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Zariski decomposition and canonical rings of elliptic threefolds

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Introduction.

A surjective holomorphic mapping $f: M \to S$ of compact complex manifolds is called an elliptic fiber space if any general fiber of f is a smooth elliptic curve. In this paper, chiefly in case $\dim M=3$, we study the graded algebra $\bigoplus_{t\geqq 0}H^0(M, tK_M)$, which is called the canonical ring of M. In particular we will prove that this is finitely generated (see (3.5) below).

As we outlined in [**F4**], we proceed as follows. §1 is devoted to a theory of Zariski decomposition in higher dimensions (cf. [**Z**], [**F1**]). In §2 we present a canonical bundle formula of Kodaira-Ueno type for elliptic fiber spaces (cf. [**Ko1**], [**U1**]). In §3, combining these two theories, we prove the main theorem with the help of a result in [**F5**] concerning "fractionally logarithmic" canonical rings of surfaces.

In the Appendix we consider the case of manifolds of general type from the view point of Zariski decomposition.

NOTATION, CONVENTION AND TERMINOLOGY. Basically we employ the customary notation in algebraic geometry. *Manifold* means a non-singular complete projective variety defined over the complex number field $\boldsymbol{C}$. A surjective morphism $f: M \to S$ is called a *fiber space* if any general fiber of f is connected. The canonical bundle of a manifold X is denoted by K_X. ω_X denotes the dualizing sheaf $\mathcal{O}_X[K_X]$. Line bundles and invertible sheaves are identified in the natural way. But tensor products of line bundles are denoted additively, while we write $\otimes$ for invertible sheaves. Thus, for example, if $\mathcal{F}$ is a coherent sheaf and if L is a line bundle, $\mathcal{F}[2L]$ denotes $\mathcal{F}\otimes\mathcal{L}\otimes\mathcal{L}$, where $\mathcal{L}$ is the invertible sheaf corresponding to L. Given a morphism $g: V \to W$ of varieties and a line bundle L on W, g^*L is often denoted by L_W. Similar notation is used for other cases in which g^* is defined.

§1. Zariski decomposition in higher dimension.

(1.1) For the sake of simplicity we shall work in the category of projective varieties defined over the complex number field $\boldsymbol{C}$. A *$\boldsymbol{Q}$-divisor* on a manifold (=non-singular variety) M is a formal linear combination $D=\sum_i \mu_i D_i$ of prime divisors D_i on M with coefficients μ_i being rational numbers. D is said to be *effective* if each $\mu_i \geqq 0$. A *$\boldsymbol{Q}$-bundle* on M is an element of $\mathrm{Pic}(M)\otimes \boldsymbol{Q}$. A $\boldsymbol{Q}$-divisor D defines naturally a $\boldsymbol{Q}$-bundle, which is denoted by $[D]$ or sometimes just by D by abuse of notation.

Intersection numbers of $\boldsymbol{Q}$-bundles are defined naturally and they are rational numbers. A $\boldsymbol{Q}$-bundle L is said to be *nef* (or numerically semipositive) if $LC \geqq 0$ for any curve C in M.

(1.2) Let L be a $\boldsymbol{Q}$-bundle on M. An effective $\boldsymbol{Q}$-divisor E on M is said to *clutch* L if $F-E$ is effective for any effective $\boldsymbol{Q}$-divisor F such that $L-F$ is nef. Given a surjective morphism $f: V \to M$, we sometimes say that E clutches L on V if f^*E clutches f^*L. The following assertion is obvious by definition.

(1.3.1) If E_1 and E_2 clutch L, then $\mathrm{Max}(E_1, E_2)$ clutches L.

(1.3.2) If E_1 clutches L and if E_2 clutches $L-E_1$, then E_1+E_2 clutches L.

(1.4) LEMMA. *Let $f: M' \to M$ be a surjective morphism and suppose that f^*E clutches f^*L. Then E clutches L.*

PROOF. If $L-F$ is nef, then so is f^*L-f^*F. Hence $f^*F-f^*E=f^*(F-E)$ is effective. So $F-E$ is effective.

(1.5) LEMMA. *Let $f: M \to V$ be a surjective morphism onto a variety V, let X be an effective $\boldsymbol{Q}$-divisor on M, let $Y=\{Y_1, \cdots, Y_r\}$ be a family of finite number of prime divisors on M and let Z be an irreducible component of $f(\mathrm{Supp}(X)) \subset V$. Assume one of the following two conditions:*

1) $\dim Z \leqq \dim V-2$.

2) $\dim Z=\dim V-1$, *every general fiber of f is connected and there is a prime divisor D on M not contained in $\mathrm{Supp}(X)$ such that $f(D)=Z$.*

Then there is an open dense subset U of Z such that, for every $u \in U$, there exists a curve C with the following properties: i) $f(C)=u$. ii) $XC<0$. iii) *C is not contained in more than one member of Y.*

PROOF. Cases 1) and 2) are treated similarly. Clearly we may assume that every component of X is a member of Y. We use the induction on $n=\dim M$. Note that $\dim M>\dim V$ in case 2).

When $n=2$, the assertion follows from the index theorem in case 1). In case 2), V is a curve and Z is a point on it. So the assertion is well-known and easy to prove.

Next we consider the case in which Z is a point. Take a general hyperplane section A of M. Since A is general, we may assume that the restrictions of Y_i and Y_j to A have no common component for every $i\neq j$. Let Y_A be the family of components of the restrictions to A of members of Y. Applying the induction hypothesis to $f_A: A\to f(A)$, X_A and Y_A, we find a curve C in $f^{-1}(Z)$ such that $XC<0$ and C is not contained in more than one member of Y_A. This curve C has the desired property with respect to f, X and Y.

Finally we consider the case $\dim Z>0$. Let U be the set of points u on Z such that there exists a subvariety W of V with the following properties a)-d): a) $\dim W=\dim V-\dim Z$ and $\dim f^{-1}(W)=n-\dim Z$. b) u is an isolated point of $Z\cap W$. c) $f^{-1}(W)$ is smooth at any point in $f^{-1}(u)$. d) For every $i\neq j$, the restrictions of Y_i and Y_j to $f^{-1}(W)$ have no common component.

Now, letting T be a non-singular model of $f^{-1}(W)$ and applying the preceding argument to $T\to W$, we find a curve C in $f^{-1}(u)$ with the desired property.

On the other hand, taking general hyperplane sections on V ($\dim Z$)-times successively, we find many subvarieties W as above and many points u on $W\cap Z$. So U is open and dense in Z. Thus we complete the proof.

(1.6) COROLLARY. *Let $\pi: M'\to M$ be a birational morphism and suppose that the strict transform E' of an effective $\boldsymbol{Q}$-divisor E on M clutches π^*L. Then E clutches L both on M' and M.*

PROOF. Set $E^*=\pi^*E=E'+\sum\delta_iD_i$ where each D_i is a prime divisor on M' such that $\operatorname{codim}(\pi(D_i))\geqq 2$. Suppose that π^*L-F is nef for some effective $\boldsymbol{Q}$-divisor F on M'. By assumption $F'=F-E'$ is effective. Write $F'=R+\sum\mu_iD_i$, where the components of R are other than D_i's. Assume that $\mu_i<\delta_i$ for some i. Set $X=\sum(\delta_i-\mu_i)D_i$, where the sum is taken over those i's with $\delta_i>\mu_i$. Applying (1.5) we find a curve C such that $XC<0$, $\pi(C)$ is a point, $RC\geqq 0$ and $D_iC\geqq 0$ for any i with $\delta_i\leqq\mu_i$. Then, since $\pi^*L\cdot C=E^*C=0$, we have $0\leqq(\pi^*L-F)C=(E^*-F)C=-(F'-\sum\delta_iD_i)C\leqq XC<0$. From this contradiction we infer that $F-E^*=R+\sum(\mu_i-\delta_i)D_i$ is effective. Thus E^* clutches π^*L. Finally, applying (1.4), we complete the proof.

