J. Math. Soc. Japan
Vol. 38, No. 1, 1986

Differential operators and congruences for Siegel modular forms of degree two

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(Received Oct. 2, 1984)

Introduction.

We study congruences between Siegel modular forms of different weights by using differential operators. As an example, we prove the following congruence between eigenvalues of Hecke operators on $\chi_{20}^{(3)}$ and on $[\Delta_{18}]$:

$$\lambda(m, \chi_{20}^{(3)}) \equiv m^2 \lambda(m, [\Delta_{18}]) \quad \bmod 7, \tag{0.1}$$

which was conjectured in Kurokawa [7]. Here $\chi_{20}^{(3)}$ is the normalized eigen cusp form of degree 2 and weight 20 which does not lie in the image of the Saito-Kurokawa lifting and $[\Delta_{18}]$ is the Eisenstein series of degree 2 and weight 18 characterized as the eigenform satisfying $\Phi[\Delta_{18}]=\Delta_{18}$ where Φ is the Siegel Φ-operator and Δ_{18} is the normalized cusp form of degree 1 and weight 18. Further, $\lambda(m, f)$ is the eigenvalue of the m-th Hecke operator on an eigenform f. For precise definitions of these two forms and some other congruences, see § 4 below.

In Kurokawa [7], congruences of eigenvalues of Hecke operators between lifted eigenforms are proved by using theory of the Saito-Kurokawa lifting and the Eisenstein lifting. Our method is different and is as follows. We denote by $M_k(\Gamma_n)$ (resp. $M_k^\infty(\Gamma_n)$) the $\boldsymbol{C}$-vector space of holomorphic modular forms (resp. C^∞-modular forms) of degree n and weight k. Let δ_k be the differential operator introduced by Maass [9] and modified as in Harris [3, 1.5.3], which sets up a map

$$\delta_k : M_k^\infty(\Gamma_2) \longrightarrow M_{k+2}^\infty(\Gamma_2).$$

However, the differential operator δ_k does not keep holomorphy, so we use holomorphic projection P_k on $M_k^\infty(\Gamma_2)$ defined by Sturm [18, Theorem 1] to obtain information on a holomorphic constituent. For a subring R of $\boldsymbol{C}$, let $M_k(\Gamma_2)_R$ be the R-module of holomorphic modular forms of degree 2 and weight k whose Fourier coefficients belong to R. Assume $(1/2)R \subset R$ in the following. We put $\delta_k^r=\delta_{k+2r-2}\cdots\delta_{k+2}\delta_k$. In Theorem 1.5, we prove a certain congruence modulo $(2w-2r-3)I$ between Fourier coefficients of fg and those of $P_w(\delta_k^p f\cdot\delta_l^q g)$

at the multiples of the unit matrix, where $f\in M_k(\Gamma_2)_R$ and $g\in M_l(\Gamma_2)_R$ with $r=p+q$, $w=k+l+2r$ and I is an ideal of R satisfying $(1/2)I\subset I$ and containing all the Fourier coefficients of g at non-zero matrices. This integrality of an analytically defined map is relevant in our method. (Cf. Remark 1.6.) We denote by $M_w^r(\Gamma_2)_R$ the R-module generated by $\delta_k^p f\cdot\delta_l^q g$ where $f\in M_k(\Gamma_2)_R$ and $g\in M_l(\Gamma_2)_R$ with $r=p+q$ and $w=k+l-2r$. In § 2, we study the condition such that holomorphic projection of an element of $M_w^r(\Gamma_2)_C$ is actually a holomorphic cusp form of weight w. (We note that an element of $M_w^r(\Gamma_2)_C$ is not necessarily of bounded growth in the sense of Sturm [**18**, (6)].) Taking a suitable element of $M_w^r(\Gamma_2)_R$, we obtain congruences modulo $(2w-2r-3)I$ of Fourier coefficients at the multiples of the unit matrix between holomorphic eigenforms f and g of weight w and $w-2r$ respectively. Here, I is an ideal of R depending on f and g. For passage to congruences of eigenvalues of Hecke operator, we study their relation in Proposition 3.3. Our method of proving congruences is gathered in Theorem 3.4. Finally, concrete examples are proved in § 4.

Our results suggest the following. Let $f\in M_k(\Gamma_n)$ and $g\in M_l(\Gamma_n)$ be eigenforms where $k-l$ is a positive even integer. Then under some additional conditions, a suitable divisor d of $k+l-(n+1)$ is likely to provide congruences of type

$$\lambda(M,\ f)\equiv r(M)^{n(k-l)/2}\lambda(M,\ g) \qquad \bmod d$$

where $r(M)$ is a multiplicator of $M\in GSp(2n,\ \boldsymbol{Z})$ and $\lambda(M,\ f)$ is the eigenvalue of Hecke operator $T(\Gamma_n M\Gamma_n)$ normalized as Andrianov [**1**, 1.3.3] on f. In our example (0.1), we have $k+l-(n+1)=20+18-3=5\cdot 7$. (The other factor 5 does not give congruences.) This also fits to degree one case (cf. Swinnerton-Dyer [**19**, p. 31, Corollary]).

We remark that there remains much to be done to obtain systematic results as the degree one case treated by Serre [**17**] and Swinnerton-Dyer [**19**], including the study of l-adic representations attached to Siegel modular forms.

The results of this paper have been announced in [**16**]. The author would like to thank Professor N. Kurokawa for encouragements.

NOTATION. 1. For complex numbers α and β, we put

$$\varepsilon(\alpha,\ \beta)=\begin{cases}\alpha(\alpha-1)\cdots(\beta+1)\beta & \text{if } \alpha-\beta \text{ is a non-negative integer,}\\ 1 & \text{otherwise,}\end{cases}$$

and

$$\eta(\alpha,\ \beta)=\begin{cases}\alpha\left(\alpha-\frac{1}{2}\right)\cdots\left(\beta+\frac{1}{2}\right)\beta & \text{if } 2(\alpha-\beta) \text{ is a non-negative integer,}\\ 1 & \text{otherwise.}\end{cases}$$

2. For a square matrix T, $|T|$ and $\mathrm{Tr}(T)$ stand for its determinant and trace respectively. We denote by $\sum_{T\geq 0}$ (resp. $\sum_{T>0}$) the summation over all

symmetric, semi-integral, positive semi-definite (resp. positive definite) matrices T (of a fixed size). For simplicity, we denote such a matrix $T=\begin{pmatrix} t_1 & t_3/2 \\ t_3/2 & t_2 \end{pmatrix}$ of size two by (t_1, t_2, t_3).

3. For each integer $n\geqq 1$, H_n denotes the Siegel upper half plane of degree n. For a C^∞-function $f(Z)$ on H_n satisfying $f(Z+S)=f(Z)$ for all $Z\in H_n$ and all symmetric $S\in M(n, \boldsymbol{Z})$, we denote its Fourier expansion as $f(Z)=\sum_T a(T, Y, f)e^{2\pi i \operatorname{Tr}(TZ)}$, where T runs over all symmetric semi-integral matrices of size n. Usually f is written in the form $f(Z)=\sum a'(T, Y, f)e^{2\pi i \operatorname{Tr}(TX)}$, but it is convenient for our purpose to write f as the former. If f is holomorphic, $a(T, Y, f)$ does not depend on Y. In this case, we write $a(T, Y, f)$ as $a(T, f)$ for simplicity.

§ 1. Differential operators and Fourier coefficients.

We study some differential operators and their effects on Siegel modular forms. For a variable $Z=\begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ on H_2, we put

$$X=\frac{1}{2}(Z+\bar{Z})=\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}, \qquad Y=\frac{1}{2i}(Z-\bar{Z})=\begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix}$$

and

$$\frac{d}{dZ}=\begin{pmatrix} \dfrac{\partial}{\partial z_1} & \dfrac{1}{2}\cdot\dfrac{\partial}{\partial z_3} \\ \dfrac{1}{2}\cdot\dfrac{\partial}{\partial z_3} & \dfrac{\partial}{\partial z_2} \end{pmatrix},$$

where $\bar{Z}$ is the complex conjugate of Z, $\partial/\partial z_j=(\partial/\partial x_j - i\partial/\partial y_j)/2$ and $i=\sqrt{-1}$. We define three differential operators on a C^∞-function f on H_2 as follows:

$$D : f \longrightarrow \left|\frac{d}{dZ}\right| f=\frac{\partial^2 f}{\partial z_1 \partial z_2}-\frac{1}{4}\frac{\partial^2 f}{\partial z_3^2},$$

$$\sigma : f \longrightarrow i\cdot\operatorname{Tr}\left(Y\frac{d}{dZ}f\right)=i\sum_{j=1}^{3} y_j\frac{\partial f}{\partial z_j},$$

$$\delta_k : f \longrightarrow |Y|^{-k+1/2}D(|Y|^{k-1/2}f).$$

Further, we set for a positive integer r,

$$\delta_k^r : f \longrightarrow \delta_{k+2r-2}\cdots\delta_{k+2}\delta_k f.$$

We understand that δ_k^0 is the identity operator. These differential operators were studied by Maass [9]. In this section, for a $T=(t_1, t_2, t_3)$ as in Notation 2, we put $B=\pi \operatorname{Tr}(TY)$ and $q^T=\exp(2\pi i \operatorname{Tr}(TZ))$.

