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On indefinite modular forms of weight one

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Introduction.

As shown in our previous papers ([3], [4]), there are deep relations between the class fields over imaginary quadratic fields and cusp forms of weight one of "neben typus" in Hecke's sense. In this paper we study a similar problem for class fields over real quadratic fields which satisfy a condition due to Shintani ([13]). The paper consists of five sections. In Section 1 we recall the definition of Hecke's indefinite modular forms of weight one which are associated to real quadratic fields ([1], [2], [10]). In Section 2 we summarize certain results of Shintani for the real quadratic problem which is transferable to the imaginary quadratic situation ([13]). In Section 3 we apply the result of Shintani to our problem and obtain the two representations for some dihedral cusp forms of weight one by positive definite theta series and indefinite theta series. Kac and Peterson in [7] gave many examples of new identities for cusp forms of weight one which arise from the Dedekind eta function. In Section 4 we shall reconstruct these examples from our point of view, by using the results of Section 3. In the final section we establish the higher reciprocity law for a defining equation of ray class fields over some real quadratic fields.

The authors would like to express their sincere thanks to Professor N. Iwahori for informing them of the work of Kac and Peterson ([7]). Our work has been particularly inspired by this work.

1. Hecke's indefinite modular forms of weight one.

In this section we shall review the definition and basic properties of the indefinite modular forms which were introduced by Hecke ([1], [2]).

Let F be a real quadratic field with discriminant D, and $\mathfrak{o}_F$ the ring of all integers in F. Let Q be a natural number and denote by $\mathfrak{u}_0$ the group of totally positive units ε of $\mathfrak{o}_F$ such that $\varepsilon \equiv 1 \bmod Q\sqrt{D}$. Let $\mathfrak{a}$ be an integral ideal of $\mathfrak{o}_F$, and put

$$|N(\mathfrak{a})| = A.$$

Then the Hecke modular form for the ideal $\mathfrak{a}$ is defined by

$$\vartheta_\kappa(\tau;\rho,\mathfrak{a},Q\sqrt{D})=\sum_{\substack{\mu\in\mathfrak{o}_F\\ \mu\equiv\rho \bmod \mathfrak{a}Q\sqrt{D}\\ \mu\in\mathfrak{o}_F/\mathfrak{u}_0,\, N(\mu)\kappa>0}}(\operatorname{sgn}\mu)q^{N(\mu)/(AQD)},$$

where $\kappa=\pm1$, $\rho\in\mathfrak{a}$, $\operatorname{Im}(\tau)>0$ and $q=e^{2\pi i\tau}$. This is a holomorphic function of τ and satisfies

$$\vartheta_\pm\left(\frac{a\tau+b}{c\tau+d};\rho,\mathfrak{a},Q\sqrt{D}\right)=\left(\frac{D}{|d|}\right)e^{\mp2\pi iab\rho\rho'/(AQD)}(c\tau+d)\vartheta_\pm(\tau;a\rho,\mathfrak{a},Q\sqrt{D})$$

for all $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(QD)$ ([1], [2]). 1) Therefore $\vartheta_\pm$ is the cusp form of weight one for a certain congruence subgroup of level QD under the condition $\vartheta_\pm\not\equiv0$. If in particular $\mathfrak{a}=\mathfrak{o}_F$, we put

$$\vartheta_\pm(\tau;\rho,Q\sqrt{D})=\vartheta_\pm(\tau;\rho,\mathfrak{o}_F,Q\sqrt{D}).$$

We are far from being able to judge whether $\vartheta_\pm$ vanishes identically or not.

2. Ray class fields over real quadratic fields.

In his paper ([13]), Shintani presented a conjecture on the generation of class fields over real quadratic fields in terms of the double gamma function and gave a proof for the case transferable to the imaginary quadratic situation. His conjecture is closely related to a general conjecture on the value of an Artin L-function at $s=1$, which was discovered independently by Stark. This section is based upon the work of Shintani ([13]) and we shall describe those parts of it applicable to our problem in the following section.

Let there be given a real quadratic field F as described in Section 1. Let $\mathfrak{f}$ be a self conjugate integral ideal of $\mathfrak{o}_F$ which satisfies the condition:

(1) For any totally positive unit ε of $\mathfrak{o}_F$, $\varepsilon+1\notin\mathfrak{f}$.

We denote by $H_F(\mathfrak{f})$ the narrow ray class group modulo $\mathfrak{f}$ of F. Then, under the condition (1), the group $H_F(\mathfrak{f})$ has a character χ of the following type:

$$\chi((x))=\operatorname{sgn}x \qquad\text{or}\qquad \chi((x))=\operatorname{sgn}x'$$

for $x-1\in\mathfrak{f}$, where x' denotes the conjugate of x. We denote the Hecke L-function of F attached to χ by

$$L_F(s,\chi)=\sum_{c\in H_F(\mathfrak{f})}\chi(c)\sum_{\substack{\mathfrak{a}\in c\\ \mathfrak{a}\subset\mathfrak{o}_F}}N(\mathfrak{a})^{-s}\qquad(\operatorname{Re}s>1).$$

1) For a general treatment of this function via the Weil representation, see [7] and [10].

Then the Γ-factor in the functional equation of $L_F(s, \chi)$ is of the form

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right).$$

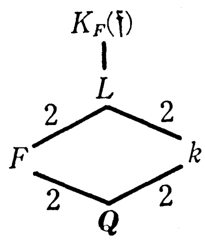
We put

$$H_F(\mathfrak{f})_0=\{c\in H_F(\mathfrak{f}) : c'=c\},$$

and assume that

$$(2) \qquad [H_F(\mathfrak{f}) : H_F(\mathfrak{f})_0]=2.$$

Let $K_F(\mathfrak{f})$ denote the maximal narrow ray class field over F corresponding to $H_F(\mathfrak{f})$ and σ denote the Artin canonical isomorphism given by class field theory. Let L be the subfield of $\sigma(H_F(\mathfrak{f})_0)$-fixed elements of $K_F(\mathfrak{f})$. Then, under the assumption (2), L is a composition of F with a suitable imaginary quadratic field k, and $K_F(\mathfrak{f})$ is an abelian extension of k ([**13**]). Therefore there exists



an integral ideal $\mathfrak{c}$ of k such that $K_F(\mathfrak{f})$ is a class field over k with conductor $\mathfrak{c}$. Let $\mathfrak{f}_\chi$ be the conductor of χ and $\tilde{\chi}$ the primitive character of $H_F(\mathfrak{f}_\chi)$ corresponding to χ. We denote by ξ_χ one of the characters of the group $H_k(\mathfrak{c})$ determined by χ in a natural manner. Let $\mathfrak{c}_\chi$ be the conductor of ξ_χ and $\tilde{\xi}_\chi$ the primitive character of $H_k(\mathfrak{c}_\chi)$ corresponding to ξ_χ. Then we have the following coincidence of two L-functions associated with the real quadratic field F and the imaginary quadratic field k ([**13**]):

$$(3) \qquad L_F(s, \tilde{\chi})=L_k(s, \tilde{\xi}_\chi).$$ [2)]