(1.7) DEFINITION. An effective $\boldsymbol{Q}$-divisor E on M is said to be *numerically fixed* by a $\boldsymbol{Q}$-bundle L on M if π^*E clutches π^*L for any birational morphism $\pi: M'\to M$.

(1.8) PROPOSITION. *Let E be an effective $\boldsymbol{Q}$-divisor on M and suppose that E is numerically fixed by a line bundle L on M. Then $\bar{E}$ is contained in the fixed part of $|L|$, where $\bar{E}$ is the smallest (usual) divisor such that $\bar{E}-E$ is effective.*

PROOF. By virtue of Hironaka's theory there is a birational morphism $\pi: M'\to M$ such that $\operatorname{Bs}|\pi^*L-F|=\emptyset$ for the fixed part F of $\pi^*|L|$. Then $F-\pi^*E$ is effective since π^*E clutches π^*L. Therefore π_*F-E is effective and hence so is $\pi_* F - \bar{E}$. Since $\pi_* F$ is the fixed part of $|L|$, this proves the assertion.

(1.9) COROLLARY. *In the above situation, the graded algebra* $G(M, L) = \bigoplus_{t \geqq 0} H^0(M, tL)$ *is isomorphic to* $\bigoplus_{t \geqq 0} H^0(M, tL - t\bar{E})$.

(1.10) PROPOSITION. *Let* $f: M \to S$ *be a surjective morphism onto another manifold* S *such that any general fiber is connected. Let* X *be an effective* $\boldsymbol{Q}$*-divisor on* M *such that* $\dim f(X) < \dim S$. *Suppose that, for every irreducible component* Z *of* $f(X)$ *with* $\dim Z = \dim S - 1$, *there is a prime divisor* D *on* M *such that* $f(D) = Z$ *and* $D \not\subset \mathrm{Supp}(X)$. *Then* X *is numerically fixed by* $X + f^* L$ *for any* $\boldsymbol{Q}$*-bundle* L *on* S.

PROOF. For any birational morphism $\pi: M' \to M$, $\pi^* X$ has the same property as X with respect to $f' = f \cdot \pi: M' \to S$. Therefore it suffices to show that X clutches $X + f^* L$.

Suppose that $\pi^* L + X - F$ is nef for some effective $\boldsymbol{Q}$-divisor F. Let $Z_1, Z_2, \cdots, Z_r$ be components of $f(X)$ of codimension one in S and write $X = X_1 + X_2 + \cdots + X_r + X'$, where the components of X_j are those mapped onto Z_j and $\mathrm{codim} f(X') \geqq 2$. Write similarly $F = F_0 + F_1 + \cdots + F_r + F'$, where the components of F_0 are not mapped into $f(X)$. First we claim that $F_j - X_j$ is effective for each $j = 1, \cdots, r$. Indeed, otherwise, $F_j - X_j = Y - \Delta$ for some effective $\boldsymbol{Q}$-divisors Y, Δ without common components. In view of (1.5; 2), we find a curve C lying over a general point u on Z_j such that $\Delta C < 0$, $YC \geqq 0$, $F_0 C \geqq 0$. Since $F_i C = X_i C = 0$ for $i \neq j$, we have $(\pi^* L + X - F)C = (\Delta - Y - F_0)C < 0$, contradicting the semipositivity. Thus we prove the claim.

Next we claim that $F' - X'$ is effective. Indeed, otherwise, $F' - X' = F'' - X''$ for some effective $\boldsymbol{Q}$-divisors F'', X'' without common components and $X'' \neq 0$. Using (1.5; 1), we find a curve C lying over a point in $f(X'')$ such that $X'' C < 0$, $F'' C \geqq 0$, $(F_j - X_j)C \geqq 0$ for each j and $F_0 C \geqq 0$. Then we have $(\pi^* L + X - F)C = (X'' - F'')C - F_0 C - \sum_{j=1}^{r} (F_j - X_j)C < 0$, contradicting the semipositivity. Therefore $F' - X'$ is effective, and hence so is $F - X$. Thus we show that X clutches $X + \pi^* L$.

(1.11) PROPOSITION. *Let* $f: M \to S$ *be a surjective morphism of manifolds and suppose that an effective* $\boldsymbol{Q}$*-divisor* E *on* S *is numerically fixed by a* $\boldsymbol{Q}$*-bundle* L *on* S. *Then* $f^* E$ *is numerically fixed by* $f^* L$.

This fact will be proved in several steps below. First we recall the following result.

(1.12) THEOREM (Hironaka [**H3**]). *Let* $f: M \to S$ *be any surjective morphism of manifolds. Then there exists a flat morphism* $g: V \to T$ *from a variety* V *onto* *a manifold T together with birational morphisms $\nu: V\to M$ and $\pi: T\to S$ such that $\pi\cdot g=f\cdot\nu$.*

Such a mapping g will be called (Hironaka's) *flat model* of f. Actually, π can be taken to be a succession of blowing-ups of non-singular centers.

(1.13) LEMMA. *Let $f: M\to S$ be a fiber space of manifolds and let $g: V\to T$ be a flat model of it. Let L be a $\boldsymbol{Q}$-bundle on S and let F be an effective $\boldsymbol{Q}$-divisor on M such that f^*L-F is nef. Then there exists an effective $\boldsymbol{Q}$-divisor D on T such that $g^*D=\nu^*F$.*

REMARK. It follows that π^*L-D is nef, because $g^*(\pi^*L-D)=\nu^*(f^*L-F)$.

PROOF OF THE LEMMA. We claim that $\dim g(\nu^*F)<\dim T$. Indeed, otherwise, $\nu^*F\cdot C>0$ for some curve C contained in a general fiber of g. Then $(f^*L-F)_V\cdot C=-\nu^*F\cdot C<0$, contradicting the semipositivity of f^*L-F. Thus we prove the claim.

Now we take the smallest effective $\boldsymbol{Q}$-divisor D on T such that $X=g^*D-\nu^*F$ is effective. If $X\neq 0$, we take a non-singular model W of V. Applying (1.5) to $W\to T$, we find a curve C contained in a fiber of g such that $XC<0$. Then $\nu^*F\cdot C>0$, which yields a contradiction as above. Thus we see $X=0$, which proves the lemma.

(1.14) COROLLARY. *Let things be as in* (1.11) *and suppose in addition that any general fiber of f is connected. Then f^*E clutches f^*L.*

PROOF. Suppose that f^*L-F is nef for some effective $\boldsymbol{Q}$-divisor F on M. Take a flat model $g: V\to T$ of f as in (1.12). Then, by (1.13), $\nu^*F=g^*D$ for some effective $\boldsymbol{Q}$-divisor D on T. Since π^*L-D is nef, $D-\pi^*E$ is effective. Therefore $g^*(D-\pi^*E)=\nu^*(F-f^*E)$ is effective, and hence so is $F-f^*E$.

