But then by Lemma 4.4, we get $\phi(xy)=\phi(x)\phi(y)$. Similarly we obtain $\phi(x+y)=\phi(x)+\phi(y)$.

Next assume $x\leqq y$. If $\phi(x)>\phi(y)$, then by Proposition 3.2, 3), we have,

$$\phi(x) > \phi(y)+(y-\phi(y))+(\phi(x)-x).$$

Then it follows,

$$y-x = y-\phi(y)+\phi(y)-\phi(x)+\phi(x)-x < 0,$$

a contradiction. Hence we get $\phi(x)\leqq\phi(y)$. This implies that ϕ is an isomorphism from L into M.

By Lemma 4.4, the following is immediate.

LEMMA 4.6. *A mapping which satisfies the condition of Lemma 4.5 is determined uniquely for given subfields L and M of K, if it exists.*

DEFINITION 4.7. The isomorphism from L into M which satisfies the condition of Lemma 4.5 will be called *the pseudo-identity* from L into M.

PROPOSITION 4.8. *Let L and M be unifinitely closed subfields of K. Then one of the following holds.*

1) *There exists the pseudo-identity from L into M.*
2) *There exists the pseudo-identity from M into L.*

PROOF. Suppose false. If, for each element x of L, there exists an element y of M such that $x-y$ is 0 or infinitely small with respect to M, then by Lemma 4.5, there exists a pseudo-identity from L into M, a contradiction. Hence we conclude that there exists an element z of L such that $z-y$ is neither 0 nor infinitely small with respect to M for any element y of M. It follows that z is not contained in M. Since M is unifinitely closed, z is not unifinite with respect to M. But then $z-y$ must be infinitely large with respect to M for some $y \in M$ by Definition 4.1. Therefore z is infinitely large with respect to M by Proposition 3.2, 2). Exchanging the role of L and M in the above argument, we can find an element w of M which is infinitely large with respect to L. However this yields $|w|>|z|$, which contradicts the choice of z. This proves Proposition 4.8.

REMARK 4.9. Proposition 4.8 suggests the importance of the notion of "unifiniteness". For example we show that unifinitely closed subfields in Proposition 4.8 cannot be replaced by such subfields that satisfy the following condition (*).

(*) *Whenever $L(x)$ is archimedean over L for $x \in K$, it follows $x \in L$.*

We should note here that a subfield L of K satisfies (*) if and only if L is maximal archimedean over L. We first show that a unifinitely closed subfield L satisfies (*). Suppose $L(x)$ is archimedean over L for $x \in K$. Then by Proposition 4.2, x is unifinite with respect to L. Hence we have $x \in L$. Thus L satisfies (*). Therefore it is reasonable to ask whether Proposition 4.8 can be extended to subfields which satisfy (*). In the following we shall show that the answer is "no". For this purpose, let w be a real transcendental number and put $E=\boldsymbol{Q}(w)$. Then by the usual ordering, E is an ordered field. Let $K=E(u)$ be a lexicographically ordered field with respect to u. Set $L_n=\boldsymbol{Q}(w^n+1/u)$, where n is a positive integer. Then the following hold (note that 4) and 6) answer the above question negatively, furthermore 2), 4) and 5) imply that L_p is maximal archimedean over $\boldsymbol{Q}$ but is not unifinitely closed in K for a prime p).

1) *If a subfield M of K is archimedean over $\boldsymbol{Q}$, then there exists the pseudo-identity from M into E.*

2) *L_n is archimedean over $\boldsymbol{Q}$.*

3) *Let M be an intermediate subfield between K and L_n, where $n \geqq 2$. If M contains $w+\varepsilon$, where ε is 0 or infinitely small with respect to $\boldsymbol{Q}$, then M is not archimedean over $\boldsymbol{Q}$.*

4) *If p is a prime, then L_p satisfies (*).*

5) *If $n \geqq 2$, then L_n is not unifinitely closed in K.*

6) *If p and q are distinct primes, then there does not exist the pseudo-identity from L_p into L_q, and vice versa.*

PROOF. 1) Since an element x of K is expressed as a quotient of polynomials in u with coefficients in E, we may set,

$$x = (au^r + \cdots)/(u^s + \cdots),$$

where $a \in E$ and r, s are non negative integers. If x is finite with respect to $\boldsymbol{Q}$, then it is easy to show that $r=s$ and $x-a$ is 0 or infinitely small with respect to E. Thus for each element x of M, we can find an element a of E such that $x-a$ is 0 or infinitely small with respect to E. Then by Lemma 4.4 and Lemma 4.5, we conclude that there exists the pseudo-identity from M into E.