LEMMA 1.1. *Let j and k be integers, and let T, B and q^T be as above. Then,* *the following operator identities hold:*

$$\delta_k |Y|^j = |Y|^j \delta_{k+j}, \tag{1.1}$$

$$D\sigma = (\sigma+1)D, \tag{1.2}$$

and

$$\sigma B^j q^T = \left(\frac{j}{2} B^j - 2B^{j+1}\right) q^T \qquad (j \geqq 0). \tag{1.3}$$

PROOF. For a C^∞-function f on H_2,

$$\delta_k |Y|^j f = |Y|^{-k+1/2} D |Y|^{k+j-1/2} f$$
$$= |Y|^j \delta_{k+j} f.$$

We have $D\sigma f - \sigma D f = Df$, hence $D\sigma = (\sigma+1)D$. Using $\partial B/\partial z_l = -i\pi t_l/2$ and $\partial q^T/\partial z_l = 2\pi i t_l q^T$, we obtain

$$\sigma B^j q^T = \sum_{l=1}^{3} i y_l \left(-\frac{i\pi j t_l}{2} B^{j-1} q^T + 2\pi i t_l B^j q^T\right)$$
$$= \left(\frac{j}{2} B^j - 2B^{j+1}\right) q^T. \qquad \text{Q. E. D.}$$

For each integer $n \geqq 1$, Γ_n denotes the Siegel modular group of degree n. We denote by $M_k(\Gamma_n)$ (resp. $M_k^\infty(\Gamma_n)$, $S_k(\Gamma_n)$) the $\boldsymbol{C}$-vector space of holomorphic Siegel modular forms (resp. space of C^∞-modular forms, space of holomorphic cusp forms) of degree n and weight k. We note that δ_k^r maps $M_k^\infty(\Gamma_2)$ to $M_{k+2r}^\infty(\Gamma_2)$, by Harris [**3**, 1.5.3]. Further, for any subring R of $\boldsymbol{C}$, we set

$$M_k(\Gamma_n)_R = \{f \in M_k(\Gamma_n) \mid a(T, f) \in R \text{ for all } T \geqq 0\}$$

and

$$S_k(\Gamma_n)_R = M_k(\Gamma_n)_R \cap S_k(\Gamma_n).$$

PROPOSITION 1.2. *Let R be a subring (not necessarily containing 1) of $\boldsymbol{C}$ satisfying $(1/2)R \subset R$ and let $f \in M_k(\Gamma_2)_R$. Then for each positive integer r, we have:*
(1) *$\delta_k^r f$ is a $\boldsymbol{Z}[1/2]$-linear combination of*

$$|Y|^{-b} \sigma^c D^d f$$

where b, c and d are integers satisfying $0 \leqq c \leqq b \leqq r$, $0 \leqq d \leqq r$ and $b+d=r$. Moreover, the coefficient of $|Y|^{-r} f$ (i.e. in case of $c=d=0$) is given by

$$\left(-\frac{1}{4}\right)^r \eta\left(k+r-1,\ k-\frac{1}{2}\right).$$

(2) *$\pi^{-2d} a(T, Y, \sigma^c D^d f)$ belongs to the ring $R[B]$ and its degree is not greater than c.*
(3) *If $c \geqq 1$ or $d \geqq 1$, $a(0, Y, \sigma^c D^d f) = 0$.*

PROOF. (1) We use induction on r. In case $r=1$, we get by a straightforward computation,

$$\delta_k^1 f=-\frac{1}{4}k\left(k-\frac{1}{2}\right)|Y|^{-1}f-\frac{1}{2}\left(k-\frac{1}{2}\right)|Y|^{-1}\sigma f+Df. \tag{1.4}$$

Hence, the assertions hold. Assume (1) for r. Since δ_{k+2r} is $\boldsymbol{C}$-linear and $\delta_k^{r+1}=\delta_{k+2r}\delta_k^r$, it is enough to prove that $\delta_{k+2r}|Y|^{-b}\sigma^c D^d f$ satisfies (1) for $r+1$ in place of r. Using Lemma 1.1 with $D\sigma^c=(\sigma+1)D\sigma^{c-1}=\cdots=(\sigma+1)^c D$, we have

$$\delta_{k+2r}|Y|^{-b}\sigma^c D^d f=-\frac{1}{4}(k+2r-b)\left(k+2r-b-\frac{1}{2}\right)|Y|^{-b-1}\sigma^c D^d f$$

$$-\frac{1}{2}\left(k+2r-b-\frac{1}{2}\right)|Y|^{-b-1}\sigma^{c+1}D^d f+|Y|^{-b}(\sigma+1)^c D^{d+1}f.$$

Moreover, if $c=d=0$, then b must be equal to r and the first term of the above expression is

$$-\frac{1}{4}(k+r)\left(k+r-\frac{1}{2}\right)|Y|^{-r-1}f.$$

Thus, we see that (1) holds in case of $r+1$, too.
(2) Since

$$a(T, D^d f)=(2\pi i)^{2d}|T|^d a(T, f), \tag{1.5}$$

it is sufficient to show that there exists an $F\in R[B]$ whose degree is not greater than c, such that

$$a(T, Y, \sigma^c q^T)=F. \tag{1.6}$$

But this is a direct consequence of (1.3).
(3) In case of $d\geqq 1$, the assertion holds by (1.5). For $c\geqq 1$, we see that the constant term of the polynomial F in (1.6) vanishes by (1.3). Therefore, setting $T=0$, we have $B=0$ and $F=0$, so (3) also holds in this case. Q. E. D.

We prepare a formula on the generalized gamma function. From now on, we set

$$U=\left\{X=\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}\in M(2, \boldsymbol{R}) \;\middle|\; -\frac{1}{2}\leqq x_j\leqq\frac{1}{2} \text{ for } j=1, 2, 3\right\},$$

$$V=\left\{Y=\begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix}\in M(2, \boldsymbol{R}) \;\middle|\; Y>0\right\},$$

$$dX=dx_1dx_2dx_3, \qquad dY=dy_1dy_2dy_3,$$

and

$$d^*Y=|Y|^{-3/2}dY.$$

It is known that the measure d^*Y is invariant under $Y\to {}^tAYA$ for $A\in GL(2, \boldsymbol{R})$.

LEMMA 1.3. *Let m_1, m_2 and m_3 be non-negative integers and $\alpha>1/2$. Put*

$$I(\alpha;\, m_1, m_2, m_3)=\int_V y_1^{m_1}y_2^{m_2}y_3^{m_3}e^{-\mathrm{Tr}(Y)}|Y|^{\alpha}d^*Y.$$

Then:

(1) *If* m_3 *is odd,* $I(\alpha;\, m_1, m_2, m_3)=0$.

(2) *If* m_3 *is even,*

$$I(\alpha;\, m_1, m_2, m_3)=\frac{\Gamma\left(m_1+\alpha+\frac{m_3}{2}\right)\Gamma\left(m_2+\alpha+\frac{m_3}{2}\right)\Gamma\left(\frac{m_3}{2}+\frac{1}{2}\right)\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{m_3}{2}\right)}.$$

PROOF. If $Y=(y_1, y_2, y_3)>0$, then $(y_1, y_2, -y_3)>0$ also. So, if m_3 is odd, $I(\alpha;\, m_1, m_2, m_3)=0$. We assume that m_3 is even in the following.

We decompose Y into a product of the lower triangular matrix $T=\begin{pmatrix} t_1 & 0 \\ t_3 & t_2 \end{pmatrix}$ and its transpose as $Y=T\cdot {}^tT$ and change variables from Y to T as in Maass [**10**, p. 77]. Then we have

$$I(\alpha;\, m_1, m_2, m_3)=\frac{\Gamma\left(m_1+\alpha+\frac{m_3}{2}\right)}{\Gamma\left(\frac{m_3}{2}+\alpha\right)}\cdot 4\int_0^{\infty}t_1^{m_3+2\alpha-1}e^{-t_1^2}dt_1$$

$$\times\int_0^{\infty}\int_{-\infty}^{\infty}t_3^{m_3}(t_2^2+t_3^2)^{m_2}e^{-t_2^2-t_3^2}t_2^{2\alpha-2}dt_3dt_2$$

$$=\frac{\Gamma\left(m_1+\alpha+\frac{m_3}{2}\right)}{\Gamma\left(\frac{m_3}{2}+\alpha\right)}\cdot\int_V y_2^{m_2}y_3^{m_3}e^{-\mathrm{Tr}(Y)}|Y|^{\alpha}d^*Y. \tag{1.7}$$