(1.15) LEMMA. *Let things be as in* (1.11) *and if S is birational to the quotient space M/G where G is a finite group acting holomorphically on M, then f^*E clutches f^*L.*

PROOF. Assuming that f^*L-F is nef for some effective $\boldsymbol{Q}$-divisor F on M, we will show that $F-f^*E$ is effective. Taking positive multiples if necessary, we may assume that L is a usual line bundle and F, E are usual divisors. Let B be the ideal theoretical intersection $\bigcap_{\sigma\in G}\sigma^*F$ in M. By virtue of Hironaka's theory (cf. [**H2**]) we can find a G-equivariant birational morphism $\pi: M'\to M$ such that $\pi^*B=D$ is an effective Cartier divisor on a manifold M'. We claim that π^*f^*L-D is nef on M'.

Indeed, if we write $\pi^*F=F'+D$, then we have $\bigcap_{\sigma\in G}\sigma^*F'=\emptyset$ by construction. So, for any curve C in M', we have $\sigma^*F'\{C\}\geqq 0$ for some $\sigma\in G$. On the other hand, $0 \leqq \sigma^*\pi^*(f^*L-F)\{C\} = \pi^*f^*L\cdot C - \sigma^*(F'+D)\cdot C$. Hence $(\pi^*f^*L-D)C \geqq 0$ because $\sigma^*D=D$.

Now, we take a flat model $g: V\to T$ of $f'=f\cdot\pi: M'\to S$. Since g is finite, G acts holomorphically on the normalization $\tilde{V}$ of V. Let W be a G-equivariant desingularization of $\tilde{V}$. The pull-back of D to $\tilde{V}$ is G-invariant, hence it is the pull-back of an effective divisor D' on T. Similarly as in (1.13), $L-D'$ is nef on T. Then, by assumption, $D'-E$ is effective on T. Similarly as in (1.14), this implies that $F-f^*E$ is effective. Thus we prove the lemma.

(1.16) LEMMA. *Let things be as in* (1.11) *and suppose in addition that the extension of function fields* $K(M)/K(S)$ *is finite and Galois. Then* f^*E *clutches* f^*L.

PROOF. Let $g: V\to T$ be a flat model of f as in (1.12). Since g is finite, $G=\mathrm{Gal}(K(M)/K(S))$ acts holomorphically on the normalization $\tilde{V}$ of V. Let $W\to\tilde{V}$ be a G-equivariant desingularization. Then, applying (1.15) to $W\to T$, we infer that E clutches L on W. So (1.4) proves our assertion.

(1.17) PROOF OF (1.11). Step 1, the case where $\dim M=\dim S$. It suffices to show that f^*E clutches f^*L for any such f. As is well-known, there is a surjective morphism $\pi: M'\to M$ such that the extension of function fields $K(M')/K(S)$ is finite and Galois. Applying (1.16) to $M'\to S$, we infer that E clutches L on M'. Applying (1.4) to π, we see that f^*E clutches f^*L.

Step 2, general case. Let $W=\mathcal{S}pec(f_*\mathcal{O}_M)$. Then f factors through W, $p: W\to S$ is a finite morphism and any general fiber of $g: M\to W$ is connected. Let W' be a non-singular model of W and let M' be a non-singular model of the graph of the rational mapping $M\to W\cdots\to W'$. Then, by Step 1, E is numerically fixed by L on W'. Next, applying (1.14) to $f': M'\to W'$, we infer that E clutches L on M', so does it on M by (1.4). This argument works on any birational model of M. Hence f^*E is numerically fixed by f^*L.

q. e. d.

(1.18) DEFINITION. We say that a $\boldsymbol{Q}$-bundle L on a manifold M admits a *Zariski decomposition* if there exist a birational morphism $\pi: M'\to M$ and an effective $\boldsymbol{Q}$-divisor N on M' such that N is numerically fixed by π^*L and $H=\pi^*L-E$ is nef. N (resp. H) is called the *negative* (resp. *semipositive*) *part* of L.

(1.19) If exists, Zariski decomposition is unique up to birational equivalence in the following sense. Suppose that we have two decompositions $\pi_1^*L=N_1+H_1$ and $\pi_2^*L=N_2+H_2$ on two birational models M_1 and M_2 of M. Then, on any manifold M' which dominates both M_1 and M_2, we have $(N_1)_{M'}=(N_2)_{M'}$ and $(H_1)_{M'}=(H_2)_{M'}$. This is almost clear by definition.

(1.20) REMARK. Any pseudo-effective $\boldsymbol{Q}$-bundle L on an algebraic surface S admits a decomposition $L=N+H$ such that

1) $N=\sum\mu_iC_i$ is an effective $\boldsymbol{Q}$-divisor and the matrix $\{(C_iC_j)\}$ of intersection numbers is negative definite (unless $N=0$).

2) H is nef and $HC_i=0$ for every component C_i of N.

This was called classically the Zariski decomposition of L. Now, it is easy to see that the above conditions imply that N is numerically fixed by L. Therefore, the classical one is a Zariski decomposition in the new sense too. Thus, our definition can be viewed as a higher dimensional version of the classical one.

(1.21) PROBLEM. Does any effective divisor admit a Zariski decomposition ?

To be more optimistic, one might suppose that a $\boldsymbol{Q}$-bundle L admits a Zariski decomposition if and only if L is pseudo-effective, i.e., $tL+A$ is represented by an effective $\boldsymbol{Q}$-divisor for any $t\geqq 0$ and any ample $\boldsymbol{Q}$-bundle A.

(1.22) LEMMA. *Suppose that an effective $\boldsymbol{Q}$-divisor E is numerically fixed by a $\boldsymbol{Q}$-bundle L. Then $L-E$ admits a Zariski decomposition if and only if so does L. Moreover, if so, the semipositive parts of them are the same.*

PROOF. Obvious by definition. Recall (1.3.2).

(1.23) PROPOSITION. *Let $L=N+H$ be a Zariski decomposition on M of a $\boldsymbol{Q}$-bundle L. Then, for any effective $\boldsymbol{Q}$-divisor F on M such that* Supp(F) $\subset$Supp(N), *F is numerically fixed by $F+H$. So, $F+H$ admits a Zariski decomposition.*

PROOF. Suppose that there is an effective $\boldsymbol{Q}$-divisor X on a manifold M' with a birational morphism $\pi: M'\to M$ such that $F'+H'-X$ is nef, where $'$ denotes π^*. Take a large integer t such that $tN-F$ is effective. Since $tN'+tH'=(tN'-F')+X+(F'+tH'-X)$ is clutched by tN' while $F'+tH'-X$ is nef, we infer that $(tN'-F'+X)-tN'=X-F'$ is effective. Thus we see that F is numerically fixed by $F+H$.

(1.24) PROPOSITION. *Let $f: M\to S$ be a surjective morphism of manifolds, let L be a $\boldsymbol{Q}$-bundle on S and let R be an effective $\boldsymbol{Q}$-divisor on M such that* $\dim f(R)\leqq\dim S-2$. *Then $L^*=f^*L+R$ admits a Zariski decomposition if and only if so does L. Moreover, the semipositive part of L^* is* (*birationally*) *the pull-back of that of L.*

PROOF. By (1.10), R is numerically fixed by L^*. So, by virtue of (1.22), we may assume that $R=0$.

Suppose first that L admits a Zariski decomposition. We have a birational morphism $\pi: S'\to S$ and an effective $\boldsymbol{Q}$-divisor N on S' such that N is numerically fixed by π^*L and $H=\pi^*L-N$ is nef. Let $f': M'\to S'$ be a birational model of f with $\nu: M'\to M$ being a birational morphism such that $f\cdot\nu=\pi\cdot f'$. Then, by (1.11), $N_{M'}$ is numerically fixed by $L_{M'}$ while $L_{M'}-N_{M'}=H_{M'}$ is nef. So $L_{M'}=N_{M'}+H_{M'}$ gives a Zariski decomposition of f^*L. Thus we prove the assertion.