2) Suppose false. Then by Proposition 3.2, 1) and 5), there exists such an element z of L_n that is infinitely small with respect to $\boldsymbol{Q}$. On the other hand, since z can be expressed as $z=h(w^n+1/u)$, where h is a rational function with coefficients in $\boldsymbol{Q}$. It follows,

$$h(w^n)-z = h(w^n)-h(w^n+1/u)$$

is infinitely small with respect to $\boldsymbol{Q}$ (since w and u are algebraically independent over $\boldsymbol{Q}$, $h(w^n)-z$ cannot be 0). Thus we see that $h(w^n)$ is also infinitely small with respect to $\boldsymbol{Q}$, contradictory to our choice of w. Therefore 2) is verified.

3) Put $z=w+\varepsilon$. We may assume $\varepsilon \neq 0$. Since ε is an element of K, there exist relatively prime polynomials f and g in w, u with coefficients in $\boldsymbol{Q}$ such that $\varepsilon=f/g$. On the other hand, it is easy to see that $w^n+1/u-z^n$ is 0 or infinitely small with respect to $\boldsymbol{Q}$. Suppose M is archimedean over $\boldsymbol{Q}$, then $w^n+1/u-z^n$ must be 0. Hence we obtain,

$$w^n+1/u = (w+\varepsilon)^n = w^n+nw^{n-1}\varepsilon+\cdots+\varepsilon^n,$$

which yields,

$$g^n = u(nw^{n-1}fg^{n-1}+\cdots+f^n).$$

This implies that f divides g. It follows that f is constant. However, in this case, g^2 cannot divides the right hand side (since ε is infinitely small with respect to $\boldsymbol{Q}$, g cannot be constant, which implies that g is a scalar multiple of u), a contradiction. Thus M is not archimedean over $\boldsymbol{Q}$.

4) Suppose L_p does not satisfy (*). Then there exists an element x of $K-L_p$ such that $L_p(x)$ is archimedean over L_p. By 2), L_p is archimedean over $\boldsymbol{Q}$, hence it follows from Proposition 3.2, 8) that $L_p(x)$ is also archimedean over $\boldsymbol{Q}$. But then by 1), there exists the pseudo-identity ϕ from $L_p(x)$ into E. Now it is easy to show that the image of L_p under ϕ is $\boldsymbol{Q}(w^p)$. On the other hand, since the extension degree of E over $\boldsymbol{Q}(w^p)$ is a prime p, we conclude that ϕ is surjective. Therefore there exists an element z of $L_p(x)$ such that $\phi(z)=w$. Thus we can set $z=w+\varepsilon$, where ε is 0 or infinitely small with respect to E (hence with respect to $\boldsymbol{Q}$). But by virtue of 3), we conclude that $L_p(x)$ is not archimedean over $\boldsymbol{Q}$, a contradiction. So we obtain 4).

5) By virtue of 2) and 3), it suffices to show that w is unifinite with respect to L_n for $n\geqq 2$. Suppose false. Then for some element x of L_n, $w-x$ is either infinitely large or infinitely small with respect to L_n, hence with respect to $\boldsymbol{Q}$. Since w is a real number, we can conclude that $w-x$ is infinitely small with respect to $\boldsymbol{Q}$. But then $w-x$ is also infinitely small with respect to E, because E is archimedean over $\boldsymbol{Q}$. By 1), there exists the pseudo-identity ϕ from L_n into E. We obtain $\phi(x)=w$ from Lemma 4.4. But this contradicts the fact $\phi(L_n)=\boldsymbol{Q}(w^n)$, since $\boldsymbol{Q}(w^n)$ does not contain w in the case $n\geqq 2$. Thus 5) is proved.