Here, we decompose Y into a product of the upper triangular matrix $T=\begin{pmatrix} t_1 & t_3 \\ 0 & t_2 \end{pmatrix}$ and its transpose. Since m_3 is even,

$$(1.7)=\frac{\Gamma\left(m_1+\alpha+\frac{m_3}{2}\right)}{\Gamma\left(\frac{m_3}{2}+\alpha\right)}\cdot 8\int_0^{\infty}t_1^{2\alpha-2}e^{-t_1^2}dt_1$$

$$\times\int_0^{\infty}t_2^{2m_2+m_3+2\alpha-1}e^{-t_2^2}dt_2\int_0^{\infty}t_3^{m_3}e^{-t_3^2}dt_3$$

$$=\frac{\Gamma\left(m_1+\alpha+\frac{m_3}{2}\right)\Gamma\left(m_2+\alpha+\frac{m_3}{2}\right)\Gamma\left(\frac{m_3}{2}+\frac{1}{2}\right)\Gamma\left(\alpha-\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{m_3}{2}\right)}.$$

Q. E. D.

We put a brief description on the holomorphic projection. For details, see Sturm [**18**]. For $f\in M_w^{\infty}(\Gamma_2)$, we put

$$P(w, T, a(T, Y, f))=M(w, T)\int_V a(T, Y, f)e^{-4\pi \mathrm{Tr}(TY)}|Y|^{w-3}dY \tag{1.8}$$

where

$$M(w, T)^{-1}=\int_V e^{-4\pi \mathrm{Tr}(TY)}|Y|^{w-3}dY$$
$$=I\left(w-\frac{3}{2};\ 0, 0, 0\right)|4\pi T|^{-w+3/2}.$$

We define

$$M_w^\infty(\Gamma_2)^c=\{f\in M_w^\infty(\Gamma_2)\mid P(w, T, |a(T, Y, f)|)\ \text{converges for all}\ T>0\}$$

and for each $f\in M_w^\infty(\Gamma_2)^c$, we put

$$P_w(f)=\sum_{T>0}P(w, T, a(T, Y, f))q^T.$$

Then, $P_w(f)$ belongs to the ring of formal power series $\boldsymbol{C}[q_3, q_3^{-1}][[q_1, q_2]]$ where $q_j=\exp(2\pi i z_j)$. Assume, moreover, that f is of bounded growth, namely,

$$\int_U\int_V|f(X+iY)||Y|^{w-3}e^{-\rho \mathrm{Tr}(Y)}dYdX<\infty$$

for any positive constant ρ. Then, $P_w(f)$ converges for all $Z\in H_2$ and belongs to $S_w(\Gamma_2)$. (See Sturm [**18**, Theorem 1].)

LEMMA 1.4. *Let E be the unit matrix and m be a positive integer. For non-negative integers b, c, c_1, c_2, c_3, d and w satisfying $c\leqq b<w-2$, we have:*

(1) *If c_3 is odd,* $P(w, mE, |Y|^{-b}\pi^{c_1+c_2+c_3}y_1^{c_1}y_2^{c_2}y_3^{c_3})=0$.

(2) *If c_3 is even,*

$$P(w, mE, |Y|^{-b}\pi^{c_1+c_2+c_3}y_1^{c_1}y_2^{c_2}y_3^{c_3})$$
$$=\pi^{2b}\mu\frac{\varepsilon\left(\frac{c_3}{2}-\frac{1}{2}, \frac{1}{2}\right)\varepsilon\left(w+\frac{c_3}{2}-b-\frac{5}{2}+c_1, w+\frac{c_3}{2}-b-\frac{3}{2}\right)}{\varepsilon(w-3, w-b-2)\,\varepsilon\left(w-\frac{5}{2}, w+c_2-b-\frac{3}{2}+\frac{c_3}{2}\right)} \tag{1.9}$$

where $\mu=(4m)^{2b-c_1-c_2-c_3}$.

(3) *If $T>0$,*

$$P(w, T, |Y|^{-b}B^c|2\pi iT|^d)=(-1)^d|4\pi T|^{b+d}4^{-d-c}$$
$$\times\sum_{c_1+c_2=c}\binom{c}{c_1}\frac{\varepsilon\left(w-b-\frac{5}{2}+c_1, w-b-\frac{3}{2}\right)}{\varepsilon\left(w-\frac{5}{2}, w-b+c_2-\frac{3}{2}\right)\varepsilon(w-3, w-b-2)}. \tag{1.10}$$

PROOF. We have (1) by Lemma 1.3 (1). Suppose that c_3 is even. Then using Lemma 1.3 (2), we have:

$$M(w, mE)\int_V \pi^{c_1+c_2+c_3}|Y|^{-b}y_1^{c_1}y_2^{c_2}y_3^{c_3}e^{-4\pi m\mathrm{Tr}(Y)}|Y|^{w-3/2}d^*Y$$

$$=\pi^{2b}\mu\frac{\Gamma\left(\frac{c_3}{2}+\frac{1}{2}\right)\Gamma(w-b-2)\Gamma\left(c_1+w-b-\frac{3}{2}+\frac{c_3}{2}\right)\Gamma\left(c_2+w-b-\frac{3}{2}+\frac{c_3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(w-2)\Gamma\left(\frac{c_3}{2}+w-b-\frac{3}{2}\right)\Gamma\left(w-\frac{3}{2}\right)}$$

$$=\pi^{2b}\mu\frac{\varepsilon\left(\frac{c_3}{2}-\frac{1}{2},\frac{1}{2}\right)\varepsilon\left(w+\frac{c_3}{2}-b-\frac{5}{2}+c_1,\ w+\frac{c_3}{2}-b-\frac{3}{2}\right)}{\varepsilon(w-3,\ w-b-2)\,\varepsilon\left(w-\frac{5}{2},\ w+c_2-b-\frac{3}{2}+\frac{c_3}{2}\right)}.$$

Now, we show (1.10). Let $U\in GL(2,\boldsymbol{R})$ be a positive definite matrix such that $T=U\cdot{}^tU$. By the substitution $Y\to(4\pi)^{-1}\,{}^tU^{-1}YU^{-1}$, we have:

$$P(w,\ T,\ |Y|^{-b}B^c|2\pi iT|^d)$$

$$=I\left(w-\frac{3}{2};\ 0,0,0\right)^{-1}(-1)^d|4\pi T|^{b+d}4^{-d-c}\int_V \mathrm{Tr}(Y)^c|Y|^{w-b-3/2}e^{-\mathrm{Tr}(Y)}d^*Y$$

$$=I\left(w-\frac{3}{2};\ 0,0,0\right)^{-1}(-1)^d|4\pi T|^{b+d}4^{-d-c}\sum_{c_1+c_2=c}\binom{c}{c_1}I\left(w-b-\frac{3}{2};\ c_1,c_2,0\right).$$

Hence, we have (1.10) by Lemma 1.3 (2). Q. E. D.

THEOREM 1.5. *Let R be a subring (not necessarily containing 1) of $\boldsymbol{C}$ satisfying $(1/2)R\subset R$. Let $f\in M_{k_1}(\Gamma_2)_R$ and $g\in M_{k_2}(\Gamma_2)_R$ with $k_1+k_2>4$. Suppose that I is an ideal of R satisfying*

(1) $(1/2)I\subset I$,

(2) $a(T,g)\in I$ *for all* $T\neq 0$.

Let r_1 be a non-negative integer and r_2 be a positive integer. We put $r=r_1+r_2$ and $w=k_1+k_2+2r$. Then for any positive integer m,

$$(2\pi i)^{-2r}\xi a(mE,\ P_w(\delta_{k_1}^{r_1}f\cdot\delta_{k_2}^{r_2}g))-\nu m^{2r}a(mE,\ fg) \tag{1.11}$$

belongs to $(2w-2r-3)I$, where $\xi=\varepsilon(w-3,\ w-r-2)\,\varepsilon(w-5/2,\ w-r-3/2)$ and $\nu=\eta(k_1+r_1-1,\ k_1-1/2)\,\eta(k_2+r_2-1,\ k_2-1/2)$.

PROOF. By Proposition 1.2 and Lemma 1.4 with $b=r$ and $c=c_1=c_2=c_3=0$, (1.11) is a $\boldsymbol{Z}[1/2]$-linear combination of

$$\pi^{-2r}\xi P(w,\ mE,\ a(T_1,f)a(T_2,g)|Y|^{-b}B_1^{e_1}B_2^{e_2}(2\pi i)^{2d}|T_1|^{d_1}|T_2|^{d_2}) \tag{1.12}$$

satisfying

$b_j+d_j=r_j$, $0\leqq e_j\leqq b_j\leqq r_j$ and $0\leqq d_j\leqq r_j$ for $j=1$ and 2,