It remains to prove the "only if" part. Suppose that $L^*=f^*L$ admits a Zariski decomposition. Replacing M by a suitable birational model if necessary, we may assume that there is an effective $\boldsymbol{Q}$-divisor N^* on M such that $H^*=L^*-N^*$ is nef and N^* is numerically fixed by L^*. We proceed in several steps as in the case of (1.11).

Step 1, the case in which $K(M)/K(S)$ is finite and Galois. Let $g: V\to T$ be a flat model of f as in (1.12). Since g is a finite morphism, $G=\mathrm{Gal}(K(M)/K(S))$ acts holomorphically on the normalization $\tilde{V}$ of V. Let W be a G-equivariant desingularization of $\tilde{V}$. L_W is G-invariant because it comes from T. So, by the uniqueness of Zariski decomposition, we infer that N_W^* is G-invariant because it is the negative part of L_W. Hence $N_{\tilde{V}}^*=D_{\tilde{V}}$ for some effective $\boldsymbol{Q}$-divisor D on T. Using (1.4), we infer that D is numerically fixed by L_T. Since $H=L_T-D$ is nef, $L_T=D+H$ is a Zariski decomposition of L.

Step 2, the case in which $\dim M=\dim S$. There is a surjective morphism $\psi: M'\to M$ such that $K(M')/K(S)$ is finite and Galois. $L_{M'}=L_{M'}^*$ admits a Zariski decomposition by the "if" part. So, by Step 1, L admits a Zariski decomposition.

Step 3, the case in which every general fiber of f is connected. Let $g: V\to T$ be a flat model of f. By (1.13), there is an effective $\boldsymbol{Q}$-divisor D on T such that $D_V=N_V^*$. Then, as in Step 1, $L_T=D+(L_T-D)$ gives a Zariski decomposition of L.

Step 4, general case. Considering the Stein factorization of f similarly as in (1.17), we find a birational model $f': M'\to S'$ of f such that f' factors through a manifold W' with $\dim W'=\dim S'$ and every fiber of $M'\to W'$ is connected. $L_{M'}=L_{M'}^*$ admits a Zariski decomposition by the "if" part. Then so does $L_{W'}$ by Step 3, and hence so does $L_{S'}$ by Step 2. This completes the proof because S' is birational to S.

(1.25) COROLLARY. *Let M, M' be manifolds birationally equivalent to each other and let K, K' be the canonical bundles of them. Then K' admits a Zariski decomposition if and only if so does K.*

PROOF. By Hironaka's theory it suffices to consider the case in which we have a birational morphism $\pi: M'\to M$. Let R be the ramification locus of π. Then $K'=\pi^*K+R$ and $\operatorname{codim}\pi(R)\geqq 2$. So (1.24) applies.

§ 2. Canonical bundle formula for elliptic fiber spaces.

(2.1) Let $f: M \to S$ be an elliptic fiber space with $\dim M = n$. Let Σ be the maximal subset of S such that f is smooth over $U = S - \Sigma$. Of course, the ideal-theoretic fiber $M_x = f^{-1}(x)$ over $x \in S$ is singular if and only if $x \in \Sigma$. So Σ is called the *singular locus* of f.

The local system $\bigcup_{x \in U} H^1(M_x; \boldsymbol{Z})$ induces a group homomorphism $\Phi: G = \pi_1(U) \to SL(2; \boldsymbol{Z})$. On the other hand, we have a multivalued holomorphic function $T: U \to H = \{\tau \in \boldsymbol{C} \mid \mathrm{Im}(\tau) > 0\}$ such that M_x is isomorphic to the complex torus $\boldsymbol{C}/(\boldsymbol{Z} + \boldsymbol{Z}T(x))$ for every $x \in U$. T gives a holomorphic mapping $\tilde{T}: \tilde{U} \to H$ on the universal covering $\tilde{U}$ of U. G acts on $\tilde{U}$ as the covering transformation group and we have $\tilde{T}(\gamma x) = \Phi(\gamma)\tilde{T}(x)$ for every $\gamma \in G$, where the action of $SL(2; \boldsymbol{Z})$ on H is the standard linear projective transformation.

Let j be the elliptic modular function on H. Then $J = j \cdot T$ turns out to be single-valued on U, and can be extended to a meromorphic function on S. Regarding J as a meromorphic mapping $S \to \boldsymbol{P}^1$, we call J the *J-invariant* of f.

(2.2) In order to describe the canonical bundle of M, we make a brief review of the theory of Kodaira and Ueno. For details and proofs, see [**Ko1**], [**Ko2**], [**U1**].

For the sake of simplicity we assume that J is a holomorphic mapping. This is harmless for many applications, because this assumption is satisfied by a suitable birational model of f. Viewed as a meromorphic function on S, J yields a meromorphic section of $J^*\mathcal{O}(-1)$, which is denoted by J by abuse of notation.

(2.3) Let $\tilde{G}$ be the semi-direct product of G and $\boldsymbol{Z} \oplus \boldsymbol{Z}$ with respect to the action of G on $\boldsymbol{Z} \oplus \boldsymbol{Z}$ given by Φ. Then, there is a standard action of $\tilde{G}$ on $\tilde{U} \times \boldsymbol{C}$ with the following properties:

a) The action is properly discontinuous and free of fixed points. So the quotient space W is non-singular.

b) There is a natural morphism $g: W \to U$ such that the mapping $\tilde{U} \times \boldsymbol{C} \to \tilde{U} \to U$ factors through g.

c) g is locally isomorphic to f_U. Namely, for every $x \in U$, there exists a neighborhood V of x such that $f^{-1}(V)$ and $g^{-1}(V)$ are isomorphic to each other as fiber spaces over V.

d) g has a global section.

Roughly speaking, W is the union of Albanese varieties of fibers of f over U. By construction one sees also that $f_*(\omega_M^{\otimes m})$ is canonically isomorphic to $g_*(\omega_W^{\otimes m})$ over U for any positive integer m.

(2.4) Let $\Delta(z)=(2\pi)^{12}\exp(2\pi iz)\{\sum_{n=1}^{\infty}(1-\exp(2\pi in))\}^{24}$ be the cusp form on H of weight six. Given any small open set V in U and holomorphic $(n-1)$-forms $\omega_1, \cdots, \omega_{12}$ on V, we set $\tilde{\Xi}(\omega_1, \cdots, \omega_{12})=\Delta(\tilde{T}(y))(\otimes_{j=1}^{12}(g^*\omega_j\wedge d\zeta))$, where ζ is the fiber coordinate of $\tilde{U}\times \boldsymbol{C}\to\tilde{U}$. This is a 12-tuple holomorphic n-form on $\pi^{-1}(V)\times\boldsymbol{C}$, where π is the covering map $\tilde{U}\to U$. Note that $\tilde{G}$ acts on $\pi^{-1}(V)\times\boldsymbol{C}$ and the quotient is $g^{-1}(V)$. By construction and by the properties of the cusp form Δ, one sees that $\tilde{\Xi}(\omega_1, \cdots, \omega_{12})$ is $\tilde{G}$-invariant, and hence gives $\Xi(\omega_1, \cdots, \omega_{12})$ $\in H^0(g^{-1}(V), \omega_W^{\otimes 12})\simeq H^0(f^{-1}(V), \omega_M^{\otimes 12})$. This assignment $(\omega_1, \cdots, \omega_{12})\to\Xi(\omega_1, \cdots, \omega_{12})$ yields a section Ξ of $\mathcal{H}om(\omega_S^{\otimes 12}, f_*(\omega_M^{\otimes 12}))\simeq f_*(\omega_{M/S}^{\otimes 12})$ defined globally on U. In the sequel we are interested in the problem whether and how Ξ extends to the whole space S.