6) Suppose there exists the pseudo-identity ϕ from L_p into L_q. Then both

$$\phi(w^p+1/u)-(w^p+1/u) \qquad \text{and} \qquad w^p+1/u-w^p$$

are 0 or infinitely small with respect to $\boldsymbol{Q}$. Therefore we can put $\phi(w^p+1/u)=w^p+\varepsilon$, where ε is 0 or infinitely small with respect to $\boldsymbol{Q}$. Taking integers r, s to satisfy $rp+sq=1$, we assert that

$$(w^p+\varepsilon)^r(w^q+1/u)^s-w$$

is 0 or infinitely small with respect to $\boldsymbol{Q}$, because both p and q are greater than 2. But then by 3), we conclude that L_q is not archimedean over $\boldsymbol{Q}$, which contradicts 2). Hence 6) is proved.

DEFINITION 4.10. Let L and M be subfields of K. If there exists the bijective pseudo-identity from L onto M, then we say L is *pseudo-identical with* M and denote it by $L\approx M$.

LEMMA 4.11. *Let L and M be subfields of K and let ϕ be the pseudo-identity from L into M. Then for each element x of L, the following hold.*

1) *$\phi(x)-x$ is 0 or infinitely small with respect to L.*
2) *$|\phi(x)|<|x|+|y|$ for any non zero element y of L.*

PROOF. 1) If $|x|<|\phi(x)-x|$, then $x\neq 0$ and we get

$$|\phi(x)| \leqq |x|+|\phi(x)-x| \leqq 2|\phi(x)-x|,$$

which contradicts our assumption, because $2|\phi(x)-x|$ is infinitely small with respect to M by Proposition 3.2, 4). Therefore we conclude $|x|\geqq|\phi(x)-x|$. Hence we have

$$|\phi(x)| \leqq |x|+|\phi(x)-x| \leqq 2|x|.$$

Now assume that 1) is false. Then there exists an element z of L such that

$$|\phi(x)-x| > |z| > 0.$$

On the other hand, since $|\phi(z)|\leqq 2|z|$ holds, it follows that

$$|\phi(x)-x| > |z| \geqq |\phi(z)|/2 > 0,$$

a contradiction. This proves 1).

2) For any non zero element y of L, we get $|\phi(x)-x|<|y|$ by 1). Hence it follows

$$|\phi(x)| \leqq |x|+|\phi(x)-x| < |x|+|y|,$$

which proves 2.

PROPOSITION 4.12. *The relation $\approx$ is an equivalence relation in the set of all subfields of K.*

PROOF. Let L, M and N be subfields of K. Since the identity mapping of L is the pseudo-identity from L onto itself, we get $L\approx L$.

Next assume $L\approx M$ and let ϕ be the pseudo-identity from L onto M. We show that the inverse mapping ϕ^{-1} of ϕ is the pseudo-identity from M onto L. To see this, it suffices to show that $y-\phi^{-1}(y)$ is 0 or infinitely small with respect to L for any element y of M. However, since $y=\phi(x)$ for some element x of L, it is already proved in Lemma 4.11, 1). Hence we obtain $M\approx L$.

Finally assume $L\approx M$ and $M\approx N$. Let ϕ and θ be the respective pseudo-identities. Then for each element x of L, we have

$$|x-\theta\phi(x)| \leqq |x-\phi(x)|+|\phi(x)-\theta(\phi(x))|.$$

Thus to prove $L\approx N$, it is sufficient to show that $x-\phi(x)$ is 0 or infinitely small with respect to N. Suppose to the contrary that

$$|x-\phi(x)| \geqq |y| > 0$$

for some element y of N. Then we obtain

$$2|x-\phi(x)| \geqq |y|+|y| \geqq |y|+|\theta^{-1}(y)-y| \geqq |\theta^{-1}(y)|,$$

which is a contradiction. Hence we conclude $L \approx N$ and the proof is completed.

