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Inverse problems for heat equations on compact intervals and on circles, I

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§ 1. Introduction.

The purpose of the present paper is to study uniqueness of certain inverse problems for heat equations.

For $p\in C^1[0, 1]$, $h\in \boldsymbol{R}$, $H\in \boldsymbol{R}$ and $a\in L^2(0, 1)$, let $(E_{p,h,H,a})$ be the heat equation

$$\frac{\partial u}{\partial t}+\left(p(x)-\frac{\partial^2}{\partial x^2}\right)u=0 \qquad (0<t<\infty\,; 0<x<1) \tag{1.1}$$

with the boundary condition

$$\left(\frac{\partial u}{\partial x}-hu\right)\Big|_{x=0}=\left(\frac{\partial u}{\partial x}+Hu\right)\Big|_{x=1}=0 \qquad (0<t<\infty) \tag{1.2}$$

and with the initial condition

$$u\,|_{t=0}=a(x) \qquad (0<x<1). \tag{1.3}$$

As is known, the solution $u=u(t, x)$ exists uniquely for given coefficients and initial value (p, h, H, a). However, let these (p, h, H, a) be unknown, and instead the values $u(t, 0)$ and $u(t, x_0)$ be observed for $t\in[T_1, T_2]$ and $x_0\in(0, 1]$, where $0\leqq T_1<T_2<\infty$. Do the data $\{u(t, 0), u(t, x_0)\,|\, T_1\leqq t\leqq T_2\}$ determine (p, h, H, a)? This kind of problem is called an inverse problem, and is formulated more precisely as follows.

Consider the mapping

$$F^1=F^1_{T_1,T_2,x_0}:\ (q, j, J, b)\longmapsto \{v(t, 0), v(t, x_0)\,|\, T_1\leqq t\leqq T_2\}, \tag{1.4.1}$$

where $v=v(t, x)$ is the solution of $(E_{q,j,J,b})$. Let $(p, h, H, a)\in C^1[0, 1]\times \boldsymbol{R}\times \boldsymbol{R}\times L^2(0, 1)$ be given and $u=u(t, x)$ be the solution of $(E_{p,h,H,a})$. Then the set

$$\boldsymbol{M}^1_{p,h,H,a,x_0}\equiv(F^1_{T_1,T_2,x_0})^{-1}(F^1_{T_1,T_2,x_0}(p, h, H, a)) \tag{1.5.1}$$

denotes the totality of equations $(E_{q,j,J,b})$ whose solutions have the same values as those of u on $\xi=0$, x_0. Namely,

$$M^1_{p,h,H,a,x_0}=\{(q,\ j,\ J,\ b)\in C^1[0,\ 1]\times\boldsymbol{R}\times\boldsymbol{R}\times L^2(0,\ 1)\mid \text{the solution } v=v(t,\ x) \text{ of the equation } (E_{q,j,J,b}) \text{ satisfies } v(t,\ \xi)=u(t,\ \xi)\ (T_1\leqq t\leqq T_2\,;\ \xi=0,\ x_0)\}.$$

M^1_{p,h,H,a,x_0} is independent of T_1 and T_2, because u and v are analytic in $t\in(0,\ \infty)$ and so the condition

$$v(t,\ \xi)=u(t,\ \xi)\qquad(T_1\leqq t\leqq T_2\,;\ \xi=0,\ x_0) \tag{1.6.1}$$

is equivalent to

$$v(t,\ \xi)=u(t,\ \xi)\qquad(0<t<\infty\,;\ \xi=0,\ x_0). \tag{1.6$'$.1}$$

It is obvious that $(p,\ h,\ H,\ a)\in\boldsymbol{M}^1_{p,h,H,a,x_0}$ holds. In the case of

$$\boldsymbol{M}^1_{p,h,H,a,x_0}=\{(p,\ h,\ H,\ a)\}, \tag{1.7.1}$$

on the other hand, these data $\{u(t,\ \xi)\mid T_1\leqq t\leqq T_2\,;\ \xi=0,\ x_0\}$ determine the unknown $(p,\ h,\ H,\ a)$, and the uniqueness of the problem holds.

However, (1.7.1) does not hold for arbitrary $(p,\ h,\ H,\ a)$. For instance, $u\equiv 0$ follows from $a\equiv 0$, hence

$$\boldsymbol{M}^1_{p,h,H,0,x_0}\supset\{(q,\ j,\ J,\ 0)\mid q\in C^1[0,\ 1],\ j\in\boldsymbol{R},\ J\in\boldsymbol{R}\}$$

for each p, h, H and x_0. Actually, Murayama [**9**] and Suzuki [**13**] proved

THEOREM 0. *In the case of* $x_0=1$, (1.7.1) *holds if and only if* $a\in L^2(0,\ 1)$ *is a generating element with respect to* $A_{p,h,H}$. *

Here, $A_{p,h,H}$ denotes the realization in $L^2(0,\ 1)$ of the differential operator $p(x)-\partial^2/\partial x^2$ with the boundary condition (1.2) and $a\in L^2(0,\ 1)$ is said to be a "generating element" with respect to $A_{p,h,H}$, if it is not orthogonal to any eigenfunction of $A_{p,h,H}$. This condition is examined by $\{u(t,\ 0)\mid T_1\leqq t\leqq T_2\}$, so that we can judge whether (1.7.1) holds or not by the data in this theorem.

In the present paper, we show that unfortunately (1.7.1) holds only if $x_0=1$. Namely,

THEOREM 1. (1.7.1) *holds if and only if* $x_0=1$ *and* a *is a generating element with respect to* $A_{p,h,H}$. *

In view of this, we next consider the mapping

$$F^2=F^2_{T_1,T_2,x_0}:\quad (q,\ j,\ J,\ b)\longmapsto\{v(t,\ 0),\ v(t,\ x_0),\ v_x(t,\ x_0)\mid T_1\leqq t\leqq T_2\} \tag{1.4.2}$$

and study when

$$\boldsymbol{M}^2_{p,h,H,a,x_0}=\{(p,\ h,\ H,\ a)\} \tag{1.7.2}$$

is satisfied, where

$$(1.5.2)\quad \begin{aligned} \boldsymbol{M}^2_{p,h,H,a,x_0} &\equiv (F^2_{T_1,T_2,x_0})^{-1}(F^2_{T_1,T_2,x_0}(p,\ h,\ H,\ a)) \\ &= \{(q,\ j,\ J,\ b)\in C^1[0,\ 1]\times\boldsymbol{R}\times\boldsymbol{R}\times L^2(0,\ 1) \mid \text{the solution} \\ &\quad v=v(t,\ x) \text{ of the equation } (E_{q,j,J,b}) \text{ satisfies} \\ &\quad v_x(t,\ x_0)=u_x(t,\ x_0),\ v(t,\ \xi)=u(t,\ \xi)\ (T_1\leqq t\leqq T_2\,;\ \xi=0,\ x_0)\}. \end{aligned}$$

We introduce a few notations to state our results. The eigenvalues and the eigenfunctions of $A_{p,h,H}$ are denoted by $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\phi_n\}_{n=0}^{\infty}$, respectively, the latter being normalized by $\|\phi_n\|_{L^2(0,1)}=1$. The number

$$(1.8)\qquad N=\#\{\phi_n \mid (a,\ \phi_n)_{L^2(0,1)}=0\}$$

is called the "degenerate number" of $a\in L^2(0,\ 1)$ with respect to $A_{p,h,H}$. It is calculated from $\{u(t,\ 0)\mid T_1\leqq t\leqq T_2\}$ by the method of [13].

Then we have

THEOREM 2. (i) *In the case of* $x_0=1$, (1.7.2) *holds if and only if* $N=0$.
(ii) *In the case of* $1/2<x_0<1$, (1.7.2) *holds whenever* $N<\infty$.
(iii) *In the case of* $x_0=1/2$, (1.7.2) *holds if and only if* $N\leqq 1$.
(iv) *In the case of* $0\leqq x_0<1/2$, *we always have* $M^2_{p,h,H,a,x_0}\supsetneqq\{(p,\ h,\ H,\ a)\}$. *

Thus, the position x_0 plays an important role as does the number N.

There are some related papers. S. Kitamura and S. Nakagiri considered in [5] the heat equation $u_t=(\alpha(x)u_x)_x-p(x)u$ $(0<t<\infty,\ 0<x<1)$ and gave a sufficient condition for $(\alpha(x),\ p(x))$ to be determined from full information of the solution: $\{u(t,\ x)\mid 0\leqq t<\infty,\ 0\leqq x\leqq 1\}$. They also studied the problem to determine $(\alpha(x),\ p(x))$ from $\{u(t,\ x_p)\mid 0<t<\infty\}$ for some $x_p\in[0,\ 1]$, assuming $\alpha(x)$ and $p(x)$ to be constant functions. T. I. Seidman considered in [12] the heat equation (1.1) with Dirichlet condition $u|_{x=0}=u|_{x=1}=0$. He showed that if $a\in L^2(0,\ 1)$ is a generating element in our notation, then the values $\{u_x(t,\ 0)\mid T_1\leqq t\leqq T_2\}$ determine $p(x)$ under the assumption of symmetry, that is, $p(1-x)=p(x)$ $(0\leqq x\leqq 1)$. The result is derived from an inverse spectral theorem by G. Borg [1]. A. Pierce considered in [11] the heat equation (1.1) with the null initial condition $u|_{t=0}=0$, with a homogeneous boundary condition of the third kind on $x=0$: $(u_x-hu)|_{x=0}=f$, and with the homogeneous boundary condition of the same kind on $x=1$: $(u_x-Hu)|_{x=1}=0$. He showed that under such a situation the values $\{u(t,\ 0)\mid 0<t<T_1\}$ and $f\neq 0$ determine $(p,\ h,\ H)$, by virtue of the inverse spectral theory of Gel'fand-Levitan [2] and Levitan-Gasymov [7]. Theorem 0, described above, by Murayama [9] and Suzuki [13] is an improvement of Suzuki-Murayama [17] for the equation $(E_{p,0,0,a})$. For other work, see the references of Suzuki [14, 15].

This paper is composed of five sections and an appendix. In § 2, we prepare some elementary propositions. In § 3, we show a key lemma, which is called "deformation formula" and is obtained by [**13**]. §§ 4 and 5 are devoted to the proof of Theorems 1 and 2, respectively. The deformation formula is applicable to some inverse spectral problems. In Appendix, we study the work [**1, 6, 3, 4**] of G. Borg, N. Levinson, H. Hochstadt and B. Lieberman, from that point of view.

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§ 2. Preliminaries.