$d_2\geqq 1$ or $e_2\geqq 1$,

$$T_1+T_2=mE, \qquad T_2\neq 0, \qquad T_1,\ T_2\geqq 0$$

where $B_1=\pi\,\mathrm{Tr}(T_1Y)$, $B_2=\pi\,\mathrm{Tr}(T_2Y)$, $b=b_1+b_2$ and $d=d_1+d_2$. Since $(1/2)R\subset R$ it is sufficient to show that

$$\pi^{-2r}\xi P(w,\ mE,\ \pi^{2d+c_1+c_2+c_3}|Y|^{-b}y_1^{c_1}y_2^{c_2}y_3^{c_3}) \tag{1.13}$$

exists and that (1.13) belongs to $(2w-2r-3)R$ when the following conditions are satisfied:

$$b+d=r, \tag{1.14}$$

$$c_1,\ c_2,\ c_3\geqq 0, \qquad 0\leqq c_1+c_2+c_3\leqq b\leqq r, \qquad c_3 \text{ is even}, \tag{1.15}$$

$$\text{at least one of } c_1,\ c_2,\ c_3,\ d \text{ is positive}. \tag{1.16}$$

By Lemma 1.4, (1.13) exists since $w-2-b\geqq 2+r>0$. If $\alpha-\beta$ and $\beta-\gamma$ are non-negative integers, by the definition of ε, we have $\varepsilon(\alpha,\gamma)\varepsilon(\alpha,\beta)^{-1}=\varepsilon(\beta-1,\gamma)$. Using Lemma 1.4 and (1.14)-(1.16), we see that (1.13) is equal to

$$\begin{aligned}&(4m)^{b'}\varepsilon\Big(\frac{c_3}{2}-\frac{1}{2},\ \frac{1}{2}\Big)\varepsilon(w-b-3,\ w-r-2)\\ &\times\varepsilon\Big(w+\frac{c_3}{2}-b+c_1-\frac{5}{2},\ w+\frac{c_3}{2}-b-\frac{3}{2}\Big)\varepsilon\Big(w+\frac{c_3}{2}-b+c_2-\frac{5}{2},\ w-r-\frac{3}{2}\Big),\end{aligned} \tag{1.17}$$

where $b'=2b-c_1-c_2-c_3\geqq 0$. By (1.14), $d>0$ is equivalent to $b<r$. Thus, if at least one of d, c_2, c_3 is positive, $w-r-3/2$ divides the fourth ε-factor of (1.17). Otherwise, by (1.14) and (1.16), we have $c_1>0$, $c_3=0$ and $b=r$. In this case, $w-r-3/2$ divides the third ε-factor of (1.17). Thus (1.13) belongs to $(w-r-3/2)R$ $\subset(2w-2r-3)R$. Q. E. D.

REMARK 1.6. The key point of the proof is that the constant μ of (1.9) is an integer under (1.15). This yields the integrality property of an analytically defined C^∞-map (differentiation followed by holomorphic projection). We note that restriction to the coefficients at the multiples of the unit matrix simplify the proof of Lemma 1.4 (2). Similar integrality seems to hold at an arbitrary half-integral positive semi-definite matrix.

§ 2. Cuspidal conditions on holomorphic projections.

If $f\in M_k^\infty(\Gamma_2)$ is of bounded growth, then $P_k(f)\in S_k(\Gamma_2)$. However we must apply P_k to general f in some cases. In this section, we show that $P_k(f)\in S_k(\Gamma_2)$ for certain types of $f\in M_k^\infty(\Gamma_2)$ constructed by using differential operators. Our method is based upon Sturm [18, § 4], where boundedness is studied for a product of a holomorphic modular form and a nonholomorphic Eisenstein series. For simplicity, we put $\partial_k^r=\varepsilon(k+r-3/2,\ k-1/2)^{-1}\delta_k^r$. We denote by $C_1, C_2, \cdots$ suitably selected positive constants independent of T and Z.

LEMMA 2.1. *For non-negative integers k and r, let $M_{k+2r}^r(\Gamma_2)$ be the $\boldsymbol{C}$-vector subspace of $M_{k+2r}^{\infty}(\Gamma_2)$ generated by*

$$\delta_{k_1}^{r_1}h_1\cdot\delta_{k_2}^{r_2}h_2$$

where $k_1+k_2=k$, $r_1+r_2=r$ $(r_1, r_2\geqq 0)$, $h_1\in M_{k_1}(\Gamma_2)$ and $h_2\in M_{k_2}(\Gamma_2)$. Let $f, g\in M_{k+2r}^r(\Gamma_2)$. Assume that $a(T, Y, g)=0$ for all $|T|=0$. Then we have:

(1) $$|f(Z)|<C_1(\lambda_1^{-r}+\lambda_1^{-r-k})(\lambda_2^{-r}+\lambda_2^{-r-k}) \tag{2.1}$$

for all $Z\in H_2$ where λ_1 and λ_2 are eigenvalues of Y.

(2) $$|g(Z)|<C_2|Y|^{-k/2-r} \tag{2.2}$$

for all $Z\in H_2$.

PROOF. Let $\Omega=\Gamma_2\backslash H_2$ be the fundamental domain such that $Z=X+iY\in\Omega$ implies $Y>C_3E$ and that Y is a reduced matrix. (Cf. Maass [**10**, p. 169].) Then we have

$$\frac{1}{2}(t_1y_1+t_2y_2)\leqq \mathrm{Tr}(TY)\leqq\frac{3}{2}(t_1y_1+t_2y_2)$$

and hence

$$|TY|\leqq(\mathrm{Tr}(TY))^2$$

for $T=(t_1, t_2, t_3)\geqq 0$. Assume that $h\in M_k(\Gamma_2)$. Using Proposition 1.1, we see that $a(T, Y, \delta_k^r h)$ is a $\boldsymbol{C}$-linear combination of

$$|Y|^{-b}\mathrm{Tr}(TY)^c|T|^d a(T, h)$$

with $b+d=r$ and $0\leqq c\leqq b\leqq r$. By the same method as Maass [**10**, pp. 184-185], we have

$$|a(T, h)|\,|TY|^d\,\mathrm{Tr}(TY)^c|q^T|\leqq C_4e^{-\pi\mathrm{Tr}(TY)}. \tag{2.3}$$

Hence, we obtain $|Y|^r|\delta_k^r h(Z)|<C_5$ for all $Z\in\Omega$. Using $r_1+r_2=r$, we see that the same is true for $f, g\in M_{k+2r}^r(\Gamma_2)$. Setting $\varphi(Z)=|Y|^{k/2+r}|f(Z)|$ and $a=k/2$ in Sturm [**18**, Proposition 2], we have (2.1). On the other hand, by (2.3), the similar method to Maass [**10**, pp. 191-192] yields

$$|g(Z)|<C_6\exp(-C_7\sqrt{|Y|})$$

for all $Z\in\Omega$. Therefore, setting $\varphi(Z)=|Y|^{k/2+r}|g(Z)|$ and $a=0$ in Sturm [**18**, Proposition 2], we have (2.2). Q.E.D.

LEMMA 2.2. *For $f\in M_k(\Gamma_2)$, we have $P_{k+2}(\delta_k f)=0$ and $P_{k+4}(\delta_k^2 f)=0$.*

PROOF. By a straightforward computation, we have (1.4) and

$$\delta_k^2 f=\frac{1}{16}\eta\left(k+1,\ k-\frac{1}{2}\right)|Y|^{-2}f+\frac{1}{4}\left(k+\frac{1}{2}\right)^2\left(k-\frac{1}{2}\right)|Y|^{-2}\sigma f$$
$$+\frac{1}{4}\left(k-\frac{1}{2}\right)\left(k+\frac{1}{2}\right)|Y|^{-2}\sigma^2 f-\frac{1}{2}(k+2)\left(k+\frac{1}{2}\right)|Y|^{-1}Df$$
$$-\left(k+\frac{1}{2}\right)|Y|^{-1}\sigma Df+D^2 f. \tag{2.4}$$

We have $a(T, D^j f)=|2\pi iT|^j a(T, f)$, $a(T, \sigma D^j f)=-2|2\pi iT|^j Ba(T, f)\ (j\geqq 0)$ and $a(T, \sigma^2 f)=(-B+4B^2)a(T, f)$ by (1.3). Using Lemma 1.4 (3), we see that $P(k+2, T, a(T, Y, \delta_k f))=0$ and $P(k+4, T, a(T, Y, \delta_k^2 f))=0$ for all $T>0$.

Q. E. D.

The next theorem gives us sufficient conditions so that $P_{k+2r}(f)$ may belong to $S_{k+2r}(\Gamma_2)$ for $f\in M_{k+2r}^r(\Gamma_2)$.