3. Positive definite and indefinite modular forms of weight one.

In this section we use the same symbols as in Section 2. We put

$$K=K_F(\mathfrak{f});$$

and assume that K/k is a cyclic extension. We denote by $\mathfrak{D}(F/\boldsymbol{Q})$ and $\mathfrak{D}(k/\boldsymbol{Q})$ the different of F over $\boldsymbol{Q}$ and that of k over $\boldsymbol{Q}$, respectively. Then we have the following relation between the conductor $\mathfrak{c}$ of the cyclic extension K/k and the finite part $\mathfrak{f}$ for the conductor of the abelian extension K/F:

2) Recently, H. Ishii proved that the coincidence (3) is equivalent to the condition (2) ([**5**]).

LEMMA 1. $\mathfrak{f}\cdot\mathfrak{D}(F/\boldsymbol{Q})=\mathfrak{c}\cdot\mathfrak{D}(k/\boldsymbol{Q})$
considered as ideals in L.

PROOF. We shall consider a triple of algebraic number fields $(\mathcal{M}, \mathcal{N}, \Omega)$ such that we have $\Omega\supset\mathcal{N}\supset\mathcal{M}$, Ω being a finite abelian extension over $\mathcal{M}$ and the degree of $\mathcal{N}$ over $\mathcal{M}$ is two. We denote by G and H the Galois group of Ω over $\mathcal{M}$ and that of Ω over $\mathcal{N}$ respectively. Let $\mathfrak{f}_{\Omega/\mathcal{M}}$ (resp. $\mathfrak{f}_{\Omega/\mathcal{N}}$) be the conductor of $\Omega/\mathcal{M}$ (resp. $\Omega/\mathcal{N}$). Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}_{\mathcal{M}}$ and $\mathfrak{P}$ a prime divisor of $\mathfrak{p}$ in $\mathcal{N}$. We denote by G_i (resp. H_i) the i-th ramification group of $\mathfrak{p}$ (resp. $\mathfrak{P}$) for $\Omega/\mathcal{M}$ (resp. $\Omega/\mathcal{N}$) and let g_i and h_i be the order of G_i and that of H_i respectively. We define the following three integers C_G, C_H and i_o by

$$C_G=\max\{i: g_i\neq 1\}, \qquad C_H=\max\{i: h_i\neq 1\} \qquad \text{and} \qquad i_o=\max\{i: G_i\not\subset H\};$$

moreover we understand $C_G=-1$ (resp. $C_H=-1$) if $\mathfrak{p}\nmid\mathfrak{f}_{\Omega/\mathcal{M}}$ (resp. $\mathfrak{P}\nmid\mathfrak{f}_{\Omega/\mathcal{N}}$) and $i_o=-1$ if $G_0\subset H$. Let $f_{\mathfrak{p}}$ (resp. $f_{\mathfrak{P}}$) be the $\mathfrak{p}$-exponent of $\mathfrak{f}_{\Omega/\mathcal{M}}$ (resp. $\mathfrak{P}$-exponent of $\mathfrak{f}_{\Omega/\mathcal{N}}$). By Hasse's theorem on conductor we have

$$f_{\mathfrak{p}}=\frac{1}{g_0}\sum_{i=0}^{C_G} g_i \qquad \text{and} \qquad f_{\mathfrak{P}}=\frac{1}{h_0}\sum_{i=0}^{C_H} h_i.$$

Furthermore, since $[\mathcal{N}:\mathcal{M}]=2$, we have

$$(*) \qquad h_i=\begin{cases} g_i/2, & \text{if } i\leqq i_o, \\ g_i, & \text{otherwise}. \end{cases}$$

Let $f_{\mathfrak{p}}\neq 0$. Since $C_G\geqq 0$ and $C_H\leqq C_G$, we have the following four possibilities for integers C_G, C_H and i_o:

(i) $i_o<0$ and $C_H=C_G$ ($\Leftrightarrow$ $\mathfrak{p}$ is unramified for $\mathcal{N}/\mathcal{M}$);
(ii) $i_o\geqq 0$ and $C_H=C_G$;
(iii) $i_o\geqq 0$, $C_H\geqq 0$ and $C_H<C_G$;
(iv) $i_o\geqq 0$ and $C_H<0\leqq C_G$ ($\Leftrightarrow$ $\mathfrak{P}$ is unramified for $\Omega/\mathcal{N}$).

By (*) we have the following corresponding to the each case of (i)-(iv):

(i)′ $f_{\mathfrak{P}}=f_{\mathfrak{p}}$;
(ii)′ $f_{\mathfrak{P}}=2f_{\mathfrak{p}}-g_{\mathfrak{P}}$;
(iii)′ $f_{\mathfrak{P}}=f_{\mathfrak{p}}-(C_G-C_H)/h_0$, $f_{\mathfrak{p}}=g_{\mathfrak{P}}$;
(iv)′ $f_{\mathfrak{P}}=0$, $f_{\mathfrak{p}}=g_{\mathfrak{P}}$,

where $g_{\mathfrak{P}}=(1/g_0)\sum_{i=0}^{i_o} g_i$, which is the $\mathfrak{P}$-exponent of the different $\mathfrak{D}(\mathcal{N}/\mathcal{M})$. If (iii) or (iv) holds, then we see easily that $i_o=C_G>C_H$,

$$(**) \qquad \#(G_{C_G})=2 \qquad \text{and} \qquad G=H\times G_{C_G} \ \text{(direct product)}.$$