Let $\Omega\subset \boldsymbol{R}^2$ be the interior of a triangle $\triangle ABC$ with $\overline{AC}=\overline{BC}$, $\angle ACB=\pi/2$, AB being parallel to either the x-axis or the y-axis, and let $r\in C^1(\bar{\Omega})$ be given. We shall state some elementary propositions on the hyperbolic equation

$$K_{xx}-K_{yy}=r(x, y)K \qquad (\text{on } \bar{\Omega}) \tag{2.1}$$

without proof. These are actually obtained by Picard's method ([**10**]). Let ν be the outer unit normal vector on $\partial\Omega$.

PROPOSITION 1. *For each* $f\in C^2(\overline{AC})$ *and* $g\in C^2(\overline{BC})$ *with* $f|_C=g|_C$, *there exists a unique* $K=K(x, y)=K(x, y\,; r, f, g)\in C^2(\bar{\Omega})$ *satisfying* (2.1) *and*

$$K|_{AC}=f, \qquad K|_{BC}=g. \tag{2.2.1}$$

Furthermore the following estimates hold, where $\tau_1: [0, \infty)\to(0, \infty)$ *is a monotone increasing continuous function:*

$$\|K(\cdot, \cdot\,; r, f, g)\|_{C^2(\bar{\Omega})}\leqq\tau_1(\|r\|_{C^1(\bar{\Omega})})(\|f\|_{C^2(\overline{AC})}+\|g\|_{C^2(\overline{BC})}). \tag{2.3.1}$$

$$\begin{aligned}&\|K(\cdot, \cdot\,; r_1, f, g)-K(\cdot, \cdot\,; r_2, f, g)\|_{C^2(\bar{\Omega})}\\ &\leqq\tau_1(\max\{\|r_1\|_{C^1(\bar{\Omega})}, \|r_2\|_{C^1(\bar{\Omega})}\})\\ &\quad\times\|r_1-r_2\|_{C^1(\bar{\Omega})}(\|f\|_{C^2(\overline{AC})}+\|g\|_{C^2(\overline{BC})}).\end{aligned} \tag{2.4.1}$$

*

PROPOSITION 2. *For each* $f\in C^2(\overline{AB})$ *and* $g\in C^1(\overline{AB})$, *there exists a unique* $K=K(x, y)=K(x, y\,; r, f, g)\in C^2(\bar{\Omega})$ *satisfying* (2.1) *and*

$$K|_{AB}=f, \qquad \frac{\partial}{\partial\nu}K\Big|_{AB}=g. \tag{2.2.2}$$

Furthermore the following estimates hold, where $\tau_2: [0, \infty)\to(0, \infty)$ *is a monotone increasing continuous function:*

$$\|K(\cdot, \cdot\,; r, f, g)\|_{C^2(\bar{\Omega})}\leqq\tau_2(\|r\|_{C^1(\bar{\Omega})})(\|f\|_{C^2(\overline{AB})}+\|g\|_{C^1(\overline{AB})}). \tag{2.3.2}$$

(2.4.2) $$\|K(\cdot,\ \cdot\ ;r_1,\ f,\ g)-K(\cdot,\ \cdot\ ;r_2,\ f,\ g)\|_{C^2(\bar{\Omega})}$$
$$\leqq \tau_2(\max\{\|r_1\|_{C^1(\bar{\Omega})},\ \|r_2\|_{C^1(\bar{\Omega})}\})$$
$$\times\|r_1-r_2\|_{C^1(\bar{\Omega})}(\|f\|_{C^2(\overline{AB})}+\|g\|_{C^1(\overline{AB})}).$$ *

PROPOSITION 3. *For each $f\in C^2(\overline{AC})$ and $g\in C^2(\overline{AB})$ with $f|_A=g|_A$, there exists a unique $K=K(x,\ y)=K(x,\ y\ ;r,\ f,\ g)\in C^2(\bar{\Omega})$ satisfying* (2.1) *and*

(2.2.3) $$K|_{AC}=f,\qquad K|_{AB}=g.$$

Furthermore the following estimates hold, where τ_3: $[0,\ \infty)\to(0,\ \infty)$ is a monotone increasing continuous function:

(2.3.3) $$\|K(\cdot,\ \cdot\ ;r,\ f,\ g)\|_{C^2(\bar{\Omega})}\leqq\tau_3(\|r\|_{C^1(\bar{\Omega})})(\|f\|_{C^2(\overline{AC})}+\|g\|_{C^2(\overline{AB})}).$$

(2.4.3) $$\|K(\cdot,\ \cdot\ ;r_1,\ f,\ g)-K(\cdot,\ \cdot\ ;r_2,\ f,\ g)\|_{C^2(\bar{\Omega})}$$
$$\leqq\tau_3(\max\{\|r_1\|_{C^1(\bar{\Omega})},\ \|r_2\|_{C^1(\bar{\Omega})}\})$$
$$\times\|r_1-r_2\|_{C^1(\bar{\Omega})}(\|f\|_{C^2(\overline{AC})}+\|g\|_{C^2(\overline{AB})}).$$ *

PROPOSITION 4. *For each $f\in C^2(\overline{AC})$, $g\in C^1(\overline{AB})$ and $h\in\boldsymbol{R}$, there exists a unique $K=K(x,\ y)=K(x,\ y\ ;r,\ h,\ f,\ g)\in C^2(\bar{\Omega})$ satisfying* (2.1) *and*

(2.2.4) $$K|_{AC}=f,\qquad \left(\frac{\partial}{\partial\nu}K+hK\right)\Big|_{AB}=g.$$

Furthermore the following estimates hold, where τ_4: $[0,\ \infty)\times[0,\ \infty)\to(0,\ \infty)$ is a monotone increasing continuous function:

(2.3.4) $$\|K(\cdot,\ \cdot\ ;r,\ h,\ f,\ g)\|_{C^2(\bar{\Omega})}$$
$$\leqq\tau_4(\|r\|_{C^1(\bar{\Omega})},\ L)(\|f\|_{C^2(\overline{AC})}+\|g\|_{C^1(\overline{AB})})\qquad(|h|\leqq L).$$

(2.4.4) $$\|K(\cdot,\ \cdot\ ;r_1,\ h_1,\ f,\ g)-K(\cdot,\ \cdot\ ;r_2,\ h_2,\ f,\ g)\|_{C^2(\bar{\Omega})}$$
$$\leqq\tau_4(\max\{\|r_1\|_{C^1(\bar{\Omega})},\ \|r_2\|_{C^1(\bar{\Omega})}\},\ L)\{\|r_1-r_2\|_{C^1(\bar{\Omega})}$$
$$+|h_1-h_2|\}(\|f\|_{C^2(\overline{AC})}+\|g\|_{C^1(\overline{AB})})\qquad(|h_1|,\ |h_2|\leqq L).$$ *

REMARK 2.1. In the proof of these propositions, the equation (2.1) with the side condition (2.2) is reduced to a certain integral equation of Volterra type. The unique existence of the solution of that integral equation holds in the class of $C^0(\bar{\Omega})$, although it is eventually shown to be a C^2-function. Suppose that $\Omega=\triangle ABC$ is divided into subdomains $\Omega_i=\triangle A_iB_iC_i$ $(1\leqq i\leqq N)$ with $\overline{A_iC_i}=\overline{B_iC_i}$ and $\angle A_iB_iC_i=\pi/2$, A_iB_i being parallel to either the x-axis or the y-axis. Suppose, furthermore, that $K\in C^0(\bar{\Omega})$ is a piecewise C^2-function and satisfies (2.1) on each $\bar{\Omega}_i$ together with the side condition (2.2.1) for example, with $f\in C^2(\overline{AC})$ and $g\in C^2(\overline{BC})$. Then, K is shown to satisfy the same integral equation as described above, so that, in particular, $K \in C^2(\bar{\Omega})$ follows. Similar facts hold for Propositions 2-4 and for Propositions 5-6 given below. *

REMARK 2.2. Let C' be the symmetric point of C with respect to the segment AB. Then, $\square ACBC'$ makes a regular tetragon, whose interior is denoted by $\hat{\Omega}$. In this case, Propositions 1-4 still hold if we replace Ω by $\hat{\Omega}$. *

The following propositions are obtained by the method of [**10**], that is, by "continuing" solutions of Propositions 1-4. In order to make statements simple, we assume $A=(0, 0)$, $B=(1, 0)$ and $C=(1/2, 1/2)$, without loss of generality. For $A'=(1, 1)$, Ω' and $\hat{\Omega}$ denote the interiors of the triangles $\triangle A'BC$ and $\triangle ABA'$, respectively. Recall that Ω denotes the interior of $\triangle ABC$.

PROPOSITION 5. *For given* $r \in C^1(\bar{\hat{\Omega}})$, $g_1 \in C^2(\overline{AB})$, $g_2 \in C^1(\overline{AB})$ *and* $f \in C^2(\overline{BA'})$, *there exists a solution* $K=K(x, y) \in C^2(\bar{\hat{\Omega}})$ *of the equation*

$$K_{xx}-K_{yy}=r(x, y)K \qquad (\text{on } \bar{\hat{\Omega}}) \tag{2.1}$$

with

$$K(x, 0)=g_1(x), \quad K_y(x, 0)=g_2(x) \qquad (0 \leqq x \leqq 1) \tag{2.2.5}$$

and

$$K(1, y)=f(y) \qquad (0 \leqq y \leqq 1), \tag{2.2.5'}$$

if and only if the compatibility condition

$$\begin{gathered} g_1(1)=f(0), \qquad g_2(1)=f'(0), \\ g_1''(1)-f''(0)=r(1, 0)g_1(1) \end{gathered} \tag{2.5.1}$$

is satisfied. Furthermore, the solution is unique. *

PROPOSITION 6. *For given* $r \in C^1(\bar{\hat{\Omega}})$, $g_1 \in C^2(\overline{AB})$, $g_2 \in C^1(\overline{AB})$, $f \in C^1(\overline{BA'})$ *and* $J \in \boldsymbol{R}$, *there exists a solution* $K=K(x, y) \in C^2(\bar{\hat{\Omega}})$ *of the equation*

$$K_{xx}-K_{yy}=r(x, y)K \qquad (\text{on } \bar{\hat{\Omega}}) \tag{2.1}$$

with

$$K(x, 0)=g_1(x), \quad K_y(x, 0)=g_2(x) \qquad (0 \leqq x \leqq 1) \tag{2.2.6}$$

and

$$K_x(1, y)+JK(1, y)=f(y) \qquad (0 \leqq y \leqq 1), \tag{2.2.6'}$$

if and only if the compatibility condition

$$g_1'(1)+Jg_1(1)=f(0), \qquad g_2'(1)+Jg_2(1)=f'(0) \tag{2.5.2}$$

is satisfied. Furthermore, the solution is unique. *

In Propositions 5 and 6, similar estimates to (2.3.1)-(2.3.4) and (2.4.1)-(2.4.4) hold for the solution K.

§3. Deformation formula.

Let $D=\{(x, y)\mid 0<y<x<1\}$. The following lemma is obtained in [**13**].