(2.5) Consider first the case in which $n=2$. So S is a curve and Σ is a finite set. For the moment, until (2.9), we assume that f is minimal, that means, there is no exceptional curve contained in a fiber of f. As we shall see later, the general case can be easily reduced to such a case.

(2.6) The types of singular fibers of minimal elliptic surfaces were classified by Kodaira [**Ko1**]. For each type we define a number μ by the following table below.

Type	${}_m\mathrm{I}_b$	I*	II	II*	III	III*	IV	IV*
μ	$1-m^{-1}$	1/2	1/6	5/6	1/4	3/4	1/3	2/3

The meaning of μ will be clarified later. m is called the *multiplicity* of fibers of type ${}_m\mathrm{I}_b$. For other types we set $m=1$. *Multiple fiber* means a fiber of type ${}_m\mathrm{I}_b$ with $m>1$.

The J-invariant has a pole at $x\in S$ if and only if $M_x=f^{-1}(x)$ is of type ${}_m\mathrm{I}_b$. The order of the pole is b.

The local monodromy Φ_x of Φ at x is determined up to conjugacy by the type of M_x. Conversely, the type of M_x is determined by Φ_x except that Φ_x does not depend on m in case of type ${}_m\mathrm{I}_b$.

(2.7) In [**U1**], Ueno has solved our extension problem of Ξ in case $m=1$. His result is summarized as follows.

If M_x is not of type ${}_m\mathrm{I}_b$, Ξ extends to a holomorphic section $\bar{\Xi}$ of $\mathcal{P}_{12}$ $=f^*(\omega_{M/S}^{\otimes 12})$ over x. Moreover, if dt is a local base of ω_S at x, the divisor of zeros of the 12-tuple holomorphic 2-form $\bar{\Xi}(dt^{\otimes 12})$ in a neighborhood of M_x is $12\mu M_x$. Note that 12μ is a positive integer.

If M_x is of type ${}_1\mathrm{I}_b$, then Ξ extends to a section $\bar{\Xi}$ of $\mathcal{P}_{12}$ over x and the divisor of zeros of $\bar{\Xi}(dt^{\otimes 12})$ is bM_x.

Thus, $J\bar{\Xi}$ gives a section of $\mathcal{P}_{12}\otimes J^*\mathcal{O}(-1)$ off the multiple fibers. Moreover, if there is no multiple fiber, $J\bar{\Xi}$ yields isomorphisms $\mathcal{P}_{12}\simeq J^*\mathcal{O}(1)\otimes\sum_x 12\mu[x]$ and $f^*\mathcal{P}_{12}\simeq\omega_{M/S}^{\otimes 12}$.

REMARK. Apparently, [**U1**] solves the problem only for "basic members", namely, in the case where f admits a global section. However, his method is completely local with respect to S. On the other hand, if M_x is not a multiple fiber, f admits a local section over a small neighborhood of x. So his method works in general.

(2.8) To study the case of multiple fibers, we use Kodaira's theory [**Ko2**; §4].

Suppose that M_x is of type ${}_m\mathrm{I}_b$. Then, replacing M_x by a fiber of type ${}_1\mathrm{I}_b$, we obtain another elliptic surface $f^*: M^*\to S$. From the converse viewpoint, M is obtained from M^* by a standard process called "logarithmic transformation". From this explicit description of M_x, it follows that any non-vanishing holomorphic 2-form on a neighborhood of M_x^* induces naturally a holomorphic 2-form on a neighborhood of M_x, the divisor of zeros of which is $(1-m^{-1})M_x$ (recall that $m^{-1}M_x$ is a usual divisor).

In view of this theory we infer that $J^m\bar{\Xi}^m$ extends to a section of $\mathcal{P}_{12}^{\otimes m}\otimes J^*\mathcal{O}(-m)$ over x, the divisor of zeros of which is $12(m-1)M_x$. So $\omega_{M/S}^{\otimes 12m}\simeq f^*(\mathcal{P}_{12}^{\otimes m}\otimes J^*\mathcal{O}(-m))\otimes[12(m-1)M_x]\simeq f^*(\mathcal{P}_{12}^{\otimes m}\otimes J^*\mathcal{O}(-m)\otimes[12(m-1)x])$ over x.

(2.9) Combining (2.7) and (2.8), we obtain the following

THEOREM. *Let $f: M\to S$ be a minimal elliptic surface and let m be a positive integer such that $k=12m$ is divisible by the multiplicities of all the singular fibers of f. Then $(J\Xi)^m$ extends to a holomorphic section of $\mathcal{F}_k=f_*(\omega_{M/S}^{\otimes k})\otimes J^*\mathcal{O}(-m)$ and gives isomorphisms $\mathcal{F}_k\simeq\mathcal{O}_S[\sum_{x\in\Sigma}k\mu_x x]$ and $\omega_{M/S}^{\otimes k}\simeq f^*(\mathcal{F}_k\otimes J^*\mathcal{O}(m))$.*

Note that $\mathcal{F}_k$ is a usual invertible sheaf because $k\mu_x\in\boldsymbol{Z}$.

(2.10) If f is not minimal, we blow down exceptional curves successively and we finally obtain a minimal model $f': M'\to S$ of f. Multiplicities and μ's of singular fibers are defined as those of f. Then, since $\mathcal{F}_k$ in (2.9) is a birational invariant, we have $\mathcal{F}_k\simeq\mathcal{O}_S[\sum_{x\in\Sigma}k\mu_x x]$ as in (2.9). Moreover, $\omega_{M/S}^{\otimes k}\simeq f^*(\mathcal{F}_k\otimes J^*\mathcal{O}(m))\otimes\mathcal{O}_M[kR]$, where R is the ramification locus of the birational morphism $M\to M'$.

(2.11) Now we consider the case in which $n=\dim M$ is general.

DEFINITION. A point x on the singular locus Σ is said to be *ordinary* if the following conditions are satisfied.

a) Σ looks like a smooth divisor in a neighborhood of x.

b) There exists a curve Z in S such that Z meets Σ at x transversally and that $f^{-1}(Z)$ is ideal-theoretically non-singular in a neighborhood of $M_x=f^{-1}(x)$.

Set $\Sigma_2=\{x\in\Sigma\mid x$ is not ordinary$\}$ and $U_2=S-\Sigma_2$. By Bertini's theorem we infer that Σ_2 is Zariski closed in S and $\dim\Sigma_2\leqq n-3$.

(2.12) Let Y be an irreducible component of Σ with $\dim Y=n-2$. Any general point x on Y is ordinary and M_x is a singular fiber of the elliptic surface $f^{-1}(Z)$ as in (2.11; b). The multiplicity and the local monodromy of M_x are independent of the choice of x and Z. So the type is determined, which we define to be the type of Y. In particular μ_Y and the multiplicity m_Y are well-defined.