THEOREM 2.3. *Let $f\in M_k(\Gamma_2)$ and $g\in M_l(\Gamma_2)$ with $w>4$ where $k+l=w$. Let r and s be non-negative integers. Then we have the following:*

(1) *$\delta_k^r f\cdot\delta_l^s g$ is of bounded growth for $r+s\geqq 3$. Especially, $P_{w+2r}(g\delta_k^r f)$ belongs to $S_{w+2r}(\Gamma_2)$ for $r\geqq 3$.*

(2) *If at least one of f and g is a cusp form, then $\delta_k^r f\cdot\delta_l^s g$ is of bounded growth for all $r, s\geqq 0$.*

(3) *$P_{w+2}(g\partial_k f+f\partial_l g)$ belongs to $S_{w+2}(\Gamma_2)$. Especially, $P_{2k+2}(f\delta_k f)$ belongs to $S_{2k+2}(\Gamma_2)$.*

(4) *$P_{w+4}(g\partial_k^2 f+2\partial_k f\cdot\partial_l g+f\partial_l^2 g)$ belongs to $S_{w+4}(\Gamma_2)$.*

PROOF. In view of Lemma 2.1, we have only to check that the integral

$$\int_U\int_V|\delta_k^r f(X+iY)||\delta_l^s g(X+iY)||Y|^{w+2(r+s)-3}e^{-\rho\mathrm{Tr}(Y)}dYdX$$

converges at $|Y|=0$ since $e^{-\rho\mathrm{Tr}(Y)}$ is a rapidly decreasing function as $|Y|\to\infty$ for any fixed $\rho>0$. Since $\delta_k^r f\cdot\delta_l^s g$ belongs to $M_{w+2(r+s)}^{r+s}(\Gamma_2)$, by Lemma 2.1 (1), we see that there exist positive constants C_8, C_9 such that

$$|\delta_k^r f(X+iY)||\delta_l^s g(X+iY)|<C_8|Y|^{-(w+r+s)}\quad \text{for}\quad Y<C_9 E. \tag{2.5}$$

Hence, the same argument as the proof of Sturm [18, Corollary 2] proves (1). (Note that $w+2(r+s)-3-(w+r+s)>-1$ for $r+s>2$.) Without loss of generality, we may assume that g is a non-zero cusp form in the proof of (2). Then, by Lemma 2.1 (2), we have

$$|\delta_k^r f(X+iY)||\delta_l^s g(X+iY)|<C_{10}|Y|^{-(k+r+s+l/2)}\quad \text{for}\quad Y<C_{11}E$$

instead of (2.5). Noting $l\geqq 10$, we see that (2) holds by the same way.

For (3), we put $F=g\partial_k f+f\partial_l g-\partial_w fg\in M_{w+2}^{\infty}(\Gamma_2)$. By (1.4), we have

$a((0,t,0),Y,F)=0$ for all $t\geqq 0$. Using $a({}^tUTU,Y,F)=|U|^w a(T,UY{}^tU,F)$ for $U\in GL(2,\boldsymbol{Z})$, we see that $a(T,Y,F)=0$ for all $|T|=0$. Hence, by (2.2), F is of bounded growth. But $P_{w+2}(\partial_w fg)=0$ by Lemma 2.2. Therefore $P_{w+2}(g\partial_k f+f\partial_l g)$ itself belongs to $S_{w+2}(\Gamma_2)$. Similarly, using (2.4), we see that $P_{w+4}(g\partial_k^2 f+2\partial_k f\cdot\partial_l g+f\partial_l^2 g-\partial_w^2 fg)$ belongs to $S_{w+4}(\Gamma_2)$. By Lemma 2.2, we have (4).

Q. E. D.

§ 3. Congruences of eigenvalues and Fourier coefficients.

If $f\in M_k(\Gamma_1)$ is a normalized elliptic eigenform, then $\lambda(m,f)=a(m,f)$ and the study of congruences for eigenvalues of Hecke operators is equivalent to the study of congruences for Fourier coefficients. In case of Siegel modular forms of degree $\geqq 2$, the situation is rather complicated, but a similar relation exists. Here we study the degree two case. For each integer $m\geqq 1$, $T(m)$: $M_k(\Gamma_n)\to M_k(\Gamma_n)$ denotes the m-th Hecke operator. If $n\leqq 2$ and f is a non-zero eigen function of all Hecke operators $T(m)$, we call f an eigenform and denote the eigenvalue of $T(m)$ by $\lambda(m,f)$.

THEOREM 3.1. *Let R be the ring of integers of an algebraic number field. Let $f\in M_k(\Gamma_2)$ be an eigenform and $g\in M_w(\Gamma_2)$ be any form with $w\geqq k$. We assume that $a(mE,f)$ and $a(mE,g)$ belong to R for all $m\geqq 1$. Let $\mathfrak{p}$ be a prime ideal of R and suppose that there exists a positive integer e such that*

$$m^{w-k}a(mE,f)\equiv a(mE,g) \mod \mathfrak{p}^e \tag{3.1}$$

for all $m\geqq 1$. Then, for all $m\geqq 1$, we have

$$m^{w-k}\lambda(m,f)a(E,f)\equiv a(E,T(m)g) \mod \mathfrak{p}^e. \tag{3.2}$$

PROOF. From Proposition 2.1.2 and Theorem 2.3.1 of Andrianov [**1**], we have for a prime power p^i and for a positive integer n which is prime to p,

$$a(nE,T(p^i)g)=\begin{cases} a(n2^iE,g)+2^{w-2}a(n2^{i-1}E,g) & \text{if } p=2,\\ a(np^iE,g)+2\sum_{j=1}^{i}p^{(w-2)j}a(np^{i-j}E,g) & \text{if } p\equiv 1 \mod 4,\\ a(np^iE,g) & \text{if } p\equiv 3 \mod 4. \end{cases} \tag{3.3}$$

The same formulas hold for f also with k instead of w.

We prove that if $(m,n)=1$, then

$$a(nE,T(m)g)\equiv(mn)^{w-k}\lambda(m,f)a(nE,f) \mod \mathfrak{p}^e. \tag{3.4}$$

We have (3.2) by setting $n=1$ in (3.4). We prove (3.4) by induction on the number of primes dividing m. In case of $m=1$, (3.4) certainly holds because of (3.1). Next, we set $m=p^im'$ with $(p,m')=1$. We note that $T(m)=T(p^i)T(m')$. Hence, using (3.3), if $p=2$ for example, we have

$$a(nE,\ T(m)g)=a(nE,\ T(2^i)T(m')g)$$
$$=a(n2^iE,\ T(m')g)+2^{w-2}a(n2^{i-1}E,\ T(m')g).$$

By the induction hypotheses and the multiplicativity of eigenvalues, this is congruent mod $\mathfrak{p}^e$ to the following:

$$(mn)^{w-k}\lambda(m',\ f)(a(n2^iE,\ f)+2^{k-2}a(n2^{i-1}E,\ f))$$
$$=(mn)^{w-k}\lambda(m',\ f)a(nE,\ T(2^i)f)$$
$$=(mn)^{w-k}\lambda(m,\ f)a(nE,\ f).$$

The same is true for other primes p also. Hence we have (3.4) for all co-prime m and n. Q. E. D.

COROLLARY 3.2. *Under the same assumptions and notations, we further assume that:*
(1) *g is also an eigenform,*
(2) $a(E,\ g)\not\equiv 0 \bmod \mathfrak{p}$.
Then,

$$m^{w-k}\lambda(m,\ f)\equiv\lambda(m,\ g) \quad \bmod\ \mathfrak{p}^e \tag{3.5}$$

for all $m\geqq 1$.

PROOF. By (3.1) with $m=1$, $a(E,\ f)\equiv a(E,\ g) \bmod \mathfrak{p}^e$, which are units in $R_{\mathfrak{p}}$ (the localization of R at $\mathfrak{p}$) by the assumption (2). Noting that $\lambda(m,\ f)$ and $\lambda(m,\ g)$ are algebraic integers by Kurokawa [8, Theorem 1 (2)], we see that (3.2) implies (3.5) as a congruence in R. Q. E. D.

Now, we study a suitable condition which leads to the above congruence (3.5). For example, when $w=k+2$, if $a(T,\ g-(1/4\pi^2)Df)$ belong to $\mathfrak{p}^e$ for all $T\geqq 0$, then (3.1) is satisfied. Hence we have the congruence (3.5) under additional assumptions. But such a condition requiring all the Fourier coefficients seems to be too restrictive for applications. So, in Theorem 3.4 below, we formulate a condition which requires the Fourier coefficients at mE $(m\geqq 1)$ only. For this purpose, we prepare a proposition.

PROPOSITION 3.3. *Let $\{f_1,\ \cdots,\ f_n\}$ $(n=\dim S_k(\Gamma_2))$ be an eigen basis of $S_k(\Gamma_2)$. Let K be an algebraic number field, O_K its ring of integers, $\mathfrak{p}$ a prime ideal of O_K and R the localization of O_K at $\mathfrak{p}$. Denote by L the composite field of K and $\boldsymbol{Q}(\lambda(m,\ f_j)\,|\,m\geqq 1)$ for $j=1,\ \cdots,\ n$. Suppose that there exist positive integers $m_1,\ \cdots,\ m_n$ such that*

$$N_{L/K}(|(\lambda(m_i,\ f_j))_{1\leq i,j\leq n}|)\not\equiv 0 \quad \bmod\ \mathfrak{p}, \tag{3.6}$$

where $N_{L/K}$ denotes the norm mapping from L to K. Let $g\in S_k(\Gamma_2)_R$ and assume that

$$a(E, T(m_i)g) \equiv 0 \quad \bmod \mathfrak{p}^e \tag{3.7}$$

for $i=1, \cdots, n$ *with an integer* $e>0$. *Then we have:*

$$a(mE, g) \equiv 0 \bmod \mathfrak{p}^e \quad \textit{for all} \quad m \geqq 1.$$

PROOF. We denote by O_L the integer ring of L.