LEMMA 1 (Deformation Formula). (i) *For given* $p, q\in C^1[0, 1]$ *and* $h, j\in \boldsymbol{R}$, *there exists a unique* $K\in C^2(\bar{D})$ *such that*

$$K_{xx}-K_{yy}+p(y)K=q(x)K \qquad (on\ \bar{D}), \tag{3.1.a}$$

$$K(x, x)=(j-h)+\frac{1}{2}\int_0^x(q(s)-p(s))ds \qquad (0\leqq x\leqq 1), \tag{3.1.b}$$

$$K_y(x, 0)=hK(x, 0) \qquad (0\leqq x\leqq 1). \tag{3.1.c}$$

(ii) *If* $\Phi=\Phi(x)\in C^2[0, 1]$ *satisfies*

$$\left(p(x)-\frac{d^2}{dx^2}\right)\Phi=\lambda\Phi \quad (0\leqq x\leqq 1), \qquad \Phi'(0)=h\Phi(0) \tag{3.2}$$

for $\lambda\in\boldsymbol{R}$, *then* $\Psi=\Psi(x)\in C^2[0, 1]$ *defined by*

$$\Psi(x)=\Phi(x)+\int_0^x K(x, y)\Phi(y)dy \qquad (0\leqq x\leqq 1) \tag{3.3}$$

satisfies

$$\left(q(x)-\frac{d^2}{dx^2}\right)\Psi=\lambda\Psi \qquad (0\leqq x\leqq 1), \tag{3.4}$$

$$\Psi(0)=\Phi(0), \qquad \Psi'(0)=j\Psi(0).$$ *

(i) is shown by Propositions 4 and 1, while (ii) is obtained in an elementary way. See also [**14, 15**], for the proof.

Gel'fand-Levitan [**2**] showed the formula for $(p, h, H)=(0, 0, 0)$, in which case we have $\Phi(x)=\text{constant}\times\cos\sqrt{\lambda}x$. Suzuki-Murayama [**17**] showed the formula in the case of $h=j=0$.

§4. Proof of Theorem 1.

Recall

(4.1) $\boldsymbol{M}^1_{p,h,H,a,x_0}=\{(q, j, J, b)\in C^1[0, 1]\times\boldsymbol{R}\times\boldsymbol{R}\times L^2(0, 1)\mid$ the solution $v=v(t, x)$ of the equation $(E_{q,j,J,b})$ satisfies the following condition (4.2)$\}$, (4.2) $$v(t, \xi)=u(t, \xi) \qquad (T_1 \leqq t \leqq T_2\,;\ \xi=0,\ x_0).$$

Here $u=u(t, x)$ is the solution of $(E_{p,h,H,a})$. We want to show

(4.3) $$\boldsymbol{M}^1_{p,h,H,a,x_0} \supsetneqq \{(p,\ h,\ H,\ a)\}$$

for each (p, h, H, a), in the case of $0<x_0<1$. Let

(4.4) $$(a,\ \phi_{n_l})=0 \quad (1\leqq l \leqq N), \qquad (a,\ \phi_n)\neq 0 \quad (n\neq n_l,\ 1\leqq l\leqq N),$$

N being finite or infinite. By the definition, N is the degenerate number of a with respect to $A_{p,h,H}$. Here and henceforth (,) denotes the L^2-inner product.

Assume first that (4.2) holds for some (q, j, J, b). Then,

(4.2′) $$v(t, \xi)=u(t, \xi) \qquad (0<t<\infty\,;\ \xi=0,\ x_0)$$

holds. Let $\{\mu_m\}_{m=0}^{\infty}$ and $\{\psi_m\}_{m=0}^{\infty}$ be the eigenvalues and the eigenfunctions of $A_{q,j,J}$, respectively, the latter being normalized by $\|\psi_m\|_{L^2(0,1)}=1$. We expand u and v in terms of $\{\phi_n\}$ and $\{\psi_m\}$, respectively, and get by (4.2′)

(4.2″) $$\sum_{n=0}^{\infty} e^{-t\lambda_n}(a,\ \phi_n)\phi_n(\xi)=\sum_{m=0}^{\infty} e^{-t\mu_m}(b,\ \psi_m)\psi_m(\xi) \qquad (0<t<\infty\,;\ \xi=0,\ x_0)$$

In the same way as in [18], we compare the behavior as $t\to\infty$ of both sides of (4.2″) and see that for each $n\neq n_l$, there exists $m(n)\in \boldsymbol{N}\equiv\{0, 1, 2, \cdots\}$ such that

(4.5) $$\lambda_n=\mu_{m(n)} \qquad (n\neq n_l,\ 1\leqq l\leqq N),$$

(4.6) $$(a,\ \phi_n)\phi_n(\xi)=(b,\ \psi_{m(n)})\psi_{m(n)}(\xi) \qquad (n\neq n_l,\ 1\leqq l\leqq N\,;\ \xi=0,\ x_0),$$

and that for each $m\notin\{m(n)\,|\,n\neq n_l\}$,

(4.7) $$(b,\ \psi_m)=0 \qquad (m\notin\{m(n)\,|\,n\neq n_l\})$$

holds. Note that λ_n and μ_m are simple $(-\infty<\lambda_0<\lambda_1<\cdots\to\infty,\ -\infty<\mu_0<\mu_1<\cdots\to\infty)$, $\phi_n(0)\neq 0$ and $\psi_m(0)\neq 0$. The equalities (4.5)-(4.7) are equivalent to (4.2) under the assumption (4.4).

Set

(4.8) $$\Psi_n(x)=c_n\psi_{m(n)}(x) \qquad (n\neq n_l,\ 1\leqq l\leqq N),$$

where

(4.9) $$c_n=(b,\ \psi_{m(n)})/(a,\ \phi_n) \qquad (n\neq n_l,\ 1\leqq l\leqq N).$$

Then, $\Psi_n(x)$ satisfies, by (4.6) for $\xi=0$,

(4.10) $$\left(q(x)-\frac{d^2}{dx^2}\right)\Psi_n=\lambda\Psi_n \qquad (0\leqq x\leqq 1),$$

$$\Psi_n(0)=\phi_n(0), \qquad \Psi'_n(0)=j\Psi_n(0),$$

for $\lambda=\lambda_n\,(=\mu_{m(n)})$. Hence

$$\Psi_n(x)=\phi_n(x)+\int_0^x K(x,\,y)\phi_n(y)dy \qquad (0\leqq x\leqq 1) \tag{4.11}$$

holds by Lemma 1, for the solution K of (3.1). The equality

$$\Psi_n'(1)+J\Psi_n(1)=0 \tag{4.12}$$

yields

$$(J-H+K(1,\,1))\phi_n(1)+\int_0^1\{K_x(1,\,y)+JK(1,\,y)\}\,\phi_n(y)dy=0 \qquad (n\neq n_l,\ 1\leqq l\leqq N). \tag{4.13}$$

The equality (4.6) for $\xi=x_0$ gives

$$\Psi_n(x_0)=\phi_n(x_0) \qquad (n\neq n_l,\ 1\leqq l\leqq N), \tag{4.14}$$

which means

$$\int_0^{x_0} K(x_0,\,y)\phi_n(y)dy=0 \qquad (n\neq n_l,\ 1\leqq l\leqq N). \tag{4.15}$$

Suppose, conversely, that there exists $(q,\,j,\,J)$ and $K\in C^2(\bar{D})$ such that (3.1), (4.13) and (4.15) hold. Then, Ψ_n defined by (4.11) satisfies (4.10), (4.12) and (4.14). We show that there exists $b\in L^2(0,\,1)$ such that $(q,\,j,\,J,\,b)\in \boldsymbol{M}^1_{p,h,H,a,x_0}$. (4.10) and (4.12) imply (4.5) and (4.8) with some $m(n)\in \boldsymbol{N}$ and $c_n\in \boldsymbol{R}\setminus\{0\}$ $(n\neq n_l,\ 1\leqq l\leqq N)$. Since (4.11) gives $\Psi_n(0)=\phi_n(0)$, we get

$$c_n=\phi_n(0)/\psi_{m(n)}(0) \qquad (n\neq n_l,\ 1\leqq l\leqq N). \tag{4.16}$$

We now show that there exists $b\in L^2(0,\,1)$ such that

$$(b,\,\psi_m)=\begin{cases} c_n(a,\,\phi_n) & (m=m(n),\ n\neq n_l) \\ 0 & (m\notin\{m(n)\,|\,n\neq n_l\}). \end{cases} \tag{4.17}$$

In fact, in the case of $\#\{m(n)\,|\,n\neq n_l\}<\infty$, the assertion is obvious. In the case of $\#\{m(n)\,|\,n\neq n_l\}=\infty$, the relation $m(n)=n$ $(n\geqq n_0;\, n\neq n_l,\ 1\leqq l\leqq N)$ follows for sufficiently large n_0, from (4.5) and the asymptotic behavior of eigenvalues:

$$\lambda_n^{1/2}=n\pi+O\Big(\frac{1}{n}\Big),\quad \mu_m^{1/2}=m\pi+O\Big(\frac{1}{m}\Big) \qquad (n,\,m\to\infty). \tag{4.18}$$

Therefore, we have

$$\begin{aligned} c_n&=\phi_n(0)/\psi_{m(n)}(0) \\ &=1+O\Big(\frac{1}{n}\Big) \qquad (n\to\infty;\ n\neq n_l,\ 1\leqq l\leqq N), \end{aligned}$$

by virtue of the asymptotic behavior of eigenfunctions:

$$(4.19)\qquad \phi_n(x)=\frac{1}{\sqrt{2}}\cos n\pi x+O\Big(\frac{1}{n}\Big),\qquad \psi_m(x)=\frac{1}{\sqrt{2}}\cos m\pi x+O\Big(\frac{1}{m}\Big)$$

$$(n,\ m\to\infty),$$

and thus, the assertion has been verified. See Levitan-Sargsjan [8] for (4.18) and (4.19), for example.