(2.13) Let m be a positive integer such that $k=12m$ is divisible by the multiplicities of all the components of Σ of dimension $n-2$. Then $(J\Xi)^m$ extends to a section of $\mathcal{F}_k=f_*(\Omega_{M/S}^{\otimes k})\otimes J^*\mathcal{O}(-m)$ over U_2 and yields an isomorphism $\mathcal{F}_k\simeq\mathcal{O}_S[\sum k\mu_Y Y]$ over U_2.

Indeed, if x is ordinary and is of multiplicity one, the extension problem over x is solved by Ueno's method. To study the case $m_Y\geqq 2$, we define the notion of "logarithmic transformation" along a smooth divisor on S in the obvious way and apply Kodaira's method as in (2.8). Thus, the extension problem is solved over U_2.

We have also the following alternate proof of the above fact when S is projectively algebraic. Clearly $\mathcal{F}_k$ is torsion free, and is invertible on U_2. So we have an invertible sheaf $\mathcal{L}$ on S such that $\mathcal{L}\simeq\mathcal{F}_k$ on U_2. $\mathcal{L}$ is the reflexive hull of $\mathcal{F}_k$. We will show that $(J\Xi)^m$ extends to a section of $\mathcal{L}$ on S. We use the induction on n since this is true when $n=2$. Take a sufficiently ample general hyperplane section H of S. Then $H^1(S,\mathcal{L}[-H])=0$ and hence $H^0(S,\mathcal{L})\to H^0(H,\mathcal{L}_H)$ is bijective. Applying the induction hypothesis to the elliptic fiber space $f^{-1}(H)\to H$, we obtain a section of $\mathcal{L}_H$. The corresponding section of $\mathcal{L}$ is obviously the desired extension. We have also $\mathcal{L}\simeq\mathcal{O}_S[\sum k\mu_Y Y]$.

(2.14) Now we study $\omega_{M/S}$. Since $\mathcal{F}_k\simeq\mathcal{L}$ on U_2, a holomorphic section of $\mathcal{L}$ gives a meromorphic section of $\omega_{M/S}^{\otimes k}\otimes f^*J^*\mathcal{O}(-m)$ whose poles are contained in $f^{-1}(\Sigma_2)$. Hence $\mathcal{L}\simeq f_*(\omega_{M/S}^{\otimes k}\otimes\mathcal{O}_M[X])\otimes J^*\mathcal{O}(-m)$ for some effective divisor X on M such that $f(X)\subset\Sigma_2$. Then we have a natural homomorphism $f^*(\mathcal{L}\otimes J^*\mathcal{O}(m))\to\omega_{M/S}^{\otimes k}[X]$. So $\omega_{M/S}^{\otimes k}\simeq f^*(\mathcal{L}\otimes J^*\mathcal{O}(m))\otimes\mathcal{O}_M[E-X]$ for some effective divisor E on M.

To study E further, take $(n-2)$ general hyperplane sections of S and let Z be the intersection of them. Then $f^{-1}(Z)\to Z$ is an elliptic surface by Bertini's theorem. Restricted to $f^{-1}(Z)$, the above morphism reduces to the one in (2.10). So $E\cap f^{-1}(Z)$ is a union of finite number of proper transforms of exceptional curves contained in fibers. From this observation we infer that $\dim f(E)\leqq n-2$. Moreover, by virtue of (1.10), E is numerically fixed by f^*L+E for any $\boldsymbol{Q}$-bundle L on S.

(2.15) Putting things together we obtain the following

THEOREM. *Let $f: M \to S$ be an elliptic fiber space with $n = \dim M$ such that the J-invariant $J: S \to \boldsymbol{P}^1$ is holomorphic. Let m be a positive integer such that $k = 12m$ is divisible by the multiplicities of all the components Y of the singular locus Σ of f with $\dim Y = n-2$. Then $\omega_M^{\otimes k} \simeq f^*(\omega_S^{\otimes k} \otimes J^*\mathcal{O}(m) \otimes \mathcal{O}_S[\sum_Y k\mu_Y Y]) \otimes \mathcal{O}_M[E-X]$ for some effective divisors E, X on M such that $f(X) \subset \Sigma_2$ and that E is numerically fixed by f^*L+E for any $\boldsymbol{Q}$-bundle L on S.*

REMARK. $X=0$ if and only if $f_*(\omega_{M/S}^{\otimes k})$ is invertible. So we would like to ask the following

(2.16) QUESTION. Does there exist a birational model f' of f such that $X'=0$ for f'? (X' corresponds to X for f.)

The answer is Yes if f admits a meromorphic section (cf. [**U1**]).

(2.17) Most arguments in this section should work for non-algebraic elliptic fiber spaces too.

§ 3. Zariski decomposition of canonical bundles of elliptic 3-folds.

(3.1) DEFINITION. A $\boldsymbol{Q}$-bundle H is said to be *semiample* if there exists a positive integer m such that mH is a usual line bundle with $\mathrm{Bs}|mH| = \emptyset$.

(3.2) THEOREM. *Let $f: M \to S$ be an elliptic threefold such that $\kappa(M) \geqq 0$. Then the canonical bundle of M admits a Zariski decomposition and the semipositive part of it is semiample.*

To prove this, we recall the following result.

(3.3) THEOREM (cf. [**F5**]). *Let D be a reduced effective $\boldsymbol{Q}$-divisor on a smooth surface S such that K_S+D is pseudo-effective. Then the semipositive part of K_S+D is semiample.*

(3.4) PROOF OF (3.2). Let $g: V \to T$ be a flat model of f as in (1.12) and let W be a non-singular model of V. Replacing T if necessary we may assume that the J-invariant of the elliptic threefold $h: W \to T$ is a holomorphic mapping $J: T \to \boldsymbol{P}^1$. Applying (2.15) to h, we obtain $\omega_W^{\otimes m} = h^*(\omega_T^{\otimes m} \otimes J^*\mathcal{O}(m/12) \otimes \mathcal{O}_T[\sum m\mu_Y Y]) \otimes \mathcal{O}_M[E-X]$ for some effective divisors E, X on W as described in (2.15). So, $mK_W = mh^*(K_T+D)+E-X$ for some reduced effective $\boldsymbol{Q}$-divisor D on T. Now we claim that $H^0(W, tmK_W) \to H^0(W, tmh^*(K_T+D)+tE)$ is bijective for every positive integer t.

Indeed, the injectivity is obvious. To show the surjectivity, let $\phi \in H^0(W, tmK_W+tX)$. ϕ is identified with a meromorphic tm-ple 3-form on W whose poles are at most tX. Since $\nu: W \to M$ is birational, ϕ is a pull-back of a meromorphic tm-ple 3-form ψ' on M, which is holomorphic off $\nu(X)$. Recall that $\dim h(X) \leqq 0$. Since g is flat, the image of X in V is at most curves, hence $\dim \nu(X) \leqq 1$. Therefore ψ' is holomorphic on M by Hartogs' theorem. Consequently ψ is holomorphic, namely, comes from $H^0(W, tmK_W)$. Thus we prove the claim.

Now, since $\kappa(M)=\kappa(K_W, W) \geqq 0$, we infer $\kappa(K_T+D, S)=\kappa(h^*(K_T+D), W) \geqq 0$ by (1.8). Let $K_T+D=N+H$ be the Zariski decomposition on S. Since H is semiample by (3.3), tmN is a usual divisor and $\mathrm{Bs}|tmH|=\emptyset$ for some positive integer t. Using (1.24), (1.22) and the property of E, we infer that $mK_W+X =mh^*(K_T+D)+E$ admits a Zariski decomposition and mh^*H is the semipositive part of it. So, by (1.8), the fixed part of $|tmK_W+tX|$ is tmh^*N+tE. On the other hand, the preceding claim implies that tX is fixed by $|tmK_W+tX|$. Therefore $t(mh^*N+E-X)$ is an effective divisor. Then $K_W=m^{-1}(mh^*N+E-X)+h^*H$ is obviously a Zariski decomposition of K_W. Finally, applying (1.25), we complete the proof.