LEMMA 4.13. *Let L and M be subfields of K. Suppose $L \approx M$. Then the following conditions on an element x of K are equivalent.*

1) *x is infinitely small (resp. large) with respect to L.*
2) *x is infinitely small (resp. large) with respect to M.*

PROOF. Let ϕ be the pseudo-identity from L onto M. First assume x is infinitely small with respect to L. For any non zero element y of M, there exists a non zero element z of L such that $\phi(z)=y/2$. Then

$$|x| < |z| \leqq |z-\phi(z)|+|\phi(z)| \leqq 2|\phi(z)| = |y|.$$

Thus we conclude that x is infinitely small with respect to M. Similarly 2) implies 1). Furthermore, by virtue of Proposition 3.2, 1), we can replace the term "small" by the term "large" in the above argument. Hence the lemma is proved.

PROPOSITION 4.14. *Let L and M be subfields of K. If L is unifinitely closed in K and $L \approx M$, then M is also unifinitely closed in K.*

PROOF. Let ϕ be the pseudo-identity from L onto M. Assume to the contrary that an element x of $K-M$ is unifinite with respect to M. Then for each element y of L, there exist u and v of M such that

$$0 < u \leqq |x-\phi(y)| \leqq v,$$

because x is unifinite with respect to M and $x-\phi(y) \neq 0$. Now take elements w and z of L to satisfy $\phi(w)=u/2$ and $\phi(z)=2v$. Then since $u/2-w$ is 0 or infinitely small with respect to M,

$$u-w = u-u/2+u/2-w$$

is positive by virtue of Proposition 3.2, 3). Hence we get $u \geqq w$. Similarly we have $z \geqq v$. So it follows

$$0 < w \leqq |x-\phi(y)| \leqq z.$$

But then

$$\begin{aligned} 0 < w/2 &\leqq |x-\phi(y)|-|\phi(y)-y| \leqq |x-y| \\ &\leqq |x-\phi(y)|+|\phi(y)-y| \leqq 2z, \end{aligned}$$

because $\phi(y)-y$ is 0 or infinitely small with respect to L by Lemma 4.11, 1). Therefore we can conclude that x is unifinite with respect to L, which yields

$x \in L$ by the assumption. Considering $x-\psi(x)$ is 0 or infinitely small with respect to M and x is unifinite with respect to M, we conclude $x=\psi(x)$. This implies $x \in M$, a contradiction. Hence we prove Proposition 4.14.

As a result of Proposition 4.14, if a $\approx$-equivalence class contains a unifinitely closed subfield, then it follows that all elements of this class are unifinitely closed.

From now on, $I_K(L)$ denotes the set of all unifinitely closed subfields of K containing a fixed subfield L. Then the restriction of the relation $\approx$ to $I_K(L)$ is also an equivalence relation in $I_K(L)$. We put $I_K(L)^* = I_K(L)/\approx$ and for each element M of $I_K(L)$, M^* denotes an element of $I_K(L)^*$ which contains M.

In the rest of this section, L is a fixed subfield of K.

LEMMA 4.15. *Let M and N be elements of $I_K(L)$. Suppose M is pseudo-identical to a subfield E of N. Then there exists the pseudo-identity from M into N.*

PROOF. Let ψ be the pseudo-identity from M onto E. Then by Lemma 4.11, 1), $x-\psi^{-1}(x)$ is 0 or infinitely small with respect to M for any $x \in E$. If there exists the pseudo-identity θ from N into M, $x-\theta(x)$ is 0 or infinitely small with respect to M. However, by Lemma 4.4, we get $\theta(x)=\psi^{-1}(x)$. Thus the image of E under θ coincides with M. It follows that $N=E$, because θ is one-to-one. But then this implies that ψ is the pseudo-identity from M onto N, which proves the lemma.

REMARK 4.16. It should be remarked here that in the above lemma, the pseudo-identity from M onto E is not necessarily the pseudo-identity from M into N. For example, let E and K be as in Remark 4.9. Then L_1 is pseudo-identical with E. But the pseudo-identity from L_1 into K must send each element of L_1 to itself by Lemma 4.4.

Now we shall define a total ordering on $I_K(L)^*$. Let M and N be elements of $I_K(L)$. If there exists the pseudo-identity from M into N, then we denote it by $M^* \leqq N^*$. First of all, we must show that the relation $\leqq$ is well defined. Suppose M_1 and N_1 be elements of M^* and N^* respectively. Since $M_1 \approx M$ and $N \approx N_1$, it follows that M_1 is pseudo-identical with a subfield of N_1. Hence by Lemma 4.15, there exists the pseudo-identity from M_1 into N_1. Thus $\leqq$ is well defined.