Now let $(\mathcal{M}, \mathcal{N}, \Omega)=(k, L, K)$. Then, since K/k is a cyclic extension of degree $4m$, the relation (**) is impossible for G. Therefore the possible case is (i) or (ii). This shows

$$\mathfrak{c}=\mathfrak{f}_{K/k}=\mathfrak{f}_{K/L}\cdot\mathfrak{D}(L/k).$$

Next we put $(\mathcal{M}, \mathcal{N}, \Omega)=(F, L, K)$. Suppose that the relation (**) holds for (F, L, K). Then we have $C_G=C_H$ since K is the maximal ray class field mod $\mathfrak{f}(\infty_1)(\infty_2)$. This implies that neither (iii) nor (iv) occurs. Therefore we have

$$\mathfrak{f}=\mathfrak{f}_{K/F}=\mathfrak{f}_{K/L}\cdot\mathfrak{D}(L/F).$$

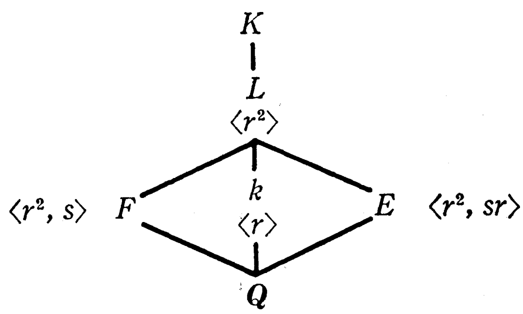
On the other hand, it is well known that

$$\mathfrak{D}(L/\boldsymbol{Q})=\mathfrak{D}(L/F)\mathfrak{D}(F/\boldsymbol{Q})=\mathfrak{D}(L/k)\mathfrak{D}(k/\boldsymbol{Q}).$$

Hence we have

$$\mathfrak{c}=\mathfrak{f}\cdot\mathfrak{D}(F/\boldsymbol{Q})\mathfrak{D}(k/\boldsymbol{Q})^{-1}.$$ Q.E.D.

Let us, temporarily, assume that $K/\boldsymbol{Q}$ is a dihedral extension. Then the Galois group $G(K/\boldsymbol{Q})$ is the dihedral group D_4 of order 8 and we have the following diagram of fields:



$G=G(K/\boldsymbol{Q})=\langle r, s\rangle$

where E denotes the imaginary quadratic field determined by F and k. The conductor $\mathfrak{c}$ of K/k is an ideal of $\boldsymbol{Z}$ by Satz 7 of Halter-Koch ([8]). Now we put

$$\mathfrak{c}=(c), \qquad c\in\boldsymbol{Z}.$$

Since $\mathfrak{f}'=\mathfrak{f}$, $(\mathfrak{f}\cdot\mathfrak{D}(F/\boldsymbol{Q}))^2$ is an ideal of $\boldsymbol{Z}$, i.e.,

$$(\mathfrak{f}\cdot\mathfrak{D}(F/\boldsymbol{Q}))^2=(q^2\cdot d),$$

where q is a positive integer and d is a positive squarefree integer. K/k being a cyclic extension by assumption, we have the following by Lemma 1.

LEMMA 2. $\quad c=q\cdot e_d^{-1}$ *and* $k=\boldsymbol{Q}(\sqrt{-d})$,

where

$$e_d=\begin{cases} 1 & \text{if } d\equiv 3 \bmod 4, \\ 2 & \text{otherwise}. \end{cases}$$

We are going to discuss how to obtain an identity between cusp forms of weight one. Take an integer μ of F such that $\mu<0$, $\mu'>0$ and $\mu\equiv 1 \bmod \mathfrak{f}$, and denote by the same letter μ the ray class modulo $\mathfrak{f}$ represented by the principal ideal (μ). Then, by the condition (1), μ is an element of order 2 of $H_F(\mathfrak{f})$, and by the condition (2), we have

$$H_F(\mathfrak{f})=H_F(\mathfrak{f})_0+H_F(\mathfrak{f})_0\mu.$$

Let $\langle\mu\mu'\rangle$ be the subgroup of $H_F(\mathfrak{f})_0$ generated by $\mu\mu'$ and let R be a complete set of representatives of $H_F(\mathfrak{f})_0 \bmod \langle\mu\mu'\rangle$. Since $\langle\mu\mu'\rangle$ is the subgroup of order 2 of $H_F(\mathfrak{f})_0$, we have

$$H_F(\mathfrak{f})=R\cup R\mu\cup R\mu'\cup R\mu\mu' \qquad \text{(disjoint)}.$$

For $c\in H_F(\mathfrak{f})$, we put

$$\zeta_F(s, c)=\sum_{\substack{\mathfrak{a}\in c\\ \mathfrak{a}\subset\mathfrak{o}_F}} N(\mathfrak{a})^{-s}.$$

Then it is easily checked that

$$\zeta_F(s, \sigma\mu)=\zeta_F(s, \sigma\mu')$$

for $\sigma\in R$. Let χ be a character of $H_F(\mathfrak{f})$ with conductor $\mathfrak{f}(\infty_1)$ satisfying the condition (1). Then the Hecke L-function of F attached to χ has the following expression

$$\begin{aligned} L_F(s, \chi)&=\sum_{\sigma\in R}\chi(\sigma)\{\zeta_F(s, \sigma)-\zeta_F(s, \sigma\mu)+\zeta_F(s, \sigma\mu')-\zeta_F(s, \sigma\mu\mu')\} \\ &=\sum_{\sigma\in R}\chi(\sigma)\{\zeta_F(s, \sigma)-\zeta_F(s, \sigma\mu\mu')\}. \end{aligned}$$

Let σ be an element of R and let $\mathfrak{a}_\sigma$ be an integral ideal of σ^{-1}. We put

$$A_\sigma^+=\{\alpha\in\mathfrak{a}_\sigma:\ \alpha\equiv 1 \bmod \mathfrak{f},\ \alpha>0,\ \alpha'>0\},$$

$$A_\sigma^-=\{\alpha\in\mathfrak{a}_\sigma:\ \alpha\equiv 1 \bmod \mathfrak{f},\ \alpha<0,\ \alpha'<0\}$$

and

$$A_\sigma=A_\sigma^+\cup A_\sigma^-.$$

Then it is easy to verify that

$$A_\sigma=\{\alpha\in\mathfrak{o}_F:\ \alpha\equiv\rho_\sigma \bmod \mathfrak{a}_\sigma\mathfrak{f},\ N(\alpha)>0\},$$

where ρ_σ denotes an element of $\mathfrak{a}_\sigma$ such that $\rho_\sigma\equiv 1 \bmod \mathfrak{f}$. Moreover, we have the following two bijections:

$$A_\sigma^+ \bmod E_\mathfrak{f}^+\ni\alpha \bmod E_\mathfrak{f}^+ \longleftrightarrow \alpha\mathfrak{a}_\sigma^{-1}\in\sigma\cap\mathfrak{o}_F$$

and

$$A_\sigma^- \bmod E_\mathfrak{f}^+\ni\alpha \bmod E_\mathfrak{f}^+ \longleftrightarrow \alpha\mathfrak{a}_\sigma^{-1}\in\sigma\mu\mu'\cap\mathfrak{o}_F,$$

where

$$E_\mathfrak{f}^+=\{\varepsilon: \text{a unit of } \mathfrak{o}_F \mid \varepsilon\equiv 1 \bmod \mathfrak{f},\ \varepsilon>0,\ \varepsilon'>0\}.$$