First we remark the following fact. Let S be a localization of O_L, and $f \in M_w(\Gamma_2)_S$ be an eigenform. Then, if $a(E, f) \equiv 0 \bmod I$ for an ideal I of S, we have $a(mE, f) \equiv 0 \bmod I$ for all $m \geqq 1$. Since $\lambda(m, f)$ are algebraic integers in O_L, this fact is obvious from the following equality:

$$\sum_{m=1}^{\infty} \frac{a(mE, f)}{m^s} = a(E, f)\zeta(2s-2k+4)\zeta_{Q(\sqrt{-1})}(s-k+2)^{-1} \sum_{m=1}^{\infty} \frac{\lambda(m, f)}{m^s}, \tag{3.8}$$

which is obtained by setting $D=-4$, χ=trivial character, $N=N_1=E$ in Theorem 2.4.1 of Andrianov [1].

Now, write $g=\sum_{j=1}^{n} c_j f_j$ with $c_j \in \boldsymbol{C}$. Let $\mathfrak{P}$ be a prime ideal of O_L lying above $\mathfrak{p}$ and h be its ramification index. Then by (3.7),

$$\sum_{j=1}^{n} \lambda(m_i, f_j) c_j a(E, f_j) \equiv 0 \quad \bmod \mathfrak{P}^{eh}. \tag{3.9}$$

By (3.6), $|(\lambda(m_i, f_j))| \not\equiv 0 \bmod \mathfrak{P}$. Therefore, (3.9) has a unique solution modulo $\mathfrak{P}^{eh}$ in the localization of O_L at $\mathfrak{P}$ and, moreover, $c_j a(E, f_j) \equiv 0 \bmod \mathfrak{P}^{eh}$. Since $c_j f_j$ is an eigenform (or is equal to 0), we have, as was remarked above, $c_j a(mE, f_j) \equiv 0 \bmod \mathfrak{P}^{eh}$ for all $m \geqq 1$. So are $a(mE, g)$. But $a(mE, g) \in R$ and this yields $a(mE, g) \equiv 0 \bmod \mathfrak{p}^e$ for all $m \geqq 1$. Q. E. D.

THEOREM 3.4. *Let K be an algebraic number field, O_K its ring of integers, $\mathfrak{p}$ its prime ideal not dividing the ideal* (2), *and R the localization of O_K at $\mathfrak{p}$. Let $f \in M_{w-2r}(\Gamma_2)_R$ and $g \in S_w(\Gamma_2)_R$ be eigenforms with $4<w-2r<w$. Suppose that all the following conditions* (1)-(6) *are satisfied:*

(1) *There exist positive integers $m_1, \cdots, m_n$ such that*

$$N_{L/K}(|(\lambda(m_i, f_j))_{1 \leq i, j \leq n}|) \not\equiv 0 \quad \bmod \mathfrak{p}$$

where $n=\dim S_w(\Gamma_2)$ and $\{f_1, \cdots, f_n\}$ is an eigen basis of $S_w(\Gamma_2)$ and L is the composite field of K and $\boldsymbol{Q}(\lambda(m, f_j) | m \geqq 1)$ for $j=1, \cdots, n$.

(2) *There exist a positive integer e and $2s$ ($s \geqq 1$) modular forms $h_{1,t} \in M_{k_{1,t}}(\Gamma_2)_R$, $h_{2,t} \in M_{k_{2,t}}(\Gamma_2)_R$ with $k_{1,t}+k_{2,t}=w-2r$, $r_{1,t} \geqq 0$, $r_{2,t} \geqq 1$ and $r_{1,t}+r_{2,t}=r$ for $t=1, \cdots, s$ such that*

$$a(mE, f) \equiv a\left(mE, \sum_{t=1}^{s} \nu_t h_{1,t} h_{2,t}\right) \quad \bmod \mathfrak{p}^e$$

for all $m \geqq 1$, where

$$\nu_t=\eta\Big(k_{1,t}+r_{1,t}-1,\ k_{1,t}-\frac{1}{2}\Big)\eta\Big(k_{2,t}+r_{2,t}-1,\ k_{2,t}-\frac{1}{2}\Big).$$

(3) $\mathfrak{p}^e$ *divides* $(2w-2r-3)I$ *where* I *is the ideal of* R *generated by* $a(T, h_{2,t})$ *for* $T\geqq 0$, $T\neq 0$ *and* $t=1, \cdots, s$.
(4) $a(E, f)\equiv a(E, g) \bmod \mathfrak{p}^e$ *and* $a(E, f)\not\equiv 0 \bmod \mathfrak{p}$.
(5) $m_i^{2r}\lambda(m_i, f)\equiv\lambda(m_i, g) \bmod \mathfrak{p}^e$ *for* $i=1, \cdots, n$.
(6) $\sum_{t=1}^{s} P_w(\delta_{k_{1,t}}^{r_{1,t}} h_{1,t}\cdot\delta_{k_{2,t}}^{r_{2,t}} h_{2,t})$ *belongs to* $S_w(\Gamma_2)$.
Then we have:

$$m^{2r}\lambda(m, f)\equiv\lambda(m, g) \mod \mathfrak{p}^e \qquad \textit{for all } m\geqq 1. \tag{3.10}$$

PROOF. For each t, put

$$h_{3,t}=\varepsilon(w-3,\ w-r-2)\,\varepsilon\Big(w-\frac{5}{2},\ w-r-\frac{3}{2}\Big)(2\pi i)^{-2r}P_w(\delta_{k_{1,t}}^{r_{1,t}} h_{1,t}\cdot\delta_{k_{2,t}}^{r_{2,t}} h_{2,t}).$$

Then, by Theorem 1.5, we see that

$$a(mE,\ h_{3,t})-\nu_t m^{2r}a(mE,\ h_{1,t}h_{2,t})$$

belongs to $(2w-2r-3)I$. (See the definition of the ideal I in (3).) We put $h_3=\sum_{t=1}^{s} h_{3,t}$. By the condition (6), h_3 belongs to $S_w(\Gamma_2)$. Using (2) and (3), we have

$$a(mE,\ h_3)\equiv m^{2r}a(mE,\ f) \mod \mathfrak{p}^e.$$

Hence, by Theorem 3.1, we have (particularly)

$$a(E,\ T(m_i)h_3)\equiv m_i^{2r}\lambda(m_i,\ f)a(E,\ f) \mod \mathfrak{p}^e.$$

Therefore, using (4) and (5), we obtain

$$a(E,\ T(m_i)(g-h_3))\equiv\lambda(m_i,\ g)a(E,\ g)-m_i^{2r}\lambda(m_i,\ f)a(E,\ f) \mod \mathfrak{p}^e$$

$$\equiv 0 \mod \mathfrak{p}^e.$$

Hene by Proposition 3.3 and the assumption (1), $a(mE, g)\equiv a(mE, h_3)\equiv m^{2r}a(mE, f) \bmod \mathfrak{p}^e$ for all $m\geqq 1$. Using Corollary 3.2 with (4), we have (3.10). Q. E. D.

REMARK 3.5. By Igusa [4], if $\mathfrak{p}$ divides neither the ideal (2) nor the ideal (3), then any element of $M_k(\Gamma_2)_R$ is an R-linear combination of $\varphi_4^a\varphi_6^b\chi_{10}^c\chi_{12}^d$ where a, b, c and d are non-negative integers and $4a+6b+10c+12d=k$. (It is convenient in numerical computation to use $4\chi_{10}$ and $12\chi_{12}$ instead of χ_{10} and χ_{12}.) If $r\geqq 3$, it is possible to put all $r_{1,t}=0$ without violating the condition (6). Suppose moreover that $\mathfrak{p}^e$ divides the ideal $(2w-2r-3)R$ and that $\mathfrak{p}$ does not divide rational primes less than or equal to $2r+21$. Then, selecting $h_{2,t}$ from φ_4, φ_6, χ_{10} and χ_{12}, we see that ν_t is a unit in R. Therefore, conditions (2) and (3) are satisfied in this case.

§ 4. Examples.

We prove some congruences between Siegel modular forms of degree two and different weights by using Theorem 3.4. For simplicity, we shall omit subscript t in case of $s=1$. We note that, in the situation of Theorem 3.4, it is enough to calculate values in $R/\mathfrak{p}^e R \cong O_K/\mathfrak{p}^e O_K$ to prove congruences modulo $\mathfrak{p}^e$. This device reduces computational complexity.

First, we recall some facts on Siegel modular forms of degree 1 and degree 2. For an even integer $k \geqq 4$, we denote by $E_k \in M_k(\Gamma_1)$ the Eisenstein series normalized to $a(0, E_k)=1$ and, for $\dim S_k(\Gamma_1)=1$, we denote by Δ_k the normalized eigen cusp form of weight k. The graded $\boldsymbol{C}$-algebra of even weight $\bigoplus_{k \geq 0} M_{2k}(\Gamma_2)$ is generated by four elements. They are $\varphi_4 \in M_4(\Gamma_2)$, $\varphi_6 \in M_6(\Gamma_2)$, $\chi_{10} \in S_{10}(\Gamma_2)$ and $\chi_{12} \in S_{12}(\Gamma_2)$. They are uniquely determined by the following normalizing conditions: $a(0, \varphi_4)=1$, $a(0, \varphi_6)=1$, $a((1, 1, 1), 4\chi_{10})=-1$ and $a((1, 1, 1), 12\chi_{12})=1$. We can calculate their Fourier coefficients using the method of Maass [**11**, Sätze 1 and 2]. In general, we denote Eisenstein series of weight k and degree 2 by φ_k. It is known that $\Phi\varphi_k = E_k$, where Φ is the Siegel Φ-operator.