THEOREM C. *The relation $\leqq$ is a total ordering on $I_K(L)^*$.*

PROOF. Let M, N and E be elements of $I_K(L)$. It is easy to show $M^* \leqq M^*$. Next assume $M^* \leqq N^*$ and $N^* \leqq M^*$. Let ψ (resp. θ) be the pseudo-identity from M (resp. N) into N (resp. M), and let M_1 be the image of the composite of ψ and θ. Then M_1 is a subfield of M which is pseudo-identical with M. But then by Lemma 4.6, the pseudo-identity from M_1 onto M must send each element of M_1 to itself. Hence we obtain $M_1=M$. Thus θ is bijective, which implies $N^*=M^*$.

Finally assume $M^*\leqq N^*$ and $N^*\leqq E^*$. Then M is pseudo-identical with a subfield of E. Therefore by Lemma 4.15, we have $M^*\leqq E^*$. This together with Proposition 4.8, completes the proof of Theorem C.

5. Unifinite closures.

By the preceding results, we can attach $\mathrm{ord}(I_K(\boldsymbol{Q})^*)$ and $\mathrm{ord}(X)$ to each ordered field K, where X is an order-basis of K over $\boldsymbol{Q}$. We note that they are invariant under isomorphisms. The purpose of this section is to show that they are intimately connected with each other in the case K is real closed.

DEFINITION 5.1. Let L be a subfield of K. If an intermediate subfield $\hat{L}$ between K and L satisfies,

1) $\hat{L}$ is unifinitely closed in K,

2) $\hat{L}$ is archimedean over L,

then $\hat{L}$ is called a *unifinite closure* of L in K.

PROPOSITION 5.2. *Let L and M be subfields of K. Suppose there exist unifinite closures $\hat{L}$ and $\hat{M}$ of L and M respectively and $L\approx M$. Then $\hat{L}\approx\hat{M}$.*

PROOF. By Proposition 4.8, we may assume that there exists the pseudo-identity ψ from $\hat{L}$ into $\hat{M}$. So it suffices to show $\psi(\hat{L})=\hat{M}$. Let θ be the pseudo-identity from L onto M. Since $\hat{M}$ is archimedean over M, any element which is infinitely small with respect to M is infinitely small with respect to $\hat{M}$. Hence $x-\theta(x)$ is 0 or infinitely small with respect to $\hat{M}$ for $x\in L$. It follows from Lemma 4.4 that $\theta(x)=\psi(x)$ for each $x\in L$. Thus $\psi(\hat{L})$ contains M. Consequently, we conclude that $\hat{M}$ is archimedean over $\psi(\hat{L})$. On the other hand, by Proposition 4.14, $\psi(\hat{L})$ is unifinitely closed in K. Hence we have $\psi(\hat{L})=\hat{M}$, which proves the assertion.

PROPOSITION 5.3. *Let M be a subring of K which contains a subfield L and let E be the field of fractions of M in K. Suppose each element of M is finite with respect to L. Then E is archimedean over L.*

PROOF. Suppose false. Then by Proposition 3.2, 5), we can choose an element x/y of E which is infinitely large with respect to L, where x and y are elements of M and $y\neq 0$. By the assumption, x and y are finite with respect to L. So there exist elements z and w such that $|x|<|z|$ and $0<|w|<|y|$. It follows $|x/y|<|z/w|$, a contradiction. Thus E is archimedean over L.

PROPOSITION 5.4. *Let X be an order-basis of K over its subfield L. Suppose there exists a unifinite closure M of $L(S)$ in K for a segment S of X. Then S is an order-basis of M over L.*

PROOF. Suppose false. Then there exists a positive element y of M such that $S\cup\{y\}$ is order-independent over L. Since M is archimedean over $L(S)$, an element which is infinitely large with respect to $L(S)$ is also infinitely large with respect to M. So each element of $X-S$ is infinitely large with respect to M. Therefore it can be easily shown that $X\cup\{y\}$ is order-independent over L, which is a contradiction. Thus the lemma is verified.