From these correspondences, it is easy to see that

$$\zeta_F(s, \sigma) = \sum_{\alpha \in A_\sigma^+ \bmod E_{\mathfrak{f}}^+} (N(\alpha)/N(\mathfrak{a}_\sigma))^{-s}$$

and

$$\zeta_F(s, \sigma\mu\mu') = \sum_{\alpha \in A_\sigma^- \bmod E_{\mathfrak{f}}^+} (N(\alpha)/N(\mathfrak{a}_\sigma))^{-s}.$$

Hence we obtain an explicit form of $L_F(s, \chi)$:

$$L_F(s, \chi) = \sum_{\sigma \in R} \chi(\sigma) \sum_{\alpha \in A_\sigma \bmod E_{\mathfrak{f}}^+} (\operatorname{sgn} \alpha)(N(\alpha)/N(\mathfrak{a}_\sigma))^{-s}$$
$$= \sum_{\sigma \in R} \chi(\sigma) \sum_{\alpha} (\operatorname{sgn} \alpha)(N(\alpha)/N(\mathfrak{a}_\sigma))^{-s},$$

where α in the summation runs over all integers of F such that $\alpha \equiv \rho_\sigma \bmod \mathfrak{a}_\sigma \mathfrak{f}$, $\alpha \bmod E_{\mathfrak{f}}^+$ and $N(\alpha) > 0$.

We apply the inverse Mellin transformation on the above L-function and obtain the following indefinite cusp form of weight one:

$$\theta_F(\tau) = \sum_{\sigma \in R} \chi(\sigma) \sum_{\alpha} (\operatorname{sgn} \alpha) q^{N(\alpha)/N(\mathfrak{a}_\sigma)} \qquad (q = e^{2\pi i \tau})$$
$$= \sum_{\sigma \in R} \chi(\sigma) \theta(QD_1\tau;\, \rho_\sigma, \mathfrak{a}_\sigma, \mathfrak{f}),$$

where $\mathfrak{f} = Q\mathfrak{f}_1$, $\mathfrak{f}_1 | \sqrt{D}$, $D_1 = N(\mathfrak{f}_1)$ and

$$\theta(\tau;\, \rho_\sigma, \mathfrak{a}_\sigma, \mathfrak{f}) = \sum_{\alpha} (\operatorname{sgn} \alpha) q^{N(\alpha)/N(\mathfrak{a}_\sigma)QD_1}.$$

In particular, if we put $\mathfrak{f}_1 = \sqrt{D}$, then the above function θ is just the Hecke indefinite modular form defined in Section 1.

On the other hand, since K/k is a cyclic extension, we can put

$$H_k(\mathfrak{c})/C = \langle \lambda \rangle,$$

where C denotes the subgroup of $H_k(\mathfrak{c})$ corresponding to K. The generator λ is an element of order $4m$. The restriction of the representation of Gal$(K/\boldsymbol{Q})$ induced from χ to Gal(K/k) is a direct sum of two distinct primitive characters ξ and ξ' of $H_k(\mathfrak{c})/C$ via the Artin map. Then we consider the Hecke L-function of k attached to ξ:

$$L_k(s, \xi) = \sum_{\mathfrak{a} \subset \mathfrak{o}_k} \xi(\mathfrak{a}) N(\mathfrak{a})^{-s}$$
$$= \sum_{j=0}^{4m-1} \xi(\lambda)^j \sum_{\substack{\mathfrak{a} \in \lambda^j \\ \mathfrak{a} \subset \mathfrak{o}_k}} N(\mathfrak{a})^{-s}.$$

For every odd j, the correspondence

$$\mathfrak{a} \in \lambda^j,\ \mathfrak{a} \subset \mathfrak{o}_k \longleftrightarrow \mathfrak{a}' \in \lambda^{(2m+1)j},\ \mathfrak{a}' \subset \mathfrak{o}_k$$

is bijective and

$$\xi(\lambda)^j = (-1)^j \xi(\lambda)^{(2m+1)j}.$$

Hence

$$L_k(s,\xi)=\sum_{j=0}^{2m-1}\xi(\lambda^2)^j\sum_{\substack{\mathfrak{a}\in\lambda^{2j}\\ \mathfrak{a}\subset\mathfrak{o}_k}}N(\mathfrak{a})^{-s}$$

$$=\sum_{j=0}^{m-1}\xi(\lambda^2)^j\Big\{\sum_{\substack{\mathfrak{a}\in\lambda^{2j}\\ \mathfrak{a}\subset\mathfrak{o}_k}}N(\mathfrak{a})^{-s}-\sum_{\substack{\mathfrak{a}\in\lambda^{2j+2m}\\ \mathfrak{a}\subset\mathfrak{o}_k}}N(\mathfrak{a})^{-s}\Big\}.$$

Applying the inverse Mellin transformation on the above L-function $L_k(s,\xi)$, we have the following positive definite cusp form of weight one:

$$\theta_k(\tau)=\sum_{j=0}^{m-1}\xi(\lambda^2)^j\{\theta_{2j}(\tau)-\theta_{2j+2m}(\tau)\},$$

where

$$\theta_j(\tau)=\sum_{\substack{\mathfrak{a}\in\lambda^{j}\\ \mathfrak{a}\subset\mathfrak{o}_k}}q^{N(\mathfrak{a})}\qquad(q=e^{2\pi i\tau}).$$

It is now clear that the above results, combined with the coincidence (3) in Section 2, prove the following identity:

$$\theta_F(\tau)=\theta_k(\tau).$$

From now on, we assume again that $K/\boldsymbol{Q}$ is a dihedral extension. Then $m=1$ and

$$\theta_F(\tau)=\theta(QD_1\tau;1,\mathfrak{o}_F,\mathfrak{f})=t^{-1}\vartheta_\kappa(QD_1\tau;\rho,Q\sqrt{D}),$$

where $\kappa=\pm1$, $N(\rho)\cdot\kappa>0$, $\mathfrak{f}\rho=(Q\sqrt{D})$ and $t=[E_\mathfrak{f}^+:\mathfrak{u}_0]$. Consequently we have

THEOREM 1. *The notation and assumptions being as above, we have the following identity between positive definite and indefinite cusp forms of weight one:*

$$t^{-1}\vartheta_\kappa(QD_1\tau;\rho,Q\sqrt{D})=\theta_0(\tau)-\theta_2(\tau).\tag{4}$$

Theorem 1 gives a number theoretic explanation of the identities discovered by Kac and Peterson ([7]).

4. Numerical examples.

In this section we shall give some numerical examples based on Lemma 2 and Theorem 1 in Section 3. As the method for making of the examples which are the focus of this section is the same for each, we shall give the details only for the first example.