Now, (4.7) follows immediately from (4.17), while (4.8), (4.14), (4.16) and (4.17) imply

$$\begin{aligned}\phi_n(0)&=c_n\psi_{m(n)}(0)\\ &=\frac{(b,\ \psi_{m(n)})}{(a,\ \phi_n)}\psi_{m(n)}(0)\qquad (n\neq n_l,\ 1\leqq l\leqq N)\end{aligned}$$

and

$$\begin{aligned}\phi_n(x_0)&=\Psi_n(x_0)\\ &=c_n\psi_{m(n)}(x_0)\\ &=\frac{(b,\ \psi_{m(n)})}{(a,\ \phi_n)}\psi_{m(n)}(x_0)\qquad (n\neq n_l,\ 1\leqq l\leqq N),\end{aligned}$$

which mean (4.6). Therefore $(q,\ j,\ J,\ b)\in \boldsymbol{M}^1_{p,h,H,a,x_0}$ holds. Furthermore, the conditions (4.6) and (4.7) determine b uniquely, and thus we have established

CLAIM 1. *Suppose* (4.4) *holds and put*

$$(4.20)\qquad \tilde{\boldsymbol{M}}^1_{p,h,H,a,x_0}\equiv\{(q,\ j,\ J)\in C^1[0,\ 1]\times\boldsymbol{R}\times\boldsymbol{R}\mid \text{there exists some } b\in L^2(0,\ 1) \text{ such that } (q,\ j,\ J,\ b)\in\boldsymbol{M}^1_{p,h,H,a,x_0}\}.$$

Then, $(q,\ j,\ J)\in\tilde{\boldsymbol{M}}^1_{p,h,H,a,x_0}$ *if and only if there exists* $K\in C^2(\bar{D})$ *satisfying* (3.1), (4.13) *and* (4.15). *Furthermere, for each* $(q,\ j,\ J)\in\tilde{\boldsymbol{M}}^1_{p,h,H,a,x_0}$, *a unique* b *satisfies* $(q,\ j,\ J,\ b)\in\boldsymbol{M}^1_{p,h,H,a,x_0}$. *

If $K\equiv 0$, $(q,\ j,\ J)=(p,\ h,\ H)$ holds by (3.1.b) and (4.13), because of $\phi_n(1)\neq 0$. If $(q,\ j)=(p,\ h)$, conversely, $K(x,\ x)=0$ $(0\leqq x\leqq 1)$ holds by (3.1.b). Put $D'=\{(x,\ y)\mid 0<1-x<y<x<1\}$. Then $K=0$ on $\overline{D\setminus D'}$ follows by Proposition 4 from (3.1.a), (3.1.c) and $K(x,\ x)=0$ $(0\leqq x\leqq 1/2)$. Now $K=0$ on $\bar{D}'$ follows by Proposition 1 from (3.1.a), $K(x,\ 1-x)=0$ $(1/2\leqq x\leqq 1)$ and $K(x,\ x)=0$ $(1/2\leqq x\leqq 1)$, hence $K\equiv 0$ holds. Therefore, the theorem has been reduced to

CLAIM 2. *In the case of* $0<x_0<1$, *there exist* $K\in C^2(\bar{D})$, $q\in C^1[0,\ 1]$, $j\in\boldsymbol{R}$ *and* $J\in\boldsymbol{R}$ *with* $K\not\equiv 0$, *satisfying* (3.1) *together with*

(4.13′.1) $J=H-K(1,1)$,

(4.13′.2) $K_x(1,y)+JK(1,y)=0 \quad (0\leqq y\leqq 1)$,

(4.15′) $K(x_0,y)=0 \quad (0\leqq y\leqq x_0)$. *

PROOF OF CLAIM 2. In view of (iv) of Theorem 2, we show the claim for the case of $1/2\leqq x_0<1$. Put $A=(0,0)$, $B=(1,1)$, $C=(1,0)$, $P=(x_0,0)$, $Q=(x_0,x_0)$ and $\rho=\overline{PC}=1-x_0>0$. On the segment PQ, we take points $P_0, P_1, \cdots, P_n$ in turn so that $\overline{PP_0}=\rho$, $\overline{P_0P_1}=\overline{P_1P_2}=\cdots=\overline{P_{n-1}P_n}=2\rho$ and $\overline{P_nQ}\leqq 2\rho$. Similarly, on the segment CB, we take points $C_1, \cdots, C_{n+1}$ in turn as $\overline{CC_1}=\overline{C_1C_2}=\cdots=\overline{C_nC_{n+1}}=2\rho$. On the line prolonged from PQ, we take P_{n+1} as $\overline{P_nP_{n+1}}=2\rho$, and the crossing of QB and $C_{n+1}P_{n+1}$ is denoted by P'_{n+1}.

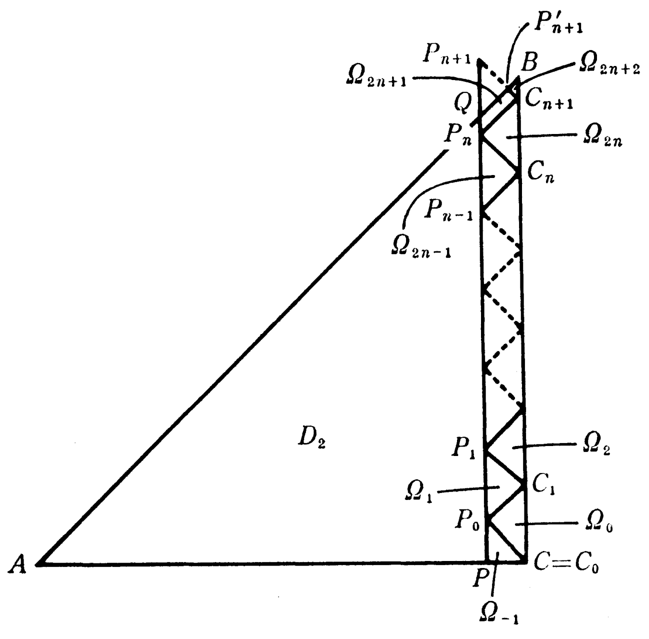

Figure 1.

Now we divide D into D_1 and D_2, where $D_2=\{(x,y)\,|\,0<y<x<x_0\}$ and $D_1=D\cap(\bar{D}_2)^c$. We furthermore divide D_1 into Ω_j $(-1\leqq j\leqq 2n+2)$, where

$\Omega_{-1}=$the interior of $\triangle PCP_0$,

$$\Omega_{2j}=\begin{cases}\text{the interior of } \triangle P_jC_jC_{j+1} & (0\leqq j\leqq n)\\ \text{the interior of } \triangle P'_{n+1}C_{n+1}B & (j=n+1),\end{cases}$$

$$\Omega_{2j-1}=\begin{cases}\text{the interior of } \triangle C_jP_{j-1}P_j & (1\leqq j\leqq n)\\ \text{the intersection of } D \text{ with the interior of } \triangle C_{n+1}P_nP_{n+1} & (j=n+1).\end{cases}$$

Henceforth we sometimes write C_0 for C.

Take $g\in C^2[x_0, 1]$ such that

$$g(x_0)=g''(x_0)=g(1)=g'(1)=0, \tag{4.21}$$

and suppose, for the monent, that $q\in C^1[0, 1]$ and $J\in \boldsymbol{R}$ are given. We shall construct $K=K(x, y)\in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c), (4.13′.2), (4.15′) and

$$K(x, 0)=g(x) \qquad (x_0\leqq x\leqq 1). \tag{4.22}$$

Firstly, by Proposition 5, there exists a unique $K_{-1}=K_{-1}(x, y)\in C^2(\bar{\Omega}_{-1})$ such that

$$\begin{cases} K_{-1xx}+K_{-1yy}+p(y)K_{-1}=q(x)K_{-1} & (\text{on } \bar{\Omega}_{-1}), \\ K_{-1}|_{PP_0}=0, \quad K_{-1}|_{PC}=g(x), \quad K_{-1y}|_{PC}=hg(x), \end{cases}$$

because of (4.21). Next, for $g_0=K_{-1}|_{CP_0}\in C^2(\overline{CP_0})$ there exists a unique $K_0=K_0(x, y)\in C^2(\bar{\Omega}_0)$ such that

$$\begin{cases} K_{0xx}-K_{0yy}+p(y)K_0=q(x)K_0 & (\text{on } \bar{\Omega}_0), \\ K_0|_{CP_0}=g_0, \quad (K_{0x}+JK_0)|_{CC_1}=0. \end{cases}$$

Similarly, setting $g_1=K_0|_{P_0P_1}$, we have $K_1=K_1(x, y)\in C^2(\bar{\Omega}_1)$ such that

$$\begin{cases} K_{1xx}-K_{1yy}+p(y)K_1=q(x)K_1 & (\text{on } \bar{\Omega}_1), \\ K_1|_{P_0C_1}=g_1, \quad K_1|_{P_0P_1}=0. \end{cases}$$

Continuing this procedure, we get $K_j=K_j(x, y)\in C^2(\bar{\Omega}_j)$ $(0\leqq j\leqq 2n)$ such that

$$K_{jxx}-K_{jyy}+p(y)K_j=q(x)K_j \qquad (\text{on } \bar{\Omega}_j),$$

with

$$\begin{cases} K_{2j}|_{C_jP_j}=K_{2j-1}|_{C_jP_j}, \quad (K_{2jx}+JK_{2j})|_{C_jC_{j+1}}=0 & (0\leqq j\leqq n), \\ K_{2j-1}|_{P_{j-1}C_j}=K_{2j-2}|_{P_{j-1}C_j}, \quad K_{2j-1}|_{P_{j-1}P_j}=0 & (1\leqq j\leqq n). \end{cases}$$

We now extend $p\in C^1[0, 1]$ to $\hat{p}\in C^1[0, 2]$ and obtain $K_{2n+1}=K_{2n+1}(x, y)\in C^2(\bar{\hat{\Omega}}_{2n+1})$, $\hat{\Omega}_{2n+1}$ being the interior of $\triangle C_{n+1}P_nP_{n+1}$, such that

$$\begin{cases} K_{2n+1\,xx}-K_{2n+1\,yy}+\hat{p}(y)K_{2n+1}=q(x)K_{2n+1} & (\text{on } \bar{\hat{\Omega}}_{2n+1}), \\ K_{2n+1}|_{P_nC_{n+1}}=K_{2n}|_{P_nC_{n+1}}, \quad K_{2n+1}|_{P_nP_{n+1}}=0. \end{cases}$$

Finally, we obtain $K_{2n+2}=K_{2n+2}(x, y)\in C^2(\bar{\Omega}_{2n+2})$ such that

$$\begin{cases} K_{2n+2\,xx}-K_{2n+2\,yy}+p(y)K_{2n+2}=q(x)K_{2n+2} & (\text{on } \bar{\Omega}_{2n+2}), \\ K_{2n+2}|_{C_{n+1}P'_{n+1}}=K_{2n+1}|_{C_{n+1}P'_{n+1}}, \\ (K_{2n+2\,x}+JK_{2n+2})|_{C_{n+1}B}=0. \end{cases}$$

Define $\tilde{K}_1\in C^0(\bar{D}_1)$ by

$$\tilde{K}_1(x, y)=K_j(x, y) \qquad ((x, y)\in\bar{\Omega}_j, \ -1\leqq j\leqq 2n+2). \tag{4.23}$$

Then, $\tilde{K}_1$ satisfies $\tilde{K}_1|_{\bar{\Omega}_j}\in C^2(\bar{\Omega}_j)$ $(-1\leqq j\leqq 2n+2)$, (4.24.a) $\tilde{K}_{1xx}-\tilde{K}_{1yy}+p(y)\tilde{K}_1=q(x)\tilde{K}_1$ (on $\bar{\Omega}_j$, $-1\leqq j\leqq 2n+2$),

(4.24.b) $\tilde{K}_1(x_0, y)=0$ $(0\leqq y\leqq x_0)$,

(4.24.c) $\tilde{K}_1(x, 0)=g(x)$, $\tilde{K}_{1y}(x, 0)=hg(x)$ $(x_0\leqq x\leqq 1)$,

and

(4.24.d) $\tilde{K}_{1x}(1, y)+J\tilde{K}_1(1, y)=0$ $(0\leqq y\leqq 1)$.