PROPOSITION 5.5. *Let L be a subfield of K. Suppose L is real closed and is maximal archimedean over L. Then L is unifinitely closed in K.*

PROOF. Suppose false. Then there exists such an element z of $K-L$ that is unifinite with respect to L. If each element of $L[z]$ is finite with respect to L, then by Proposition 5.3, $L(z)$ is archimedean over L, contradictory to our assumption. So there exists an element w of $L[z]$ which is not finite with respect to L. We note that $w=f(z)$ for some $f\in L[u]$, where $L[u]$ is a polynomial ring in u with coefficients in L. Since each term of $f(z)$ cannot be infinitely large with respect to L, we conclude $f(z)$ is infinitely small with respect to L. Set

$$I=\{h\in L[u] \mid h(z) \text{ is } 0 \text{ or infinitely small with respect to } L\}.$$

Then I is a non zero ideal of $L[u]$, because it contains f. Let g be the monic polynomial which generates I. We note that $g(z)\neq 0$, because L is algebraically closed in K. Furthermore, it is easy to see that g is irreducible. Since z is unifinite with respect to L and L is real closed, it follows that the degree of g is 2. Let $x\pm y\sqrt{-1}$ be roots of g in $L(\sqrt{-1})$, where $x, y\in L$ and $y\neq 0$. Then we have $g(u)=(u-x)^2+y^2$, which implies

$$g(z)=(z-x)^2+y^2\geqq y^2>0\,.$$

This contradicts the choice of g. Thus we have proved Proposition 5.5.

In the sequel, K is a real closed field.

THEOREM D. *Let L be a subfield of a real closed field K and M be an extension of L in K which is maximal archimedean over L. Then M is unifinitely closed in K.*

PROOF. By the assumption and Proposition 3.2, 8), there is no archimedean extension of M in K except for M itself. Hence by Lemma 3.6, M is algebraically closed in K. It follows that M is real closed, because K is real closed (see for example, Corollary of Theorem 6 in Chap. VI of [8]). Therefore by Proposition 5.5, the conclusion is obtained.

REMARK 5.6. In the case that K is not real closed, there does not always exist a unifinite closure of L. For example, let K and L_p be as in Remark 4.9, where p is a prime. Then by 4) and 5) in Remark 4.9, we can conclude that there is not a unifinite closure of L_p in K.

To prove Theorem E, we need the following lemma.

LEMMA 5.7. *Let X be an order-basis of K over its subfield L and let E be an extension of L in K. Suppose E is unifinitely closed in K. Then there exists a segment of X such that E is pseudo-identical with a unifinite closure of $L(S)$ in K.*

PROOF. Set

$$S=\{x\in X \mid x<y \text{ for some } y\in E\}.$$

We note that S is a segment of X and each element of $X-S$ is infinitely large with respect to E. By Theorem D, there exists a unifinite closure M of $L(S)$ in K. We shall show that $E\approx M$. By Theorem C, either $E^*\leqq M^*$ or $E^*\geqq M^*$ occurs, where E^* and M^* are as defined in Section 4.

First assume $E^*\leqq M^*$. Let ψ be the pseudo-identity from E into M. Now suppose $\psi(E)\neq M$. By Proposition 4.14, $\psi(E)$ is unifinitely closed in K. So M is not archimedean over $\psi(E)$. Hence there exists such an element y of M that is infinitely large with respect to $\psi(E)$. Both E and M contain L, we have $\psi(L)=L$ by Lemma 4.4. This implies M is not archimedean over L. Hence S is not empty. By Proposition 5.4, S is an order-basis of M over L. So we obtain $|y|<x^n$ for some $x\in S$ and a suitable integer n by virtue of Lemma 3.11. However it asserts that x is infinitely large with respect to $\psi(E)$. But then, Lemma 4.13 implies that x is also infinitely large with respect to E, which contradicts the choice of x. Thus we obtain $E\approx M$ in this case.