1. For the first example we set $F=\boldsymbol{Q}(\sqrt{3})$ and $\mathfrak{f}=(2\sqrt{3})$. The fundamental unit of F is totally positive and is given by $\varepsilon=2+\sqrt{3}$. It is easy to see that $\varepsilon^2\equiv1 \bmod \mathfrak{f}$. Put $\mu=(7-6\sqrt{3})$. Then the group $H_F(\mathfrak{f})$ is an abelian group of type (2, 2):

$$H_F(\mathfrak{f})=\{1,\ \mu,\ \mu',\ \mu\mu'\}\ ;$$

and

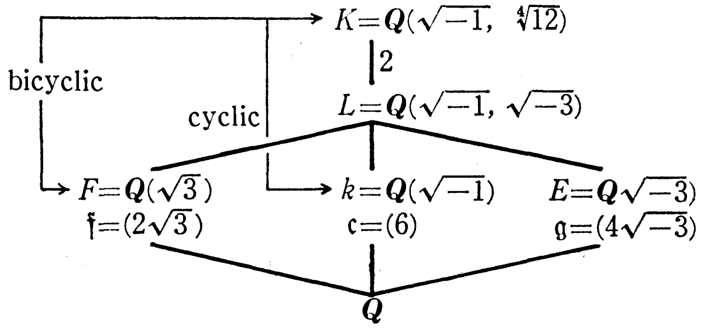
$$H_F(\mathfrak{f})_0=\{1,\ \mu\mu'\}.$$

Hence the field F and the conductor $\mathfrak{f}$ satisfy the conditions (1) and (2) in Section 2. By Lemma 2 we know that

$$k=\boldsymbol{Q}(\sqrt{-1}) \quad \text{and} \quad \mathfrak{c}=(6).$$

Furthermore, since $H_k(\mathfrak{c})$ is a group of order 4, we have $C=\{1\}$, and so

$$H_k(\mathfrak{c})=\langle\lambda\rangle, \qquad \lambda=(2+\sqrt{-1}).$$



In the following we shall look for the explicit forms of θ_k and θ_F. First we treat the function $\theta_k(\tau)$. It is easy to see that

$$\begin{cases} \mathfrak{a}\in(1) & \Longleftrightarrow \quad \mathfrak{a}=(\alpha),\ \alpha\equiv 1 \mod 6, \\ \mathfrak{a}\in\lambda^2 & \Longleftrightarrow \quad \mathfrak{a}=(\alpha),\ \alpha\equiv 2+3\sqrt{-1} \mod 6. \end{cases}$$

Hence, if $\alpha=x+3\sqrt{-1}\,y$ $((x,3)=1)$, then we have

$$(\alpha)\in(1) \quad \Longleftrightarrow \quad x\equiv 1 \mod 2 \ \text{ and } \ y\equiv 0 \mod 2,$$

$$(\alpha)\in\lambda^2 \quad \Longleftrightarrow \quad x\equiv 0 \mod 2 \ \text{ and } \ y\equiv 1 \mod 2.$$

Therefore

$$\theta_k(\tau)=\frac{1}{2}\sum_{\substack{x,y\in\boldsymbol{Z} \\ (x,3)=1,\ x\not\equiv y \bmod 2}}(-1)^y q^{x^2+9y^2}=\eta^2(12\tau).$$

Next, for the function $\theta_F(\tau)$,

$$\begin{cases} \mathfrak{a}\in(1) & \Longleftrightarrow \quad \mathfrak{a}=(\alpha),\ \alpha\gg 0 \text{ and } \alpha\equiv 1 \mod 2\sqrt{3}, \\ \mathfrak{a}\in\mu\mu' & \Longleftrightarrow \quad \mathfrak{a}=(\alpha),\ \alpha\gg 0 \text{ and } \alpha\equiv -1 \mod 2\sqrt{3}. \end{cases}$$

Therefore, if $\alpha=x+2\sqrt{3}\,y$ $(\gg 0,\ x\equiv\pm 1 \mod 6)$, we have

$$(\alpha)\in(1) \quad \Longleftrightarrow \quad x\equiv 1 \mod 3,$$

$$(\alpha)\in\mu\mu' \quad \Longleftrightarrow \quad x\equiv -1 \mod 3.$$

Since $\alpha\varepsilon^{\pm 2}=(7x\pm 24y)+(14y\pm 4x)\sqrt{3}$, we have the following as a fundamental domain:

$$x \geqq 4|y|,$$

so that

$$\theta_F(\tau)=\vartheta_+(12\tau;1,\sqrt{12})=\sum_{\substack{x,y\in\mathbf{Z}\\ x\geq 4|y|,(x,6)=1}}\left(\frac{x}{3}\right)q^{x^2-12y^2}.$$ [3)]

Another form of $\theta_F(\tau)$ is obtained as follows: Let ρ be any positive integer in F. Then it is easy to see that

$$\theta_F(\tau)=\sum_{\beta}(\operatorname{sgn}\beta)q^{N(\beta)/N(\rho)},$$

where β in the sum runs over all integers of F such that $\beta\equiv\rho \bmod \mathfrak{f}\rho$, $\beta \bmod E_{\mathfrak{f}}^+$ and $N(\beta)\cdot N(\rho)>0$. Now we set $\rho=1+\sqrt{3}$. Put

$$\beta=\begin{cases} x+y\sqrt{3}, & \text{if } \beta>0,\\ x-y\sqrt{3}, & \text{if } \beta<0\end{cases}$$

for rational integers x and y. Then, for the case $\beta>0$,

$$y>0,\quad x\equiv 1 \bmod 6 \quad\text{and}\quad x\equiv y \bmod 4.$$

Therefore we can put

$$x=6l+1,\quad y=2k+1 \quad\text{and}\quad k\equiv l \bmod 2$$

for rational integers k and l. Since $\beta\varepsilon^{\pm 2}=(7x\pm 12y)+(7y\pm 4x)\sqrt{3}$, we have

$$7y\pm 4x\geqq y,\quad \text{i.e.,}\quad 3y\geqq 2|x|,$$

and hence

$$k\geqq 2|l|.$$

For the case $\beta<0$, we have

$$y>0,\quad x\equiv 1 \bmod 6 \quad\text{and}\quad x\equiv y+2 \bmod 4.$$

Hence we put

$$x=6l+1,\quad y=2k+1 \quad\text{with}\quad k\not\equiv l \bmod 2$$

for rational integers k and l. Since $\beta\varepsilon^{\pm 2}=(7x\mp 12y)+(-7y\pm 4x)\sqrt{3}$, we also have the following as a fundamental domain:

$$k\geqq 2|l|.$$

Therefore we obtain the following expression of $\theta_F(\tau)$:

$$\theta_F(\tau)=\sum_{\substack{k,l\in\mathbf{Z}\\ k\geq 2|l|}}(-1)^{k+l}q^{(3(2k+1)^2-(6l+1)^2)/2}.$$

For comparison, we write down the expression of the above right-hand side by

3) For instance, cf. Hecke [**1**, pp. 425–426].