Put

$$f(y)=\tilde{K}_{1x}(x_0, y) \quad (0\leqq y\leqq x_0).$$

By Proposition 6, there exists a unique $\tilde{K}_2=\tilde{K}_2(x, y)\in C^2(\bar{D}_2)$ such that

(4.25.a) $\tilde{K}_{2xx}-\tilde{K}_{2yy}+p(y)\tilde{K}_2=q(x)\tilde{K}_2$ (on $\bar{D}_2$),

(4.25.b) $\tilde{K}_2(x_0, y)=0$, $\tilde{K}_{2x}(x_0, y)=f(y)$ $(0\leqq y\leqq x_0)$

and

(4.25.c) $\tilde{K}_{2y}(x, 0)=h\tilde{K}_2(x, 0)$ $(0\leqq x\leqq x_0)$,

because the compatibility condition

$$\begin{aligned} f'(0)&=\tilde{K}_{1xy}(x_0, 0)\\ &=hg'(x_0) \quad \text{(in fact, (4.23.c))}\\ &=h\tilde{K}_{1x}(x_0, 0) \quad \text{(in fact, (4.23.c))}\\ &=hf(0) \end{aligned}$$

is satisfied.

We define $K\in C^0(\bar{D})$ as

(4.26) $$K(x, y)=\begin{cases} \tilde{K}_1(x, y) & ((x, y)\in\bar{D}_1)\\ \tilde{K}_2(x, y) & ((x, y)\in\bar{D}_2)\end{cases}$$

and show $K\in C^2(\bar{D})$. Then, K satisfies the desired relations (3.1.a), (3.1.c), (4.13′.2), (4.15′) and (4.22). To this end, we have only to prove

(4.27) $$\hat{g}=\hat{g}(x)\equiv K(x, 0)\in C^2[0, 1].$$

In fact, if (4.27) holds, then there exists a unique $\tilde{K}=\tilde{K}(x, y)\in C^2(\bar{D})$ such that

(4.28.a) $\tilde{K}_{xx}-\tilde{K}_{yy}+p(y)\tilde{K}=q(x)\tilde{K}$ (on $\bar{D}$),

(4.28.b) $\tilde{K}(x, 0)=\hat{g}(x)$, $\tilde{K}_y(x, 0)=h\hat{g}(x)$ $(0\leqq x\leqq 1)$,

and

(4.28.c) $\tilde{K}_x(1, y)+J\tilde{K}(1, y)=0$ $(0\leqq y\leqq 1)$,

because the compatibility condition

$$\hat{g}'(1)+J\hat{g}(1)=g'(1)+Jg(1)=0$$

is satisfied by (4.21). On the other hand, $K\in C^0(\bar{D})$ is piecewise in C^2-class on $\bar{D}$ and satisfies (4.28.a) almost everywhere as well as (4.28.b) and (4.28.c). Hence by Remark 2.1, $K\equiv\tilde{K}\in C^2(\bar{D})$ follows.

In order to prove (4.27), we take a point C' on the segment PA as $\overline{PC'}=\rho$ $(=\overline{PC})$. S and S' denote the middle points of the segments P_0C and P_0C' respectively. We then obtain a regular tetragon $\square P_0S'PS$ whose interior is denoted by $\tilde{\Omega}_{-1}$. Let Ω'_{-1} and $\hat{\Omega}_{-1}$ be the interior of $\triangle PC'P_0$ and $\triangle P_0CC'$, respectively.

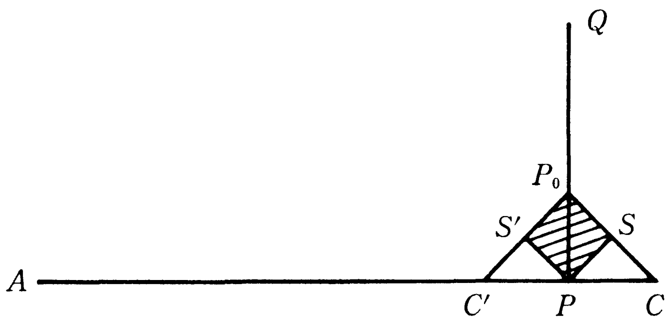

Figure 2.

Because of the definition of $\tilde{K}_2$ by (4.25), K defined by (4.26) satisfies $K|_{\bar{\tilde{\Omega}}_{-1}}\in C^2(\bar{\tilde{\Omega}}_{-1})$, on account of Remark 2.2 and the uniqueness assertion of Proposition 2. On the other hand, $K\in C^2(\bar{\Omega}_{-1})$ and $K\in C^2(\bar{\Omega}'_{-1})$ have been verified, hence $K\in C^2(\hat{\Omega}_{-1})$ follows. Therefore, $\hat{g}\in C^2(\overline{C'C})$ holds true, while $\hat{g}\in C^2(\overline{AP})$ follows from $\tilde{K}_2\in C^2(\bar{D}_2)$. Thus, (4.27) has been proved.

In this way, we have constructed $K\in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c), (4.13'.2), (4.15') and (4.22) for each $g\in C^2[x_0, 1]$ with (4.21), and for each $q\in C^1[0, 1]$ and $J\in \boldsymbol{R}$. Now we can consider the mapping

(4.29) $$T_g : C^1[0, 1] \longrightarrow C^1[0, 1];$$

$$(q, J) \longmapsto \left(2\frac{d}{dx}K(x, x)+p(x),\ H-K(1, 1)\right),$$

for each $q\in C^2[x_0, 1]$ with (4.21).

$X=C^1[0, 1]\times\boldsymbol{R}$ is a Banach space with the norm $\|(q, J)\|_X=\|q\|_{C^1[0,1]}+|J|$. Set $U_B\equiv\{(q, J)\,|\,\|(q, J)\|_X\leqq B\}$ $(B>0)$. In view of the construction of K, we get by combining the estimates (2.3.1)-(2.3.4) and (2.4.1)-(2.4.4) a monotone increasing continuous function $\tau: [0, \infty)\to(0, \infty)$ such that

(4.30) $$\|T_g(q, J)\|_X\leqq\tau(B)\|g\|_{C^2[x_0,1]}+M \qquad ((q, J)\in U_B)$$

and

(4.31)
$$\|T_g(q_1, J_1)-T_g(q_2, J_2)\|_X \leqq \tau(B)\|(q_1, J_1)-(q_2, J_2)\|_X\|g\|_{C^2[x_0, 1]} \qquad ((q_1, J_1), (q_2, J_2)\in U_B)$$

for each $B>0$, with a positive constant M depending on (p, h, H) and x_0. Therefore, for each $B>M$, there exists a positive constant δ such that $\|g\|_{C^2[x_0, 1]}\leqq\delta$ implies that T_g is a strict contraction mapping on U_B, so that it has a fixed point on U_B denoted by $(q(g), J(g))$. Construct $K_g=K_g(x, y)\in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c), (4.13′.2), (4.15′) and (4.22) for $q=q(g)$ and $J=J(g)$ as before, and set

(4.32)
$$j(g)=h+K_g(0, 0).$$

Then, $(q(g), j(g), J(g))$ and K_g satisfy (3.1.b) and (4.13′.1), while $K_g\not\equiv 0$ holds if $g\not\equiv 0$. Thus, by taking $g\in C^2[x_0, 1]$ with $g\not\equiv 0$, (4.21) and $\|g\|_{C^2[x_0, 1]}\leqq\delta$, Claim 2 has been established. □

§ 5. Proof of Theorem 2.

Recall

(5.1)
$$\boldsymbol{M}^2_{p, h, H, a, x_0}\equiv\{(q, j, J, b)\in C^1[0, 1]\times\boldsymbol{R}\times\boldsymbol{R}\times L^2(0, 1) \mid \text{the solution } v=v(t, x) \text{ of the equation } (E_{q, j, J, b}) \text{ satisfies the following condition (5.2)}\},$$

(5.2)
$$v_x(t, x_0)=u_x(t, x_0), \quad v(t, \xi)=u(t, \xi) \qquad (T_1\leqq t\leqq T_2\,;\ \xi=0, x_0).$$

Assume (4.4) and (5.2) hold. In the same way as in § 4, we expand u and v in terms of $\{\phi_n\}$ and $\{\psi_m\}$, respectively, compare both sides of (5.2) and see that for each $n\neq n_l$, there exists $m(n)\in\boldsymbol{N}$ such that

(5.3)
$$\lambda_n=\mu_{m(n)} \qquad (n\neq n_l,\ 1\leqq l\leqq N),$$

(5.4.1)
$$(a, \phi_n)\phi_n'(x_0)=(b, \psi_{m(n)})\psi_{m(n)}'(x_0) \qquad (n\neq n_l,\ 1\leqq l\leqq N),$$

(5.4.2)
$$(a, \phi_n)\phi_n(\xi)=(b, \psi_{m(n)})\psi_{m(n)}(\xi) \qquad (n\neq n_l,\ 1\leqq l\leqq N\,;\ \xi=0, x_0),$$

and that for $m\notin\{m(n) \mid n\neq n_l\}$,

(5.5)
$$(b, \psi_m)=0 \qquad (m\notin\{m(n) \mid n\neq n_l\})$$

holds. (5.3)-(5.5) are equivalent to (5.2) under (4.4). The conditions (5.3) and (5.4) are expressed in terms of K in Lemma 1, and we have

CLAIM 3. *Suppose* (4.4) *holds and put*

(5.6) $$\tilde{\boldsymbol{M}}^2_{p,h,H,a,x_0} \equiv \{(q, j, J) \in C^1[0, 1] \times \boldsymbol{R} \times \boldsymbol{R} \mid \text{there exists some } b \in L^2(0, 1) \text{ such that } (q, j, J, b) \in \boldsymbol{M}^2_{p,h,H,a,x_0}\}.$$

Then, $(q, j, J) \in \tilde{\boldsymbol{M}}^2_{p,h,H,a,x_0}$ *if and only if there exists* $K \in C^2(\bar{D})$ *satisfying* (3.1),

(5.7) $$(J-H+K(1, 1))\phi_n(1)+\int_0^1 \{K_x(1, y)+JK(1, y)\}\phi_n(y)dy=0 \quad (n \neq n_l,\ 1 \leqq l \leqq N),$$

(5.8) $$\int_0^{x_0} K(x_0, y)\phi_n(y)dy=0 \quad (n \neq n_l,\ 1 \leqq l \leqq N)$$

and

(5.9) $$K(x_0, x_0)\phi_n(x_0)+\int_0^{x_0} K_x(x_0, y)\phi_n(y)dy=0 \quad (n \neq n_l,\ 1 \leqq l \leqq N).$$

$(q, j, J)=(p, h, H)$ *if and only if* $K \equiv 0$. *For each* $(q, j, J) \in \tilde{\boldsymbol{M}}^2_{p,h,H,a,x_0}$, *a unique* b *satisfies* $(q, j, J, b) \in \boldsymbol{M}^2_{p,h,H,a,x_0}$, *hence in particular* $\boldsymbol{M}^2_{p,h,H,a,x_0}=\{(p, h, H, a)\}$ *if and only if* $\tilde{\boldsymbol{M}}^2_{p,h,H,a,x_0}=\{(p, h, H)\}$. *

Note that (5.9) follows from (5.4.1).