(3.5) COROLLARY. *Let things be as in* (3.2). *Then the canonical ring* $\bigoplus_{t \geq 0} H^0(M, tK_M)$ *is a finitely generated* $\boldsymbol{C}$*-algebra.*

PROOF. Similar as in [**F5**; (1.5)].

(3.6) COROLLARY. *Let M be an algebraic threefold such that $\kappa(M) \neq 3$. Then the canonical ring of M is finitely generated.*

PROOF. This is obvious if $\kappa \leqq 0$. If $\kappa=1$, [**F2**; Appendix] applies. If $\kappa=2$, a birational model of M has a structure of an elliptic threefold. So (3.5) applies.

(3.7) Our method works for higher dimensional elliptic fiber spaces too, provided that we have a generalization of (3.3). The conjecture (A2) in the Appendix is enough for this purpose.

Appendix.

(A1) DEFINITION. A $\boldsymbol{Q}$-bundle L on a manifold M is said to be *big* if $\kappa(L) =\dim M$. This is equivalent to saying that $L-E$ is ample for some effective $\boldsymbol{Q}$-divisor E.

An effective $\boldsymbol{Q}$-divisor $D=\sum \delta_i D_i$ on M is said to be *negligible* if $\delta_i<1$ for each coefficient δ_i and if the support of D has no singularity other than normal crossings.

(A2) Here we are interested in the following

CONJECTURE. *Let D be a negligible divisor on a manifold M with canonical bundle K such that $K+D$ is big. Then $K+D$ admits a Zariski decomposition and* *the semipositive part of it is semiample.*

(A3) THEOREM. *Let L be a line bundle and let E be a $\boldsymbol{Q}$-divisor on a manifold M such that $L+E$ is big. Set $L(t)=tL+\underline{tE}\in\mathrm{Pic}(M)$. Then the graded algebra $\bigoplus_{t\geq 0}H^0(M, L(t))$ is finitely generated if and only if $L+E$ admits a Zariski decomposition whose semipositive part is semiample.*

PROOF. The "if" part is standard (cf., e.g., [**F5**; (1.5)]), so we consider the "only if" part. Let $\phi_1, \cdots, \phi_r$ be a system of homogeneous generators of the graded algebra. Set $d_j=\deg\phi_j$ and take a positive integer m divided by all the d_j's such that mE is integral. We take a birational model $\pi: M_1\to M$ to eliminate $\mathrm{Bs}|L(m)|$. Namely, if F is the fixed part of $|L(m)|_1$, where the lower index 1 denotes the pull-back to M_1, we have $|L(m)|_1=F+\Lambda$ for a linear system Λ with $\mathrm{Bs}\Lambda=\emptyset$. We claim that tF is the fixed part of $|L(tm)|_1$ for every positive integer t.

To see this, let Δ_j be the divisor of zeros of ϕ_j and set $D_j=(\Delta_j+d_jE-\underline{d_jE})/d_j$. Clearly $[D_j]=L+E$ and ϕ_j^{m/d_j} gives $mD_j\in|L(m)|_1$. So mD_j-F is effective. For any non-negative integers $a_1, \cdots, a_r$ with $a_1d_1+\cdots+a_rd_r=tm$, the $\boldsymbol{Q}$-divisor $\sum_j a_jd_j(D_j-F/m)=-tF+\sum_j a_jd_jD_j$ is effective. Since $\phi_1^{a_1}\cdots\phi_r^{a_r}$ induces $\sum_j a_jd_jD_j\in|L(tm)|_1$ and $H^0(M, L(tm))$ is generated by such monomials by assumption, we infer that tF is in the fixed part of $|L(tm)|_1$. This proves the claim.

Next we claim that F is numerically fixed by $L(m)_1$. To see this, suppose that there is a birational morphism $M_2\to M_1$ and an effective $\boldsymbol{Q}$-divisor X such that $L(m)_2-X$ is nef, where the lower index 2 denotes the pull-back to M_2. Take a positive integer k such that kX is a usual Cartier divisor and let B be the ideal theoretical intersection $kX\cap kF_2$. Take a birational morphism $g: M_3\to M_2$ such that $D=g^*B$ is an effective Cartier divisor. Write $kX_3=D+kX_3'$ and $kF_3=D+kF_3'$. Then $X_3'\cap F_3'=\emptyset$. Since both $(L(m)-F)_3$ and $(L(m)-X)_3$ are nef, we infer that $P=L(m)_3-k^{-1}D$ is nef. Set $H=[\Lambda]$. Then $P-H_3=F_3-k^{-1}D=F_3'$ is effective and H is big. So P is big, too. By virtue of [**F3**; (6.13)], P is almost base point free in the sense of Goodman. On the other hand, by the first claim, tkF_3 is the fixed part of $|tkL(m)_3|$ for every positive integer t. This implies that $t(kF-D)_3=tkF_3'$ is the fixed part of $|t(kL(m)_3-D)|=|tkP|$. Combining them we infer $F_3'=0$. This implies that $X-F_2$ is effective. Thus we prove the claim.

Now, since $H=[\Lambda]$ is nef, $mL_1=F+H$ is a Zariski decomposition. The semipositive part of L is $m^{-1}H$, which is clearly semiample. Thus we prove the "only if" part.

(A4) THEOREM. *Let F be a line bundle on a manifold M with canonical bundle K and let D, E be effective $\boldsymbol{Q}$-divisors such that D is negligible and E is* *usual. Suppose that* $mF+E-K-D$ *is nef and big for any* $m\gg 0$ (*hence* F *itself is nef*). *Then* $H^0(M, tF+E)\neq 0$ *for every* $t\gg 0$.

For a proof, see [**S**] (or [**Ka2**]).

(A5) THEOREM. *Let* L *be a* $\boldsymbol{Q}$*-bundle admitting a Zariski decomposition* $L=N+H$ *on a manifold* M. *Suppose that* $L-K-D$ *is nef and big for some negligible* $\boldsymbol{Q}$*-divisor* D, *where* K *denotes the canonical bundle of* M. *Then, for any line bundle* F *which is numerically equivalent to* qH *for some* $q>0$, *there exists an integer* k *such that* $\mathrm{Bs}|tF|=\emptyset$ *for any* $t\geqq k$.

Proof is almost the same as that of [**Ka2**; Theorem 2.6]. Here we sketch the outline.

We will first show $\kappa(F)\geqq 0$. Let $\{N\}$ be the fractional part of N and take a birational morphism $\pi: M'\to M$ such that $\pi^{-1}(\mathrm{Supp}(\{N\})\cup\mathrm{Supp}(D))$ is a divisor having no singularity other than normal crossings. Let R be the ramification divisor of π and let K' be the canonical bundle of M'. Then $\pi^*(L-K-D)$ $=\pi^*H+\pi^*N+R-K'-\pi^*D$ is nef and big. Since D is negligible, the upper integral hull of $R-\pi^*D$ is effective. Therefore $\pi^*N+R-\pi^*D=E-D'$ for some effective Cartier divisor E and a negligible $\boldsymbol{Q}$-divisor D'. Now, applying (A4), we infer that $H^0(M', tF+E)\neq 0$ for any $t\gg 0$. On the other hand, R is numerically fixed by $L'=R+\pi^*L$ by (1.10). So, by (1.3.2), $R+\pi^*N$ is numerically fixed by L'. Since $\mathrm{Supp}(E)\subset\mathrm{Supp}(R+\pi^*N)$, (1.23) implies that E is numerically fixed by $E+s\pi^*H$ for any $s>0$. Therefore $H^0(M', tF)\neq 0$ for any $t\gg 0$ by (1.8).