Next assume $E^*\geqq M^*$. Let θ be the pseudo-identity from M into E. Again we suppose $\theta(M)\neq E$. Since $\theta(M)$ is unifinitely closed in K, we conclude that E is not archimedean over $\theta(M)$. Hence there exists a positive element z of E which is infinitely large with respect to $\theta(M)$. By Lemma 4.13, z is infinitely large with respect to M. This implies $S\cup\{z\}$ is order-independent over L. On the other hand, since each element of $X-S$ is infinitely large with respect to E, $X\cup\{z\}$ is order-independent over L, a contradiction, which yields $M\approx E$. This completes the proof of the lemma.

THEOREM E. *Let L be a subfield of a real closed field K, and let X be an order-basis of K over L. Then* $\mathrm{ord}(I_K(L)^*)=\mathrm{ord}(\tilde{X})$, *where $\tilde{X}$ is the set of all segments of X.*

PROOF. Let σ be a mapping from $\tilde{X}$ to $I_K(L)^*$, which assigns each segment S of X to a pseudo-identical class containing a unifinite closure of $L(S)$. We shall show that σ is an order isomorphism. By Lemma 5.7, $\sigma(\tilde{X})=I_K(L)^*$. Let S_1 and S_2 be elements of $\tilde{X}$ with $S_1 \subsetneqq S_2$. Then it remains to show that $\sigma(S_1) \lneqq \sigma(S_2)$. Let M_1 and M_2 be unifinite closures of $L(S_1)$ and $L(S_2)$ respectively. Assume to the contrary that $M_1^* \geqq M_2^*$ and let ϕ be the pseudo-identity from M_2 into M_1. Since M_2 contains $L(S_1)$, the restriction of ϕ to $L(S_1)$ coincides with the identity. But then each element of $\phi(S_2-S_1)$ is infinitely large with respect to $L(S_1)$, whereas M_1 is archimedean over $L(S_1)$, a contradiction. Hence we conclude $M_1^* \lneqq M_2^*$. Thus the proof is obtained.

REMARK 5.8. The assumption that K is real closed in Theorem E cannot be removed. For example, let $N=\boldsymbol{Q}(\sqrt{2})$ and let $E=N(u)$ be a lexicographically ordered field with respect to u. Put $K=\boldsymbol{Q}(\sqrt{2}+1/u)$ be a subfield of E. Then K is a purely transcendental extension of $\boldsymbol{Q}$. Since $\sqrt{2}+1/u$ is unifinite with respect to $\boldsymbol{Q}$, $\boldsymbol{Q}$ is not unifinitely closed in K. Suppose a subfield $M \neq \boldsymbol{Q}$ is unifinitely closed in K. Then by Lüroth's theorem, K is algebraic over M. Hence it follows from Lemma 3.6 that $M=K$. Thus we get $I_K(\boldsymbol{Q})=\{K\}$. On the other hand, it is not so difficult to show that an order-basis X of K over $\boldsymbol{Q}$ consists of one element, which implies that $\tilde{X}$ consists of two elements. Therefore the conclusion of Theorem E cannot hold in this case.

References

[1] E. Artin, Kennzeichnung des Körpers der reellen algebraischen Zahlen, Abh. Math. Sem. Univ. Hamburg, **3** (1924), 319-323.
[2] E. Artin and O. Schreier, Algebraische Konstruktion reeller Körper, ibid., **5** (1926), 85-99.
[3] E. Artin, Über die Zerlegung definiter Funktionen in Quadrate, ibid., **5** (1927), 100-115.
[4] E. Artin and O. Schreier, Eine Kennzeichnung der reell abgeschlossenen Körper, ibid., **5** (1927), 225-231.
[5] P.M. Cohn, Algebra, Vol. 2, John Wiley, Chichester - New York - Brisbane - Toronto, 1979.
[6] P. Erdös, L. Gillman and M. Henriksen, An isomorphism theorem for real closed fields, Ann. of Math., **61** (1955), 542-554.
[7] G. Fujisaki, Fields and Galois Theory III (in Japanese), Iwanami, Tokyo, 1978.
[8] N. Jacobson, Lectures in Abstract Algebra III, Van Nostrand, Princeton, 1964.
[9] M. Nagata, Field Theory, Dekker. New York, 1977.

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