By Claim 3, Theorem 2 is reduced to

CLAIM 4. (α) *In the cases*

(αi) $x_0=1$, $N=0$ (αii) $1/2<x_0<1$, $N<\infty$ (αiii) $x_0=1/2$, $N \leqq 1$,

the relations (3.1) *and* (5.7)-(5.9) *imply* $K \equiv 0$.

(β) *In the case of* $x_0=1$ *and* $1 \leqq N$, *there exist* $K \in C^2(\bar{D})$, $q \in C^1[0, 1]$, $j \in \boldsymbol{R}$ *and* $J \in \boldsymbol{R}$ *with* $K \not\equiv 0$, *satisfying* (3.1) *together with* $J=H$,

(5.10) $$\int_0^1 K(1, y)\phi_n(y)dy=\int_0^1 K_x(1, y)\phi_n(y)dy=0 \quad (n \neq n_l,\ 1 \leqq l \leqq N)$$

and

(5.11) $$K(1, 1)=0.$$

(γ) *In the cases*

(γi) $x_0=1/2$, $2 \leqq N$ (γii) $0<x_0<1/2$,

there exist $K \in C^2(\bar{D})$, $q \in C^1[0, 1]$, $j \in \boldsymbol{R}$ *and* $J \in \boldsymbol{R}$ *with* $K \not\equiv 0$, *satisfying* (3.1),

(5.7′.1) $$J=H-K(1, 1),$$

(5.7′.2) $$\int_0^1 \{K_x(1, y)+JK(1, y)\}\phi_n(y)dy=0 \quad (n \neq n_l,\ 1 \leqq l \leqq N),$$

(5.8′) $$K(x_0, y)=0 \qquad (0\leqq y\leqq x_0)$$

and

(5.9′) $$K_x(x_0, y)=0 \qquad (0\leqq y\leqq x_0). \qquad *$$

REMARK 5.1. In (β), we have only to show the assertion for $N=1$. Similarly, in (γ) we have only to show the assertion for the cases of

(γi′) $$x_0=1/2, \qquad N=2$$

and

(γii′) $$0<x_0<1/2, \qquad N=0$$

instead of (γi) and (γii), respectively. *

REMARK 5.2. If $N<\infty$, (5.7) is equivalent to (5.7′.1) and (5.7′.2). *

In fact, $a_n=\int_0^1\{K_x(1, y)+JK(1, y)\}\phi_n(y)dy$ satisfies $\sum_{n=0}^{\infty}a_n^2<\infty$ because of $K\in C^2(\bar{D})$, hence $a_n\to 0$ as $n\to\infty$. On the other hand, $\phi_n(1)=(-1)^n/\sqrt{2}+O(1/n)$ $(n\to\infty)$ holds by (4.19). Therefore, (5.7) with $N<\infty$ implies (5.7′.1) and (5.7′.2). □

PROOF OF CLAIM 4 FOR THE CASE OF (αi). In this case, (5.8) and (5.9) give (5.8′) and (5.9′), because $\{\phi_n\}_{n=0}^{\infty}$ is complete in $L^2(0, 1)$. Therefore, $K\equiv 0$ follows from (3.1.a), (3.1.c), (5.8′) (with $x_0=1$) and (5.9′) (with $x_0=1$) by Proposition 6. □

PROOF OF CLAIM 4 FOR THE CASE OF (β). We assume $N=1$. Then (5.10) means

(5.10′) $$K(1, y)=c\phi_{n_1}(y), \quad K_x(1, y)=d\phi_{n_1}(y) \qquad (0\leqq y\leqq 1)$$

for some $c, d\in\boldsymbol{R}$, while (5.11) means

(5.11′) $$c=0.$$

Let $g=g(x)\in C^2[0, 1]$ satisfy

(5.12) $$\frac{d^2}{dx^2}g=\Big(2\frac{d}{dx}(\phi g_{n_1})(x)+p(x)-\lambda_{n_1}\Big)g,$$

$$g(1)=0, \qquad g'(1)=d.$$

Such $g\not\equiv 0$ exists if $|d|$ is small. Set

(5.13.1) $$K(x, y)=g(x)\phi_{n_1}(y)$$

and

(5.13.2) $$q(x)=2\frac{d}{dx}(g\phi_{n_1})(x)+p(x), \qquad j=h+g(0)\phi_{n_1}(0), \qquad J=H.$$

Then, $K\in C^2(\bar{D})$ and $(q, j, J)\in C^1[0, 1]\times\boldsymbol{R}\times\boldsymbol{R}$ satisfy $K\not\equiv 0$, (3.1.a), (3.1.c), $J=H$ and (5.10′) with $c=0$. On the other hand, (3.1.b) is shown as

$$(j-h)+\frac{1}{2}\int_0^x (q(s)-p(s))ds=g(0)\phi_{n_1}(0)+\int_0^x \frac{d}{ds}(g(s)\phi_{n_1}(s))ds$$
$$=g(x)\phi_{n_1}(x)=K(x, x).$$

Thus, the claim has been verified. □

In order to proceed to the case of $0<x_0<1$, we prepare

LEMMA 2. *If $N<\infty$, then $\{\phi_n \mid n\neq n_l,\ 1\leqq l\leqq N\}$ is complete in $L^2(a, b)$ for each subdomain $(a, b)\subsetneqq(0, 1)$.* *

In fact, if $f\in L^2(a, b)$ satisfies

$$\int_a^b f(x)\phi_n(x)dx=0 \qquad (n\neq n_l,\ 1\leqq l\leqq N),$$

then $\hat{f}(x)\in L^2(0, 1)$ defined by

$$\hat{f}(x)=\begin{cases} f(x) & (x\in(a, b)) \\ 0 & (\text{otherwise}) \end{cases}$$

satisfies

$$\int_0^1 \hat{f}(x)\phi_n(x)dx=0 \qquad (n\neq n_l,\ 1\leqq l\leqq N),$$

so that

$$\hat{f}(x)=\sum_{l=1}^N c_l\phi_{n_l}(x) \qquad (x\in(0, 1)) \tag{5.14}$$

holds for some $c_l\in\boldsymbol{R}$ $(1\leqq l\leqq N)$. Since $\hat{f}(x)=0$ on $[0, 1]\setminus(a, b)$, which is open, we operate $(p(x)-d^2/dx^2)^s$ $(0\leqq s\leqq N-1)$ there and get

$$\sum_{l=1}^N c_l(\lambda_{n_l})^s\phi_{n_l}(x)=0 \qquad (x\in[0, 1]\setminus(a, b);\ 0\leqq s\leqq N-1).$$

Recalling $\lambda_{n_1}<\cdots<\lambda_{n_N}$, we have

$$c_l\phi_{n_l}(x)=0 \qquad (x\in[0, 1]\setminus(a, b);\ 1\leqq l\leqq N)$$

and so $c_l=0$ $(1\leqq l\leqq N)$ again by the openness of $[0, 1]\setminus(a, b)$. Hence $f=0$ on (a, b) holds by (5.14). □

PROOF OF CLAIM 4 FOR THE CASE OF (αii). In this case, (5.7) implies (5.7′.1) and (5.7′.2) by Remark 5.2. Also, (5.8) and (5.9) yield (5.8′) and (5.9′), respectively, by Lemma 2.

Now, let us recall the notations in §4. (3.1.a), (3.1.c), (5.8′) and (5.9′) give $K=0$ on $\bar{D}_2$ by Proposition 6. Similarly, $K=0$ on $\bar{\Omega}_{-1}$ and $K=0$ on $\bar{\Omega}_{2j-1}$

$(1 \leqq j \leqq n)$ follow from Propositions 6 and 2, respectively. Take B_1 and Q_1 on the segments BC and QP, respectively, as $\overline{BB_1}=\overline{QQ_1}=2\rho$. Let the crossing of Q_1B_1 and P_nC_n be B_2. Denote the interior of $\triangle B_1QQ_1$ and $\triangle B_2B_1C_n$ by $\tilde{\Omega}_{2n+1}$ and $\tilde{\Omega}_{2n}$, respectively.

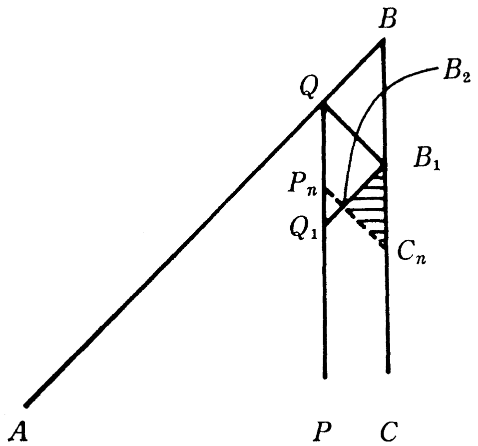

Figure 3.

Then, Proposition 2 again gives $K=0$ on $\overline{\tilde{\Omega}}_{2n+1}$. Therefore, now Proposition 1 implies $K=0$ on $\bar{\Omega}_{2j}$ $(0 \leqq j \leqq n-1)$ and $K=0$ on $\overline{\tilde{\Omega}}_{2n}$.