Now, for any given integer b with $|bF|\neq\emptyset$ and $\mathrm{Bs}|bF|\neq\emptyset$, we will show $\mathrm{Bs}|tbF|\subsetneqq\mathrm{Bs}|bF|$ for any $t\gg 0$. Our theorem follows from this assertion by a standard argument using a Noetherian induction (cf. [**Ka2**]).

To prove the assertion, take an effective divisor Δ on M such that $L-K-D-\delta\Delta$ is ample for any small positive $\delta\in\boldsymbol{Q}$. Take a birational morphism $\pi: M'\to M$ satisfying the following conditions:

1) Let R be the ramification divisor of π. Then $R\cup\pi^*(N+D+\Delta)\cup\pi^{-1}\mathrm{Bs}|bF|$ is supported on a divisor E having no singularity other than normal crossings.

2) Let $\sum_i r_iE_i$ be the fixed part of $\pi^*|bF|$. Then $\pi^*|bF|=\Lambda+\sum_i r_iE_i$ for some linear system Λ with $\mathrm{Bs}\Lambda=\emptyset$.

Set $R=\sum\rho_iE_i$, $\pi^*N=\sum\nu_iE_i$, $\pi^*D=\sum\varepsilon_iE_i$ and we choose $\delta_i\in\boldsymbol{Q}$ with $0\leqq\delta_i\ll 1$ such that $\pi^*(L-K-D)-\sum\delta_iE_i$ is ample on M'. Set $a_i=\rho_i+\nu_i-\varepsilon_i$ and $c_i=(a_i+1-\delta_i)/r_i$ for each i with $r_i>0$. Then $c_i>0$ for every such i since D is negligible and δ_i is small. Modifying δ_i slightly if necessary, we may assume that the minimum of c_i's is attained at exactly one value of i, say 0. Then $-cr_0+a_0-\delta_0=-1$ and $-cr_i+a_i-\delta_i>-1$ for $i\neq 0$, where $c=c_0$. Denoting by K' the canonical bundle of M', we see that $s\pi^*F-K'+\sum_i(-cr_i+a_i-\delta_i)E_i$

is numerically equivalent to $sF-K'+c(\Lambda-bF)+R+N-D=((s-bc)q-1)H+c[\Lambda]+\pi^*(L-K-D)-\sum_i\delta_iE_i$ and is ample for any $s\gg 0$. Therefore, by Kawamata-Viehweg's vanishing theorem, $H^0(M', s\pi^*F+A)\to H^0(B, [s\pi^*F+A]_B)$ is surjective, where A is the upper integral hull of $\sum_{i\neq 0}(-cr_i+a_i-\delta_i)E_i$ and $B=E_0$. Note that A is an effective Cartier divisor. Similarly as in [**Ka2**], $H^0(B, [s\pi^*F+A]_B)\neq 0$ by (A4). On the other hand, we have $\mathrm{Supp}(A)\subset\mathrm{Supp}(R+\pi^*N)$ since $a_i>0$ implies $\rho_i>0$ or $\nu_i>0$. So, similarly as in the first step, we infer that A is numerically fixed by $s\pi^*F+A$. Hence $H^0(M', s\pi^*F)\simeq H^0(M', s\pi^*F+A)$ for any $s>0$. Combining these observations we infer that $H^0(M', s\pi^*F)\to H^0(B, s\pi^*F)$ is not a zero-map for any $s\gg 0$. This implies $\pi(B)\not\subset\mathrm{Bs}|sF|$ for $s\gg 0$, so $\mathrm{Bs}|tbF|\subsetneqq\mathrm{Bs}|bF|$ for any $t\gg 0$.

REMARK. There is nothing new in this theorem (A5), except possibly the notion of Zariski decomposition.

(A6) DEFINITION. Let V be a normal variety and let D be a $\boldsymbol{Q}$-Weil divisor on V. The pair (V, D) is said to have only *negligible singularities* if there exists a non-singular model $\pi: M\to V$ and effective $\boldsymbol{Q}$-divisors D^*, R on M satisfying the following conditions:
1) D^* is negligible and $\pi_*(D^*)=D$.
2) $\mathrm{codim}\,\pi(R)\geqq 2$.
3) $(K+D^*-R)C=0$ for any curve C in any fiber of π, where K is the canonical bundle of M.

If (V, D) has only negligible singularities, the $\boldsymbol{Q}$-bundle $K+D^*-R$ is determined uniquely up to birational equivalence. Namely, if $\pi_1: M_1\to V$ is another nonsingular model with effective $\boldsymbol{Q}$-divisors D_1^* and R_1 on it as above, then the pull-backs of $K+D^*-R$ and $K_1+D_1^*-R_1$ to any manifold dominating M and M_1 over V are the same. This $\boldsymbol{Q}$-bundle will be denoted by $K(V, D)$, or symbolically by K_V+D. When it admits a Zariski decomposition, the semipositive part of it is well-defined.

REMARK. If in addition $K(V, D)$ comes from a $\boldsymbol{Q}$-bundle on V, (V, D) has only log-terminal singularities in the sense of Kawamata [**Ka2**]. We say that V has only negligible singularities if so does $(V, 0)$. Any canonical singularity in the sense of Reid is negligible in this sense.

(A7) THEOREM. *Let (V, D) be a pair having only negligible singularities and suppose that $K(V, D)$ admits a Zariski decomposition. Then the semipositive part of it is semiample if $K(V, D)$ is big.*

PROOF. We have a non-singular model $\pi: M\to V$ and effective $\boldsymbol{Q}$-divisors D^*, R on M as in (A6). Changing the model if necessary, we may assume that we have a Zariski decomposition $K+D^*-R=K(V, D)=N+H$ on M. Set $L=N+R+tH$ for some $t\gg 0$. Then $L-K-D^*=(t-1)H$ is nef and big. So, by virtue of (A5), it suffices to show that tH is the semipositive part of L. This is equivalent to saying that $N+R$ is numerically fixed by L.

By the property (A6; 3) of $K(V, D)$ and by (1.5), we infer that R is numerically fixed by $T=R+tK(V, D)=R+tN+tH$. Using (1.3.2) we see that $R+tN$ is numerically fixed by T. So (1.23) applies.

(A8) COROLLARY. *Let D be a negligible $\boldsymbol{Q}$-divisor on a manifold M such that $K+D$ is big. If $K+D$ admits a Zariski decomposition, then the graded algebra $\oplus_t H^0(M, tK+\underline{tD})$ is finitely generated.*

(A9) Final Comment. The essential problem is whether $K+D$ admits a Zariski decomposition or not. The answer will be Yes if we have a good theory of "minimal model". Moreover, other approach might be possible because it seems that any big $\boldsymbol{Q}$-bundle admits a Zariski decomposition.

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Note added in proof.

Recently, S. Cutkosky found a counter-example to (1.21). We have $\dim M = \kappa(L) = 3$ in his example, which shows that, in general, the negative part of the Zariski decomposition may be an $\boldsymbol{R}$-divisor whose coefficients are irrational numbers.