There are two liftings from degree 1 to degree 2 for each even integer $k \geqq 4$. The one is Eisenstein lifting [ ]: $M_k(\Gamma_1) \to M_k(\Gamma_2)$, which is defined by the generalized Eisenstein series attached to elliptic modular forms. If $f \in M_k(\Gamma_1)$ is an eigenform, $[f]$ is uniquely determined by the conditions that $\Phi[f]=f$ and that $[f]$ is an eigenform. In this case, we have $\lambda(p, [f])=(1+p^{k-2})\lambda(p, f)$ for a rational prime p. The other is the Saito-Kurokawa lifting $\sigma_k: M_{2k-2}(\Gamma_1) \to M_k(\Gamma_2)$ constructed by Maass [**12, 13, 14**] and Andrianov [**2**]. As for eigenvalues, we have $\lambda(p, \sigma_k(f))=p^{k-2}+p^{k-1}+\lambda(p, f)$ for an eigenform $f \in M_{2k-2}(\Gamma_1)$. As to $M_k(\Gamma_1)$, we know $M_k(\Gamma_1)=E_k(\Gamma_1) \oplus S_k(\Gamma_1)$ with $E_k(\Gamma_1)=\boldsymbol{C}E_k$. In the degree two case, these two liftings give rise to the following decomposition:

$$M_k(\Gamma_2)=E_k^I(\Gamma_2) \oplus E_k^{II}(\Gamma_2) \oplus S_k^I(\Gamma_2) \oplus S_k^{II}(\Gamma_2),$$

where $E_k^I(\Gamma_2)=[E_k(\Gamma_1)]=\boldsymbol{C}\cdot\varphi_k$, $E_k^{II}(\Gamma_2)=[S_k(\Gamma_1)]$, $S_k^I(\Gamma_2)=\sigma_k(S_{2k-2}(\Gamma_1))$ and $S_k^{II}(\Gamma_2)=S_k^I(\Gamma_2)^{\perp}$ (orthogonal complement of $S_k^I(\Gamma_2)$ in $S_k(\Gamma_2)$ with respect to the Petersson inner product). We may call an element of $S_k^{II}(\Gamma_2)$ "a generic form" since it does not lie in the image of above two liftings. It is shown by Kurokawa [**5**, § 5] that

$$S_{20}^I(\Gamma_2)=\boldsymbol{C}\chi_{20}^{(1)} \oplus \boldsymbol{C}\chi_{20}^{(2)},$$

$$S_{20}^{II}(\Gamma_2)=\boldsymbol{C}\chi_{20}^{(3)}$$

where $\chi_{20}^{(j)}$ $(j=1, 2$ and $3)$ are eigenforms defined by

$$\chi_{20}^{(1)}=1840\chi_{10}\varphi_4\varphi_6-12(7699+\sqrt{D})\chi_{12}\varphi_4^2-16588800(8021+\sqrt{D})\chi_{10}^2,$$

$$\chi_{20}^{(2)}=1840\chi_{10}\varphi_4\varphi_6-12(7699-\sqrt{D})\chi_{12}\varphi_4^2-16588800(8021-\sqrt{D})\chi_{10}^2,$$

and

$$\chi_{20}^{(3)}=4\chi_{10}\varphi_4\varphi_6-12\chi_{12}\varphi_4^2+28569600\chi_{10}^2$$

with $D=63737521$. We note that $\chi_{20}^{(3)}$ has the minimal weight 20 among generic forms.

THEOREM 4.1. *The following congruence holds for all* $m\geqq 1$:

$$\lambda(m, \chi_{20}^{(3)})\equiv m^2\lambda(m, [\Delta_{18}]) \mod 7. \tag{4.1}$$

REMARK. This congruence seems to be valid with mod 49, as was conjectured by Kurokawa [**7**]. In the following, some computations are done in modulo 49 to clear the situation.

PROOF. Since $\Phi([\Delta_{18}]-[\Delta_{12}]\varphi_6)=0$, there exist $\alpha, \beta\in \boldsymbol{C}$ such that

$$7[\Delta_{18}]=7[\Delta_{12}]\varphi_6+\alpha f_{18}+\beta g_{18}, \tag{4.2}$$

where $f_{18}=4\chi_{10}\varphi_4^2$ and $g_{18}=12\chi_{12}\varphi_6$. We use the method of Kurokawa [**6**] for calculating α and β. Let S be (1, 1, 1). We apply $T(2)$ on (4.2) and compare the Fourier coefficients at E and S. Then, we have:

$$\alpha(a(2S, f_{18})-\lambda a(S, f_{18}))+\beta(a(2S, g_{18})-\lambda a(S, g_{18}))$$
$$+(a(2S, 7[\Delta_{12}]\varphi_6)-\lambda a(S, 7[\Delta_{12}]\varphi_6))=0,$$
$$\alpha(a(2E, f_{18})-\mu a(E, f_{18}))+\beta(a(2E, g_{18})-\mu a(E, g_{18}))$$
$$+(a(2E, 7[\Delta_{12}]\varphi_6)-\mu a(E, 7[\Delta_{12}]\varphi_6))=0,$$

where

$$\lambda=\lambda(2, [\Delta_{18}])=-34603536,$$
$$\mu=\lambda-2^{16}=-34669072.$$

By numerical values listed in Kurokawa [**5**] and Resnikoff-Saldaña [**15**], we have the following table.

T	E	S	$2E$	$2S$
$a(T, f_{18})$	2	−1	263008	−24240
$a(T, g_{18})$	10	1	1902560	32016
$a(T, 7[\Delta_{12}]\varphi_6)$	−5814	92	−667329696	3432000

Hence we have

$$\alpha=80136/143\equiv 7 \mod 49,$$
$$\beta=66960/143\equiv 18 \mod 49.$$

Here we consider these congruences in $\boldsymbol{Z}_{(7)}$. Observing these values, we put, in Theorem 3.4, as follows: $K=\boldsymbol{Q}$, $\mathfrak{p}=(7)$, $e=1$, $w=20$, $f=23\cdot 7[\Delta_{18}]$, $g=12\chi_{20}^{(3)}$, $s=2$, $r_{1,t}=0$, $r_{2,t}=1\,(t=1, 2)$, $k_{1,1}=k_{2,2}=12$, $h_{1,1}=h_{2,2}=7[\Delta_{12}]+18\cdot 12\chi_{12}$, $k_{1,2}=k_{2,1}=6$, $h_{1,2}=11\varphi_6$ and $h_{2,1}=23\varphi_6$. Since all the Fourier coefficients of φ_6, $12\chi_{12}$, $7[\Delta_{12}]$, f_{18}, g_{18} and $\chi_{20}^{(3)}$ are rational integers by Igusa [4, Theorem 1] and Kurokawa [6], all the Fourier coefficients of f, g, $h_{1,t}$ and $h_{2,t}$ $(t=1, 2)$ belong to $R=\boldsymbol{Z}_{(7)}$.

Now we can verify that all the conditions (1)-(6) of Theorem 3.4 are satisfied.
(1) We take $m_1=1$, $m_2=2$, $m_3=9$ and $f_j=\chi_{20}^{(j)}$ for $j=1$, 2 and 3. We put $D=63737521$. Then using Kurokawa [5, § 7], we have

$$|(\lambda(m_i, f_j))_{1\leq i,j\leq 3}|\equiv\begin{vmatrix}1 & 1 & 1\\ 5+6\sqrt{D} & 5+\sqrt{D} & 6\\ 2+\sqrt{D} & 2+6\sqrt{D} & 2\end{vmatrix}\mod 7$$

$$\equiv 2\sqrt{D}\mod 7.$$

Since, $N_{\boldsymbol{Q}(\sqrt{D})/\boldsymbol{Q}}(2\sqrt{D})\equiv -4D\not\equiv 0 \mod 7$, (1) is satisfied.
(2) By the definition of $h_{1,t}$ and $h_{2,t}$, we have

$$\sum_{t=1}^{2}\nu_t h_{1,t}h_{2,t}\equiv 2277(7[\Delta_{12}]\varphi_6+18\cdot 12\chi_{12}\varphi_6)\mod 49$$

$$\equiv 23(7[\Delta_{18}]-\alpha f_{18})\mod 49.$$

Using $\alpha\equiv 0 \mod 7$ and $f_{18}\in M_{18}(\Gamma_2)_{\boldsymbol{Z}}$, we see that (2) is satisfied.
(3) In our case, $2w-2r-3=35=5\cdot 7$. Since $h_{2,t}\in M_{k_{2,t}}(\Gamma_2)_R$ for $t=1$ and 2, we have $(2w-2r-3)I\subset 7R$.
(4) From values of α and β, we have $a(E, 23\cdot 7[\Delta_{18}])\equiv 2 \mod 49$, which is congruent to $a(E, 12\chi_{20}^{(3)}) \mod 49$.
(5) By values in Kurokawa [5], we have $\lambda(m, \chi_{20}^{(3)})-m^2\lambda(m, [\Delta_{18}])\equiv 0 \mod 49$ for $m=2$ and 9 (cf. Remark 4.2 below). So condition (5) is satisfied (for mod 49 also).
(6) We have $h_{1,1}\delta_6 h_{2,1}+h_{1,2}\delta_{12}h_{2,2}=(253/2)(\varphi_6\partial_{12}h_{1,1}+h_{1,1}\partial_6\varphi_6)$. Using Theorem 2.3 (3), we see that the condition (6) is satisfied.