In this way, we have derived

(5.15) $$K=0 \qquad (\text{on } \overline{D(x_0)}),$$

where

(5.16) $$D(x_0)=D \cap \{(x, y) \mid x+y<2x_0\}.$$

In particular, (5.7′.2) gives by $1/2<x_0<1$

$$\int_{2x_0-1}^{1} \{K_x(1, y)+JK(1, y)\} \phi_n(y)dy=0 \qquad (n \neq n_l, 1 \leqq l \leqq N),$$

hence

(5.17) $$K_x(1, y)+JK(1, y)=0 \qquad (2x_0-1 \leqq y \leqq 1)$$

by Lemma 2. Now (3.1.a), (5.15) and (5.17) give

(5.18) $$K=0 \qquad (\text{on } \overline{D \setminus D(x_0)})$$

by Proposition 4, and thus $K=0$ on $\bar{D}$ has been verified. □

REMARK 5.4. Similarly, in the case of $N<\infty$ and $0<x_0 \leqq 1/2$, (5.7)-(5.9) are also reduced to (5.7′.1), (5.7′.2) and (5.15) by virtue of Lemma 2 and Proposition 6. Furthermore, in this case (5.15) is equivalent to

(5.19) $$K(x, 0)=0 \qquad (0 \leqq x \leqq 2x_0)$$

under (3.1.a) and (3.1.c), by Proposition 2. *

PROOF OF CLAIM 4 FOR THE CASE OF (αiii). In this case, (5.7)-(5.9) are reduced to (5.7′.1), (5.7′.2) with $N \leqq 1$ and (5.19) with $x_0=1/2$ by Remarks 5.2 and 5.4. (5.7′.2) with $N \leqq 1$ implies

(5.20) $$K_x(1, y)+JK(1, y)=g(y) \qquad (0 \leqq y \leqq 1)$$

with

(5.21) $$g(y)=c\phi_{n_1}(y) \qquad (0 \leqq y \leqq 1)$$

for some $c \in \boldsymbol{R}$.

(5.22) $$g(0)=0$$

follows from (5.19) (with $x_0=1/2$), hence $g \equiv 0$ holds. Now (3.1.a), (3.1.c), (5.19) and (5.20) with $g \equiv 0$ imply $K \equiv 0$ by Proposition 6. □

PROOF OF CLAIM 4 FOR THE CASE OF (γi′). In this case (5.7′.2) means (5.20) with

(5.21′) $$g(y)=\sum_{j=1}^{2} c_j \phi_{n_j}(y) \qquad (0 \leqq y \leqq 1)$$

for some $c_1, c_2 \in \boldsymbol{R}$.

Suppose, for the moment, that $q \in C^1[0, 1]$ and $J \in \boldsymbol{R}$ are given. Take g as (5.21′) with (5.22). By Proposition 6, there exists a unique $K \in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c), (5.19) (with $x_0=1/2$) and (5.20), because the compatibility condition

$$g(0)=0, \qquad g'(0)=hg(0)=0$$

is satisfied. We consider the mapping

$$T_g : C^1[0, 1] \times \boldsymbol{R} \longrightarrow C^1[0, 1] \times \boldsymbol{R};$$

$$(q, J) \longmapsto \left(2\frac{d}{dx}K(x, x)+p(x),\ H-K(1, 1)\right).$$

In the same way as in §4, we can show that T_g has a fixed point in $X=C^1[0, 1] \times \boldsymbol{R}$ if $\|g\|_{C^2[0,1]} \leqq \delta$ is satisfied for a small $\delta>0$. Noting (5.21′), we can take such $g \not\equiv 0$ with (5.22) because of $\phi_n(0) \neq 0$ $(n=0, 1, 2, \cdots)$. Therefore, in the same way as in the proof of Claim 2, we obtain (q, j, J) and $K \not\equiv 0$ satisfying (3.1), (5.7′.1), (5.7′.2) and (5.15) with $x_0=1/2$. □

PROOF OF CLAIM 4 FOR THE CASE OF (γii′). We show that there exist (q, j, J) and $K \not\equiv 0$ satisfying (3.1), (5.7′.1),

(5.7″.2) $$K_x(1, y)+JK(1, y)=0 \qquad (0 \leqq y \leqq 1)$$

and (5.19) if $0<x_0<1/2$. We take $f\in C^2[0, 1]$ such that

(5.23) $$f(1)=f'(1)=0, \qquad f(x)=0 \quad (0\leqq x\leqq 2x_0<1).$$

Then, for each $q\in C^1[0, 1]$ and $J\in \boldsymbol{R}$, there exists a unique $K\in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c), (5.7''.2) and

(5.24) $$K(x, 0)=f(x) \qquad (0\leqq x\leqq 1)$$

by Proposition 6. Therefore, we can consider the mapping

$$T_f : C^1[0, 1] \longrightarrow C^1[0, 1];$$
$$(q, J) \longmapsto \left(2\frac{d}{dx}K(x, x)+p(x),\ H-K(1, 1)\right),$$

which has a fixed point in $X=C^1[0, 1]\times \boldsymbol{R}$ if $\|f\|_{C^2[0,1]}$ is small. In the same way as in the proof of Claim 2, we obtain (q, j, J) and $K\not\equiv 0$ satisfying (3.1), (5.7'.1), (5.7''.2) and (5.15) with $0<x_0<1/2$. □

Appendix. Uniqueness theorems in inverse spectral problems.

Here we want to describe some applications of the deformation formula to the inverse Sturm-Liouville problem investigated by [**1, 6, 3, 4**]. Although our results are stated only for $p\in C^1[0, 1]$, it is possible to state them for $p\in L^1(0, 1)$ as in [**1, 6, 3, 4**], by generalizing the notion of the solution of (3.1).

Let $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_m\}_{m=0}^{\infty}$ be the eigenvalues of $A_{p,h,H}$ and $A_{q,j,J}$, respectively, where $(p, h, H), (q, j, J)\in C^1[0, 1]\times\boldsymbol{R}\times\boldsymbol{R}$. $\{\phi_n\}_{n=0}^{\infty}$ and $\{\psi_m\}_{m=0}^{\infty}$ denote the eigenfunctions of $A_{p,h,H}$ and $A_{q,j,J}$, respectively, normalized by $\|\phi_n\|_{L^2(0,1)}=\|\psi_m\|_{L^2(0,1)}=1$.

THEOREM I (Hochstadt-Lieberman [4]). *Suppose $p(x)=q(x)$ $(0\leqq x\leqq 1/2)$ and $h=j$. Suppose, furthermore, that for each $n\neq n_1$ there exists $m(n)\in \boldsymbol{N}$ such that*

(A.1) $$\lambda_n=\mu_{m(n)} \qquad (n\neq n_1).$$

Then, $p(x)=q(x)$ $(0\leqq x\leqq 1)$ and $H=J$ hold. *

REMARK I. In [4], $H=J$ and

(A.1') $$\lambda_n=\mu_{m(n)} \qquad (n=0, 1, 2, \cdots)$$

are assumed besides $p(x)=q(x)$ $(0\leqq x\leqq 1/2)$, in deriving $p=q$. *

PROOF. In terms of K in Lemma 1 in § 3, (A.1) means

$$(J-H+K(1, 1))\phi_n(1)+\int_0^1\{K_x(1, y)+JK(1, y)\}\phi_n(y)dy=0 \qquad (n\neq n_1)$$

by the argument in § 4, the equation which is equivalent to

$$J=H-K(1,1) \tag{A.2.a}$$

and

$$\int_0^1 \{K_x(1, y)+JK(1, y)\}\phi_n(y)dy=0 \qquad (n\neq n_1) \tag{A.2.b}$$

by Remark 5.2. On the other hand, $p(x)=q(x)$ $(0\leqq x\leqq 1/2)$ and $h=j$ mean $K(x, x)=0$ $(0\leqq x\leqq 1/2)$ by (3.1.b). Therefore,

$$K=0 \qquad (\text{on } \overline{D(1/2)}) \tag{A.3}$$

holds by Proposition 6, where $D(x_0)$ $(0<x_0<1)$ is the domain defined by (5.16). Now, $K=0$ on $\overline{D\setminus D(1/2)}$ follows from (A.2.b) and (A.3) in the same way as in the proof of Claim 4 for the case of (αiii). Hence $K\equiv 0$ holds, which is equivalent to $(p, h, H)=(q, j, J)$ under (3.1) and (A.2.a). □

In Theorem I, the conditions $p(x)=q(x)$ $(0\leqq x\leqq 1/2)$ and (A.1) are necessary for the uniqueness $p(x)=q(x)$ $(0\leqq x\leqq 1)$ and $J=H$ to hold. Namely, we have

THEOREM I′. (i) *For each* (p, h, H) *and* x_0 *in* $0<x_0<1/2$, *there exist* $q\neq p$, j *and* J *such that*

$$p(x)=q(x) \quad (0\leqq x\leqq x_0), \qquad \lambda_n=\mu_{m(n)} \quad (n=0, 1, 2, \cdots), \qquad h=j. \tag{A.4}$$

(ii) *For each* (p, h, H) *and* $n_1\neq n_2$, *there exist* $q\neq p$, j *and* J *such that*

$$p(x)=q(x) \quad (0\leqq x\leqq 1/2), \qquad \lambda_n=\mu_{m(n)} \quad (n\neq n_1, n_2), \qquad h=j. \tag{A.5}$$

*

PROOF. In the same way as in the proof of Theorem I, (A.4) is shown to be equivalent to (A.2.a) and

$$\begin{cases} K_x(1, y)+JK(1, y)=0 & (0\leqq y\leqq 1), \\ K=0 & \text{on } D(x_0), \end{cases} \tag{A.6}$$

and (A.5) is shown to be equivalent to (A.2.a) and

$$\begin{cases} \int_0^1 \{K_x(1, y)+JK(1, y)\}\phi_n(y)dy=0 & (n\neq n_1, n_2), \\ K=0 & \text{on } \overline{D(1/2)}. \end{cases} \tag{A.7}$$

Therefore, we have only to show that there exist (q, j, J) and $K\in C^2(\bar{D})$ with $K\not\equiv 0$ satisfying (3.1) and (A.6), and with $K\not\equiv 0$ satisfying (3.1) and (A.7) to prove (i) and (ii), respectively. However, these have been already done in the proof of Claim 4 for the cases of (γii′) and (γi′), respectively. □

Let $\{\lambda_n^*\}_{n=0}^\infty$ and $\{\mu_m^*\}_{m=0}^\infty$ be the eigenvalues of A_{p,h,H^*} and A_{q,j,J^*}, respectively, where $H\neq H^*$. $\{\phi_n^*\}_{n=0}^\infty$ and $\{\psi_m^*\}_{m=0}^\infty$ denote the eigenfunctions of A_{p,h,H^*} and A_{q,j,J^*}, respectively, normalized by $\|\phi_n^*\|_{L^2(0,1)}=\|\psi_m^*\|_{L^2(0,1)}=1$.