Hecke's modular form:

$$\vartheta_{-}(12\tau;\ 1+\sqrt{3},\ (1+\sqrt{3}),\ \sqrt{12})=\sum_{\substack{k,l\in \boldsymbol{Z}\\ k\geqq 2|l|}}(-1)^{k+l}q^{(3(2k+1)^2-(6l+1)^2)/2}.$$

By combining the above results and the identity (4), we have the following remarkable identities:

$$\begin{aligned}\theta_F(\tau)&=\vartheta_{+}(12\tau;\ 1;\ \sqrt{12})=\sum_{\substack{x,y\in \boldsymbol{Z}\\ x\geqq 4|y|,\ (x,6)=1}}\left(\frac{x}{3}\right)q^{x^2-12y^2}\\ &=\sum_{\substack{k,l\in \boldsymbol{Z}\\ k\geqq 2|l|}}(-1)^{k+l}q^{(3(2k+1)^2-(6l+1)^2)/2}\\ &=\theta_k(\tau)=\frac{1}{2}\sum_{\substack{x,y\in \boldsymbol{Z}\\ (x,3)=1,\ x\not\equiv y \bmod 2}}(-1)^y q^{x^2+9y^2}=\eta^2(12\tau),\end{aligned}$$

where $\eta(\tau)$ is Dedekind's eta function.

REMARK 1. In exactly the same way as for $\theta_k(\tau)$, we obtain

$$\theta_E(\tau)=\sum_{k,l\in \boldsymbol{Z}}(-1)^{k+l}q^{(6k+1)^2+12l^2}=\eta(24\tau)\vartheta_0(24\tau),$$

where

$$\vartheta_0(\tau)=\sum_{m\in \boldsymbol{Z}}(-1)^m e^{\pi i m^2\tau}.$$

2. We set $F=\boldsymbol{Q}(\sqrt{2})$ and $\mathfrak{f}=(4)$. The fundamental unit of F is given by $\varepsilon=1+\sqrt{2}$ and satisfies $N(\varepsilon)=-1$ and $\varepsilon^4\equiv 1 \bmod \mathfrak{f}$. Thus, in the same way as for the first example, we have

$$\begin{cases} k=\boldsymbol{Q}(\sqrt{-2}), & \mathfrak{c}=(4),\\ E=\boldsymbol{Q}(\sqrt{-1}), & \mathfrak{g}=(4(1+\sqrt{-1})),\\ K=k(\sqrt{\varepsilon}); \end{cases}$$

and obtain the following identities:

$$\begin{aligned}\theta_F(\tau)&=\vartheta_{+}(8\tau;\ 2+\sqrt{2},\ 2\sqrt{8})\\ &=\sum_{\substack{x,y\in \boldsymbol{Z}\\ x\geqq 6|y|,\ (x,2)=1}}\left(\frac{-1}{x}\right)q^{x^2-32y^2}=\sum_{\substack{m,n\in \boldsymbol{Z}\\ n\geqq 3|m|}}(-1)^n q^{(2n+1)^2-32m^2}\\ &=\theta_k(\tau)=\sum_{\substack{x,y\in \boldsymbol{Z}\\ x\equiv 1 \bmod 4}}(-1)^y q^{x^2+8y^2}\\ &=\sum_{m,n\in \boldsymbol{Z}}(-1)^n q^{(4m+1)^2+8n^2}=\eta(8\tau)\eta(16\tau)\\ &=\theta_E(\tau)=\sum_{m,n\in \boldsymbol{Z}}(-1)^{m+n}q^{(4m+1)^2+16n^2}.\end{aligned}$$

3.

$$\begin{cases} F=\boldsymbol{Q}(\sqrt{5}), & \mathfrak{f}=(4); \qquad \varepsilon=\dfrac{1+\sqrt{5}}{2}, \quad N(\varepsilon)=-1, \quad \varepsilon^6\equiv 1 \bmod \mathfrak{f}, \\ k=\boldsymbol{Q}(\sqrt{-5}), & \mathfrak{c}=(2), \\ E=\boldsymbol{Q}(\sqrt{-1}), & \mathfrak{g}=(10), \\ K=k(\sqrt{\varepsilon}). \end{cases}$$

$$\theta_F(\tau)=\frac{1}{2}\vartheta_+(4\tau;\,(5+\sqrt{5})/2,\,4\sqrt{5})$$

$$=\sum_{\substack{x,y\in\boldsymbol{Z}\\ x\geq 5|y|,\,x:\text{ odd}}}(-1)^{y+(x-1)/2}q^{x^2-20y^2}$$

$$=\sum_{\substack{k,l\in\boldsymbol{Z}\\ 2k\geq l\geq 0}}(-1)^k q^{(5(2k+1)^2-(2l+1)^2)/4}$$

$$=\theta_k(\tau)=\frac{1}{2}\sum_{\substack{x,y\in\boldsymbol{Z}\\ x\not\equiv y \bmod 2}}(-1)^y q^{x^2+5y^2}.$$

The second expression of $\theta_k(\tau)$ is obtained as follows: It is clear that $H_k(\mathfrak{c})$ is a cyclic group of order 4 and

$$H_k(\mathfrak{c})=\langle\lambda\rangle, \qquad \lambda=(3,\,1+\sqrt{-5}).$$

By the result in Section 3, we have also

$$L_k(s,\xi)=\sum_{\substack{\mathfrak{a}\in(1)\\ \mathfrak{a}\subset\mathfrak{o}_k}}N(\mathfrak{a})^{-s}-\sum_{\substack{\mathfrak{a}\in\lambda^2\\ \mathfrak{a}\subset\mathfrak{o}_k}}N(\mathfrak{a})^{-s}.$$

In the following we shall calculate the right-hand side of this equality. We can put

$$\mathfrak{a}=(\mu), \qquad \mu=a+b\sqrt{-5} \quad (a,\,b\in\boldsymbol{Z}).$$

Thus

$$\begin{cases} \mathfrak{a}\in(1) \iff \mu\equiv 1 \bmod 2 \iff a\equiv 1 \text{ and } b\equiv 0 \bmod 2, \\ \mathfrak{a}\in\lambda^2 \iff \mu\equiv 2-\sqrt{5} \bmod 2 \iff a\equiv 0 \text{ and } b\equiv 1 \bmod 2. \end{cases}$$