Thus, by Theorem 3.4, the congruence (4.1) is proved. Q. E. D.

REMARK 4.2. We have: $\lambda(m, \chi_{20}^{(3)})-m^2\lambda(m, [\Delta_{18}])=2^6\cdot 3\cdot 7^3\cdot 2089$, $2^4\cdot 3^4\cdot 7^2\cdot 26140973$, $-2^{12}\cdot 3^3\cdot 7^2\cdot 20287\cdot 92333$ and $2^6\cdot 3^8\cdot 7^2\cdot 139\cdot 5814268161029177$ for $m=2$, 3, 4 and 9, respectively. Hence modulo 49 version of the congruence (4.1) holds for all $m=2^a 3^b$ with non-negative integers a and b.

Our next examples can be proved by applying Theorem (B) of Kurokawa [7], which uses the theory of the Saito-Kurokawa lifting σ_k, to the following congruences:

$$\lambda(m, \Delta_{18}) \equiv m^2 \lambda(m, E_{14}) \quad \bmod 5,$$

$$\lambda(m, \Delta_{22}) \equiv m^4 \lambda(m, E_{14}) \quad \bmod 17,$$

$$\lambda(m, \Delta_{26}) \equiv m^6 \lambda(m, E_{14}) \quad \bmod 19.$$

Here we prove the corresponding congruences independently from the above congruences.

THEOREM 4.3. *The following congruences hold for all* $m \geqq 1$:

$$\lambda(m, \chi_{10}) \equiv m^2 \lambda(m, \varphi_8) \quad \bmod 5, \tag{4.3}$$

$$\lambda(m, \chi_{12}) \equiv m^4 \lambda(m, \varphi_8) \quad \bmod 17, \tag{4.4}$$

$$\lambda(m, \chi_{14}) \equiv m^6 \lambda(m, \varphi_8) \quad \bmod 19. \tag{4.5}$$

PROOF. In the proof of (4.3), we use Theorem 1.5 and Theorem 3.4 with a slight modification. We put $K=\boldsymbol{Q}$, $\mathfrak{p}=(5)$, $R=\boldsymbol{Z}_{(5)}$, $s=1$, $h_1=\varphi_4$ and $h_2=5^{-1}\varphi_4$. Then, h_2 does not belong to $M_4(\Gamma_2)_R$. Taking into account that 5 divides $a(T, \varphi_4)$ for $T \neq 0$, we see that $a(T_1, h_1)a(T_2, h_2) \in \boldsymbol{Z}$ for $T_1+T_2=mE$ under $m \geqq 1$ in (1.12). Hence we see that (1.11) belongs to $(2w-2r-3)R$ and consequently we have Theorem 3.4. Further we put $f=14 \cdot 5^{-1}\varphi_8$, $g=7 \cdot 4\chi_{10}$, $w=10$, $r=1$, $r_1=0$, $r_2=1$, $e=1$, and $k_1=k_2=4$. Noting $\dim S_{10}(\Gamma_2)=1$, we see that conditions (1) and (5) are satisfied with $m_1=1$. Using $\varphi_4^2=\varphi_8$, we see that $f=14h_1h_2$, hence we have (2). In our case, $2w-2r-3=15$ and $a(E, f) \equiv a(E, g) \equiv 4 \bmod 5$, which prove (3) and (4). The condition (6) holds because of Theorem 2.3 (3). Therefore, (4.3) is proved.

In the proofs of (4.4) and (4.5), there is no need of modification as above. We put $K=\boldsymbol{Q}$ and $k_{j,t}=4$ for $j=1, 2$ and all t. We note that $\varphi_4^2=\varphi_8 \in M_8(\Gamma_2)_{\boldsymbol{Z}}$ and $a(E, \varphi_8)=175680$. For (4.4) we put $s=2$, $\mathfrak{p}=(17)$, $w=12$, $r=2$, $r_{1,1}=0$, $r_{2,1}=2$, $r_{1,2}=r_{2,2}=1$, $h_{1,1}=h_{1,2}=\varphi_4$, $h_{2,1}=7\varphi_4$, $h_{2,2}=9\varphi_4$, $f=8\varphi_8$ and $g=5 \cdot 12\chi_{12}$, then we have $5 \cdot (9/2) \cdot 4 \cdot (7/2) \equiv (4 \cdot (7/2))^2 \equiv 9 \bmod 17$ and $a(E, f) \equiv a(E, g) \equiv 16 \bmod 17$. Also we see that

$$\sum_{t=1}^{2} \delta_4^{r_{1,t}} h_{1,t} \delta_4^{r_{2,t}} h_{2,t} = \frac{441}{4} (\varphi_4 \partial_4^2 \varphi_4 + (\partial_4 \varphi_4)^2).$$

For (4.5) we put $s=1$, $r=3$, $r_1=0$, $r_2=3$, $\mathfrak{p}=(19)$, $w=14$, $h_1=h_2=\varphi_4$, $f=2\varphi_8$ and $g=6 \cdot 4\chi_{14}$, then we have $6 \cdot (11/2) \cdot 5 \cdot (9/2) \cdot 4 \cdot (7/2) \equiv 2 \bmod 19$ and $a(E, f) \equiv a(E, g) \equiv 12 \bmod 19$. Hence, in both cases, conditions (2) and (4) are satisfied. For (3), we have $\mathfrak{p}=(2w-2r-3)$. Again observing $\dim S_{12}(\Gamma_2)=\dim S_{14}(\Gamma_2)=1$, we have (1) and (5) by taking $m_1=1$. Using Theorem 2.3 (4) and (1), we see that (6) holds. Therefore all the conditions (1)-(6) of Theorem 3.4 are satisfied.

Thus, the congruences (4.3)-(4.5) are proved. Q. E. D.

References

[1] A. N. Andrianov, Euler products corresponding to Siegel modular forms of genus 2, Russ. Math. Surveys, **29** (1974), 45-116.

[2] A. N. Andrianov, Modular descent and the Saito-Kurokawa conjecture, Invent. Math., **53** (1979), 267-280.

[3] M. Harris, Special values of zeta functions attached to Siegel modular forms, Ann. Sci. École Norm. Sup., **14** (1981), 77-120.

[4] J. Igusa, On the ring of modular forms of degree two over $\boldsymbol{Z}$, Amer. J. Math., **101** (1979), 149-193.

[5] N. Kurokawa, Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two, Invent. Math., **49** (1978), 149-165.

[6] N. Kurokawa, Congruences between Siegel modular forms of degree two, Proc. Japan Acad. Ser. A, **55** (1979), 417-422.

[7] N. Kurokawa, Congruences between Siegel modular forms of degree two, II, Proc. Japan Acad. Ser. A, **57** (1981), 140-145.

[8] N. Kurokawa, On Siegel eigenforms, Proc. Japan Acad. Ser. A, **57** (1981), 45-50.

[9] H. Maass, Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen, Math. Ann., **126** (1953), 44-68.

[10] H. Maass, Siegel's modular forms and Dirichlet series, Lecture Notes in Math., **216**, Springer, 1971.

[11] H. Maass, Lineare Relationen für die Fourierkoeffizienten einiger Modulformen zweiten Grades, Math. Ann., **232** (1978), 163-175.

[12] H. Maass, Über eine Spezialschar von Modulformen zweiten Grades, Invent. Math., **52** (1979), 95-104.

[13] H. Maass, Über eine Spezialschar von Modulformen zweiten Grades (II), Invent. Math., **53** (1979), 249-253.

[14] H. Maass, Über eine Spezialschar von Modulformen zweiten Grades (III), Invent. Math., **53** (1979), 255-265.

[15] H. L. Resnikoff and R. L. Saldaña, Some properties of Fourier coefficients of Eisenstein series of degree two, J. Reine Angew. Math., **265** (1974), 90-109.

[16] T. Satoh, On differential operators and congruences for Siegel modular forms of degree two, Proc. Japan Acad. Ser. A, **60** (1984), 339-342.

[17] J.-P. Serre, Congruences et forms modulaires, Séminaire Bourbaki, Exp. 416 (June 1972), Lecture Notes in Math., **317**, Springer, 1973, 319-339.

[18] J. Sturm, The critical values of zeta functions associated to the symplectic group, Duke Math. J., **48** (1981), 327-350.

[19] H. P. F. Swinnerton-Dyer, On l-adic representations and congruences for coefficients of modular forms, Lecture Notes in Math., **350**, Springer, 1973, 1-55.

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