THEOREM II (Borg [1], Levinson [6], Hochstadt [3]). (i) *Assume that for each $n\in\boldsymbol{R}$ there exist $m(n)\in\boldsymbol{R}$ and $l(n)\in\boldsymbol{R}$ such that*

(A.8) $$\lambda_n=\mu_{m(n)}, \qquad \lambda_n^*=\mu_{l(n)}^* \qquad (n=0, 1, \cdots).$$

Then $p=q$, $h=j$, $H=J$ and $H^*=J^*$ hold.
(ii) *Assume that for each $n\neq n_1$ there exists $m(n)\in\boldsymbol{N}$ and that for each $n\in\boldsymbol{N}$ there exists $l(n)\in\boldsymbol{N}$ such that*

(A.9) $$\lambda_n=\mu_{m(n)} \quad (n\neq n_1), \qquad \lambda_n^*=\mu_{l(n)}^* \quad (n=0, 1, 2, \cdots).$$

Assume, furthermore, either $H=J$ or $H^*=J^*$. Then, $p=q$, $h=j$, $H=J$ and $H^*=J^*$ hold. *

REMARK II.1. In [1, 6, 3], (A.9) with $n_1=0$, $h=j$, $H=J$ and $H^*=J^*$ are assumed in deriving $p=q$. Levitan-Gasymov [7] reconstructed p, h, H and H^* from $\{\lambda_n, \lambda_n^* \mid n=0, 1, 2, \cdots\}$ under suitable conditions. *

PROOF OF (i). In terms of K, (A.8) means

$$(J-H+K(1, 1))\phi_n(1)+\int_0^1\{K_x(1, y)+JK(1, y)\}\phi_n(y)dy=0 \qquad (n=0, 1, 2, \cdots)$$

and

$$(J^*-H^*+K(1, 1))\phi_n^*(1)+\int_0^1\{K_x(1, y)+J^*K(1, y)\}\phi_n^*(y)dy=0 \qquad (n=0, 1, 2, \cdots),$$

which is equivalent to

(A.10) $$J=H-K(1, 1), \qquad K_x(1, y)+JK(1, y)=0 \quad (0\leqq y\leqq 1)$$

and

(A.11) $$J^*=H^*-K(1, 1), \qquad K_x(1, y)+J^*K(1, y)=0 \quad (0\leqq y\leqq 1),$$

respectively. In particular, $J\neq J^*$ holds by $H\neq H^*$. Therefore, $K_x(1, y)=K(1, y)=0$ $(0\leqq y\leqq 1)$ follows, so that $K\equiv 0$ by Proposition 6. □

PROOF OF (ii). In the same way, (A.9) implies (A.11) and

$$J=H-K(1, 1), \qquad \int_0^1\{K_x(1, y)+JK(1, y)\}\phi_n(y)dy=0 \quad (n\neq n_1),$$

which means

(A.12.a) $$J-H=J^*-H^*=-K(1, 1)$$

and

(A.12.b) $$K(1, y)=c\phi_{n_1}(y), \qquad K_x(1, y)=-J^*c\phi_{n_1}(y) \qquad (0\leqq y\leqq 1)$$

for some $c\in\boldsymbol{R}$, by $H\neq H^*$. Now either $J=H$ or $J^*=H^*$ gives $K(1, 1)=0$, hence $c=0$. Therefore, $K(1, y)=K_x(1, y)=0$ $(0\leqq y\leqq 1)$, so that $K\equiv 0$ by Proposition 6. □

In (i) of Theorem II, the condition (A.8) is necessary for the uniqueness $(p, h, H, H^*)=(q, j, J, J^*)$. In (ii) of Theorem II, the conditions $H=J$ (or $H^*=J^*$) and (A.9) are necessary for the uniqueness $(p, h, H, H^*)=(q, j, J, J^*)$. Namely, we have

THEOREM II′. (i) *For each* (p, h, H), *there exist* $(q, j)\neq(p, h)$, $J\neq H$ *and* $J^*\neq H^*$ *such that* (A.9) *holds.*
(ii) *For each* (p, h, H) *and* $n_1\neq n_2$, *there exist* $(q, j)\neq(p, h)$, J *and* J^* *such that* $J=H$, $J^*=H^*$ *and*

$$\text{(A.13)}\qquad \lambda_n=\mu_{m(n)}\quad (n\neq n_1, n_2),\qquad \lambda_n^*=\mu_{l(n)}^*\quad (n=0, 1, 2, \cdots).$$

(iii) *For each* (p, h, H), n_1 *and* n_2, *there exist* $(q, j)\neq(p, h)$, J *and* J^* *such that* $J=H$, $J^*=H^*$ *and*

$$\text{(A.14)}\qquad \lambda_n=\mu_{m(n)}\quad (n\neq n_1),\qquad \lambda_n^*=\mu_{l(n)}^*\quad (n\neq n_2).$$

PROOF OF (i). As we have seen above, (A.9) is equivalent to (A.12) for some $c\in\boldsymbol{R}$. We show that there exist (q, j, J, J^*), $K\not\equiv 0$ and c satisfying (3.1) and (A.12) to prove the theorem.

For each $q\in C^1[0, 1]$, $J\in\boldsymbol{R}$, $J^*\in\boldsymbol{R}$ and $c\in\boldsymbol{R}$, there exists a unique $K\in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c) and (4.12.b) by Proposition 6, because the compatibility condition is satisfied by $\phi_{n_1}'(0)-h\phi_{n_1}(0)=0$. Therefore, we can consider the mapping

$$T_c : C^1[0, 1]\times\boldsymbol{R}\times\boldsymbol{R}\longrightarrow C^1[0, 1]\times\boldsymbol{R}\times\boldsymbol{R};$$

$$(q, J, J^*)\longmapsto\Big(2\frac{d}{dx}K(x, x)+p(x),\ H-K(1, 1),\ H^*(1, 1)\Big).$$

By means of the estimates (2.3.1)-(2.3.4) and (2.4.1)-(2.4.4), T_c is shown to be a strict contraction mapping on a certain bounded closed ball in $\hat{X}=C^1[0, 1]\times\boldsymbol{R}\times\boldsymbol{R}$, provided that $c\in\boldsymbol{R}$ is small. Therefore, in the same way as in §§4 and 5, the assertion is verified. □

PROOF OF (ii). In terms of K, (A.13), $J=H$ and $J^*=H^*$ are equivalent to

$$\text{(A.15.a)}\qquad K(1, 1)=0$$

and

$$\int_0^1\{K_x(1, y)+HK(1, y)\}\phi_n(y)dy=0\qquad (n\neq n_1, n_2),$$

$$\int_0^1\{K_x(1, y)+H^*K(1, y)\}\phi_n^*(y)dy=0\qquad (n=0, 1, 2, \cdots).$$

The latter means

$$\text{(A.15.b)}\qquad K(1, y)=g(y),\quad K_x(1, y)=-H^*g(1, y)\qquad (0\leqq y\leqq 1)$$

with

(A.16) $$g(y)=\sum_{j=1}^{2} c_j\phi_{n_j}(y)$$

for some $c_1, c_2\in \boldsymbol{R}$ by $H\neq H^*$. We show that there exist (q, j, J), $K\not\equiv 0$ and c_1, c_2, satisfying (3.1), (A.15) with (A.16).

For each $q\in C^1[0, 1]$ and $c_1, c_2\in \boldsymbol{R}$ with

(A.17) $$g(1)=\sum_{j=1}^{2} c_j\phi_{n_j}(1)=0,$$

there exists a unique $K\in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c) and (A.15.b), because the compatibility condition

$$g'(0)-hg(0)=0$$

is satisfied. We now consider the mapping

$$T_{c_1,c_2} : C^1[0, 1] \longrightarrow C^1[0, 1];$$

$$q \longmapsto 2\frac{d}{dx}K(x, x)+p(x).$$

For each sufficiently small $(c_1, c_2)\neq(0, 0)$ with (4.17), T_{c_1,c_2} has a fixed point, which proves the assertion in the same way as in §§ 4 and 5. Note that (A.15.a) follows from (A.17). □

PROOF OF (iii). In the same way, (A.14), $J=H$ and $J^*=H^*$ are equivalent to

(A.18.a) $K(1, 1)=0$

and

(A.18.b) $K(1, y)=g_1(y), \quad K_x(1, y)=g_2(y) \qquad (0\leqq y\leqq 1)$

with

(A.19) $$\begin{cases} g_1(y)=c\phi_{n_1}(y)+d\phi^*_{n_2}(y), \\ g_2(y)=-H^*c\phi_{n_1}(y)-Hd\phi^*_{n_2}(y) \end{cases}$$

for some $c, d\in \boldsymbol{R}$. We show that there exist (q, j, J), $K\not\equiv 0$ and c, d satisfying (3.1) and (A.18) with (A.19).

For each $q\in C^1[0, 1]$ and $c, d\in \boldsymbol{R}$ with

(A.20) $$g_1(1)=c\phi_{n_1}(1)+d\phi^*_{n_2}(1)=0,$$

there exists a unique $K\in C^2(\bar{D})$ satisfying (3.1.a), (3.1.c) and (A.19), because the compatibility condition

$$g_1'(0)-hg_1(0)=g_2'(0)-hg_2(0)=0$$

is satisfied. We now consider the mapping

$$T_{c,d}\,:\,C^1[0,\,1]\longrightarrow C^1[0,\,1]\,;$$
$$q\longmapsto 2\frac{d}{dx}K(x,\,x)+p(x)\,.$$

For a sufficiently small $c, d\in\boldsymbol{R}$ with (A.20), $T_{c,d}$ has a fixed point. Since $\phi_n(1)\neq 0$ and $\phi_n^*(1)\neq 0$, we can take such $(c, d)\neq(0, 0)$ and the assertion is proved in the same way as in §§ 4 and 5. □

REMARK II.2. (i) and (ii) of Theorem II′ can be generalized as the following Theorem II″. For the proof, see [**18**]. Hochstadt [**3**] studied the same problem. The nonlinear equation (A.21) is a generalization of (5.12). See also [**13**]. □

THEOREM II″. *Let N be finite and set*

$$\boldsymbol{G}=\left\{G\in C^2([0,\,1]\to\boldsymbol{R}^N)\;\middle|\;G\ \text{satisfies}\ \frac{d^2}{dx^2}G=\left[\left(2\frac{d}{dx}(G\cdot\Phi)+p\right)I-\Lambda\right]G\right\},$$

where $\cdot$ and I denote the inner product and the unit matrix in $\boldsymbol{R}^N$, and where $\Phi=\Phi(x)={}^T(\phi_{n_1}(x),\,\cdots,\,\phi_{n_N}(x))$ *and* $\Lambda=\begin{pmatrix}\lambda_{n_1} & & 0\\ & \ddots & \\ 0 & & \lambda_{n_N}\end{pmatrix}$. *Then, (q, j, J, J^*) satisfies*

(A.21) $$\lambda_n=\mu_{m(n)}\quad(n\neq n_i,\ 1\leqq i\leqq N),\qquad \lambda_n^*=\mu_{l(n)}^*\quad(n=0,\,1,\,2,\,\cdots)$$

if and only if there exists $G\in\boldsymbol{G}$ with

(A.22) $$G'(1)+(H^*-(G\cdot\Phi)(1))G(1)=0$$

such that

(A.23) $$q(x)=p(x)+2\frac{d}{dx}(G\cdot\Phi)(x),\qquad j=h+(G\cdot\Phi)(0),$$
$$J=H-(G\cdot\Phi)(1),\qquad J^*=H^*-(G\cdot\Phi)(1)\,.$$
*

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