The contribution of ideals $\mathfrak{a}$ divided by λ to the first sum cancels that to the second sum. Therefore we may consider the ideals $\mathfrak{a}$ with $(\mathfrak{a},\lambda)=1$ in the above sum. Hence, if we put $\mu=(2a+1)+2b\sqrt{-5}$ $(a,\,b\in\boldsymbol{Z})$, we have

$$2(a-b)+1\not\equiv 0 \bmod 3.$$

On the other hand,

$$(1-\sqrt{-5})\mu=(2a+10b+1)+(2(b-a)-1)\sqrt{-5}.$$

Put $s=b-a$ and $t=a+5b$, then $t\equiv 5s \bmod 6$. Therefore we put

$$s=u+6m \qquad \text{and} \qquad t=v+6n.$$

Then

$$v\equiv 5u \mod 6 \qquad (0\leqq u,\ v\leqq 5).$$

Hence

$$2(b-a)\equiv 1 \mod 3 \iff 2u-1\equiv 0 \mod 3 \iff u=2,\ 5.$$

Therefore

$$(u,\ v)=(0,\ 0),\ (1,\ 5),\ (3,\ 3) \text{ and } (4,\ 2);$$

and

$$N(\mu)=\{(12n+2v+1)^2+5(12m+2u-1)^2\}/6.$$

Now we obtain

$$\sum_{\substack{\mathfrak{a}\in(1)\\ \mathfrak{a}\subset\mathfrak{o}_k\\ (\mathfrak{a},\lambda)=1}} N(\mathfrak{a})^{-s}=\frac{1}{2}\left\{\sum_{m,n\in \boldsymbol{Z}} 2\left(\frac{(12n+7)^2+5(12m+7)^2}{6}\right)^{-s}\right.$$

$$\left.+\sum_{m,n\in \boldsymbol{Z}} 2\left(\frac{(12n+1)^2+5(12m+1)^2}{6}\right)^{-s}\right\}$$

$$=\sum_{\substack{m,n\in \boldsymbol{Z}\\ m\equiv n \bmod 2}} (-1)^{m+n}\left(\frac{(6n+1)^2+5(6m+1)^2}{6}\right)^{-s}.$$

In the same way as above, we obtain

$$\sum_{\substack{\mathfrak{a}\in\lambda^2\\ \mathfrak{a}\subset\mathfrak{o}_k\\ (\mathfrak{a},\lambda)=1}} N(\mathfrak{a})^{-s}=\sum_{\substack{m,n\in \boldsymbol{Z}\\ m+n\equiv 1 \bmod 2}} \left(\frac{(6n+1)^2+5(6m+1)^2}{6}\right)^{-s}.$$

Therefore we have

$$L_k(s,\ \xi)=\sum_{m,n\in \boldsymbol{Z}}(-1)^{m+n}\left(\frac{(6n+1)^2+5(6m+1)^2}{6}\right)^{-s}.$$

Hence

$$\theta_k(\tau)=\sum_{m,n\in \boldsymbol{Z}}(-1)^{m+n}q^{((6n+1)^2+5(6m+1)^2)/6}$$

$$=\eta(4\tau)\eta(20\tau).$$

4.

$$\begin{cases} F=\boldsymbol{Q}(\sqrt{21}), & \mathfrak{f}=\left(\dfrac{3+\sqrt{21}}{2}\right); \qquad \varepsilon=\dfrac{5+\sqrt{21}}{2}\equiv 1 \bmod \mathfrak{f}, \\ k=\boldsymbol{Q}(\sqrt{-7}), & \mathfrak{c}=(3), \\ E=\boldsymbol{Q}(\sqrt{-3}), & \\ K=k(\sqrt{\alpha}), & \alpha=\dfrac{3+\sqrt{21}}{2}. \end{cases}$$

$$\theta_F(\tau)=\sum_{\substack{x,y\in \boldsymbol{Z}\\ x\geqq 7|y|,\ x\equiv y \bmod 2}} \left(\frac{-x}{3}\right)q^{(x^2-21y^2)/4}=\frac{1}{2}\vartheta_+(3\tau;\ (7+\sqrt{21})/2,\ \sqrt{21})$$

$$=\theta_k(\tau)=\frac{1}{2}\sum_{\substack{x,y\in \boldsymbol{Z}\\ x\equiv y \bmod 2}} \sigma(x,\ y)q^{(x^2+7y^2)/4},$$

where

$$\sigma(x, y)=\begin{cases} 1, & \text{if } 3\mid y \text{ and } 3\nmid x, \\ -1, & \text{if } 3\mid x \text{ and } 3\nmid y, \\ 0, & \text{otherwise}. \end{cases}$$

On the other hand, after a similar computation to that in Example 3, we find

$$\theta_k(\tau)=\sum_{m,n\in\boldsymbol{Z}}(-1)^{m+n}q^{((6m+1)^2+7(6n-1)^2)/8}.$$

It is well known that the right-hand side of the above identity is just the function $\eta(3\tau)\eta(21\tau)$.

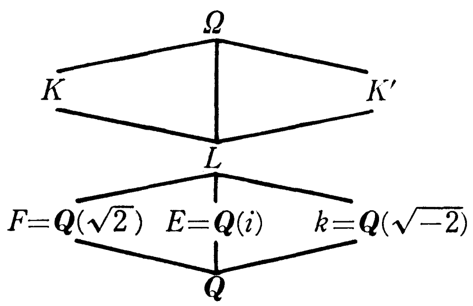
REMARK 2. Examples 1-3 were discovered by Kac-Peterson [7] by using the general theory of string functions for affine Lie algebras. A similar result was obtained for some other cases (Jimbo-Miwa [6]).

REMARK 3. $\eta(\tau)\eta(23\tau)$, $\eta(2\tau)\eta(22\tau)$ and $\eta(6\tau)\eta(18\tau)$ are of D_3-type and hence can not be expressed by the indefinite theta series.[4)]

REMARK 4. Biquadratic residue mod p and cusp forms of weight one. In Example 2, we have obtained the following identity

$$\sum_{m,n\in\boldsymbol{Z}}(-1)^n q^{(4m+1)^2+8n^2}=\sum_{m,n\in\boldsymbol{Z}}(-1)^{m+n}q^{(4m+1)^2+16n^2}, \tag{5}$$

by intermediating the function $\theta_F(\tau)$. This identity appeared for the first time in Jacobi's memoir and gives a generalization of the equivalence of Gauss' two criteria for the biquadratic residuacity of 2. In the following, we shall discuss more precisely this fact from our point of view. Consider the following diagram:



$K=\boldsymbol{Q}(i, \sqrt{\varepsilon})$, $\varepsilon=1+\sqrt{2}$

$K'=\boldsymbol{Q}(i, \sqrt[4]{2})$, $i=\sqrt{-1}$

Then, at the same time, Ω is the maximal ray class field over F mod $4\sqrt{2}(\infty_1)(\infty_2)$, over k mod $4\sqrt{-2}$ and over E mod 8. Let p and r be distinct primes such that $p\equiv r\equiv 1 \bmod 4$. We write $(r/p)_4=1$ or -1, according as r is or is not a fourth-power residue mod p. Then it is easily checked that

4) For cusp forms of weight one obtained from η, see [3] and [9].

$$\begin{aligned} p \text{ splits completely in } L &\iff (-1/p)=(-2/p)=1 \iff p\equiv 1 \bmod 8 \\ &\iff p=(4a+1)^2+8b^2 \\ &\iff p=(4\alpha+1)^2+16\beta^2; \end{aligned}$$

and moreover

$$\begin{aligned} (\varepsilon/p)=1 &\iff p \text{ splits completely in } K \\ &\iff b\equiv 0 \bmod 2 \iff \alpha+\beta\equiv 0 \bmod 2 \end{aligned} \tag{6}$$

and

$$\begin{aligned} (2/p)_4=1 &\iff p \text{ splits completely in } K' \\ &\iff a\equiv 0 \bmod 2 \iff \beta\equiv 0 \bmod 2. \end{aligned} \tag{7}$$

The above identity (5) gives a generalization of the equivalence (6); and the following identity gives a generalization of (7):

$$\frac{1}{2}\vartheta_2(8\tau)\vartheta_0(32\tau)=\sum_{\alpha,\beta\in Z}(-1)^{\beta}q^{(4\alpha+1)^2+16\beta^2}=\sum_{a,b\in Z}(-1)^{a}q^{(4a+1)^2+8b^2},$$

where

$$\vartheta_2(\tau)=\sum_{m\equiv 1 \bmod 2} e^{\pi i m^2\tau/4}.$$

We plan to discuss a more general case in a subsequent paper.

5. Higher reciprocity laws for some real quadratic fields.

Let F be a real quadratic field satisfying the conditions (1) and (2). Then there exists an imaginary quadratic field k, and two L-functions associated with F and k are coincident. Suppose that K/k is a cyclic extension and $K/\boldsymbol{Q}$ a dihedral extension. Let $f(x)$ be a defining polynomial with integer coefficients of $K/\boldsymbol{Q}$ through the real quadratic field F. Let $\mathrm{Spl}\{f(x)\}$ be the set of all primes p such that $f(x) \bmod p$ factors into a product of distinct linear polynomials over the p-elements field F_p. Then we have the following

THEOREM 2. $\mathrm{Spl}\{f(x)\}=\{p: \textit{prime} \mid p \nmid \Delta_f,\ a(p)=2\}$,

where Δ_f denotes the discriminant of f, and $a(p)$ denotes the p-th Fourier coefficient of Hecke's indefinite modular form $\theta_F(\tau)$ associated with F.

PROOF. We put

$$\theta_k(\tau)=\sum_{\mathfrak{a}\subset\mathfrak{o}_k}\xi(\mathfrak{a})q^{N(\mathfrak{a})}=\sum_{n=1}^{\infty}b(n)q^n.$$

Let $\mathfrak{p}$ be any prime ideal of k unramified for K/k. Then we know that

(i) $\xi(\mathfrak{p})=1 \Longleftrightarrow \mathfrak{p}\in(1) \Longleftrightarrow \mathfrak{p}$ splits completely in K;

(ii) $\xi(\mathfrak{p})=-1 \Longleftrightarrow \mathfrak{p}\in\lambda^2 \Longleftrightarrow \mathfrak{p}$ splits completely in L/k and remains prime in K/L;

(iii) $\xi(\mathfrak{p})=i$ or $-i \Longleftrightarrow \mathfrak{p}\in\lambda$ or $\mathfrak{p}\in\lambda^3 \Longleftrightarrow \mathfrak{p}$ remains prime in K.

Let p be a prime number and

$$p=\mathfrak{p}\mathfrak{p}' \quad \text{in} \quad k,$$

where $\mathfrak{p}'$ denotes the conjugate of a prime ideal $\mathfrak{p}$. Then

$$\mathfrak{p}\in(1) \Longrightarrow b(p)=2;$$

and vice versa. Let $F(x)$ be a defining polynomial with integer coefficients of K/k. Then it is easy to see that

$$\mathrm{Spl}\{F(x)\}=\{p \mid b(p)=2,\ p\nmid \Delta_F\},$$

where Δ_F denotes the discriminant of F. On the other hand,

$$\mathrm{Spl}\{f(x)\}\cup\{p \mid p \text{ unramified},\ p\mid\Delta_f\}=\mathrm{Spl}\{F(x)\}\cup\{p \mid p \text{ unramified},\ p\mid\Delta_F\},$$

and by Theorem 1,

$$b(p)=a(p)$$

for all p. Hence we obtain

$$\mathrm{Spl}\{f(x)\}=\{p \mid a(p)=2,\ p\nmid \Delta_f\}.$$

EXAMPLE 1′. We shall use the same symbols as in Example 1. Then we have the following defining equation of K/k:

$$F_1(x)=x^4-12 \quad \text{or} \quad F_2(x)=x^4-6x^2-3.$$

On the other hand a defining equation of K/F is given by

$$f_1(x)=x^4-4(1+\sqrt{3})x^2+4(2+\sqrt{3})^2.$$

Therefore the following is a defining equation of $K/\boldsymbol{Q}$ through the field F:

$$\begin{aligned} f(x)&=f_1(x)\cdot f_1(x)' \\ &=x^8-8x^6+24x^4+160x^2+16. \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{Spl}\{F_1(x)\}=\mathrm{Spl}\{F_2(x)\}&=\mathrm{Spl}\{f(x)\} \\ &=\{p \mid a(p)=2\} \\ &=\{p \mid p=u^2+v^2,\ u\equiv 0 \bmod 6\ (u, v\in\boldsymbol{Z})\}, \end{aligned}$$

where

$$\theta_F(\tau)=\vartheta_+(12\tau\,;\,1,\,\sqrt{12})=\sum_{n=1}^{\infty}a(n)q^n\,.$$

REMARK 5. For the defining polynomial $f(x)$ in Theorem 2, the following assertions hold:

(i) $f(x) \bmod p$ has exactly 2 distinct quartic factors over $\boldsymbol{F}_p$

$\Longleftrightarrow$ $a(p)=0$ and $a(p^2)=-1$;

(ii) $f(x) \bmod p$ has exactly 4 distinct quadratic factors over $\boldsymbol{F}_p$

$\Longleftrightarrow$ '$a(p)=-2$' or '$a(p)=0$ and $a(p^2)=1$'.

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