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Equivariant cobordism, vector fields, and the Euler characteristic

By Katsuhiro KOMIYA

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Introduction.

Throughout this paper G always denotes a finite group, and a G-manifold means a smooth manifold with smooth G-action. Two n-dimensional closed G-manifolds M and N are *G-cobordant*, if there exists an $(n+1)$-dimensional compact G-manifold L with $\partial L=M+N$, where $+$ denotes the disjoint union. Such a manifold L is called a *G-cobordism* between M and N. If L admits a nonzero G-vector field which is inward normal on M and outward normal on N, then, following Reinhart [7], M and N are called *Reinhart G-cobordant*, and L a *Reinhart G-cobordism* between M and N. The aim of this paper is to obtain a necessary and sufficient condition for the existence of a Reinhart G-cobordism between two given G-cobordant closed G-manifolds.

Given a G-manifold M and a subgroup H of G, M^H denotes the H-fixed point set of M and $M^{=H}$ denotes the union of those components of M^H on which H is the minimal isotropy subgroup. If V is a representation of H containing no direct summand of trivial representation, $M^{(H,V)}$ denotes the union of those components of $M^{=H}$ for which the normal representation is isomorphic to V. Then we will obtain

THEOREM 0.1. *Let M and N be two G-cobordant closed G-manifolds of dimension n. Suppose that n is even and G is of odd order, or that G is of order 2. Then there exists a Reinhart G-cobordism between them if and only if $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ for any pair (H, V) of a subgroup H of G and a representation V of H, where $\chi(\ )$ denotes the Euler characteristic.*

In case H is normal in G, and V is invariant under conjugation, a G-vector bundle $E\to X$ over a G-manifold X is *of type* (H, V) if for any $x\in X$, the isotropy subgroup G_x at x is H, and the fibre E_x over x is isomorphic to V as representations of H. Let $E_1\to X_1$ and $E_2\to X_2$ be G-vector bundles of type (H, V) over k-dimensional closed G-manifolds X_1 and X_2. They are called

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Reinhart G-cobordant, if there exists a G-vector bundle $F\to Y$ of type (H, V) over a $(k+1)$-dimensional compact G-manifold Y such that (i) Y is a Reinhart G-cobordism between X_1 and X_2, and (ii) $F|X_1$ and $F|X_2$ are isomorphic as G-vector bundles to E_1 and E_2, respectively.

Given a G-manifold M with $M^{(H,V)}$ nonempty, let K be the subgroup of G whose action keeps $M^{(H,V)}$ invariant. We see that $H\subset K\subset N(H)$, the normalizer of H in G, and that K is determined only by (H, V) and independent of M. So we denote K by $G_{(H,V)}$. We note that if a (real) representation of an odd order group contains no direct summand of trivial representation, then it has a complex structure, and hence it is even dimensional. If G is of odd order and $\dim M^{(H,V)}=1$, then H is the only isotropy subgroup on $M^{(H,V)}$, and the normal bundle $\nu(M^{(H,V)})\to M^{(H,V)}$ is a $G_{(H,V)}$-vector bundle of type (H, V).

THEOREM 0.2. *Let M and N be two G-cobordant closed G-manifolds of odd dimension. Suppose that G is of odd order. Then there exists a Reinhart G-cobordism between them if and only if for any pair (H, V) for which* $\dim(M+N)^{(H,V)}=1$, *the normal bundles $\nu(M^{(H,V)})\to M^{(H,V)}$ and $\nu(N^{(H,V)})\to N^{(H,V)}$ are Reinhart $G_{(H,V)}$-cobordant as $G_{(H,V)}$-vector bundles of type (H, V).*

If G is abelian and of odd order, we will show that the above condition for the normal bundles always follows. Thus we will obtain

COROLLARY 0.3. *Let M and N be two G-cobordant closed G-manifolds of odd dimension. If G is abelian and of odd order, then there exists a Reinhart G-cobordism between them.*

REMARK. When G is of order 2, Stong [8] already showed Theorem 0.1 in case either n is even, or $n=1$, or M and N have no isolated fixed point. When G is an abelian group of odd order, there is a study of Heithecker [2] on *oriented* Reinhart G-cobordism. There is also a notion of *controllable cutting and pasting* (*SKK-equivalence*) of G-manifolds. This notion is closely related to Reinhart G-cobordism. See Heithecker [1] and Prevot [4, 5, 6] for this notion and related results.

This paper will proceed as follows. In § 1 we will give a characterization of Reinhart G-cobordism in terms of the Euler characteristic. In § 2 we will introduce G-surgery and G-connected sum as technical preliminaries. In § 3 and § 4 we will show that a G-cobordism satisfying certain conditions may be altered to a Reinhart G-cobordism by G-surgery and G-connected sum. In § 5 we will prove the results mentioned above.

§1. Vector fields.

PROPOSITION 1.1. *Let L be a compact connected manifold with boundary $\partial L=M+N$, the disjoint union of closed manifolds M and N. Then L admits a nonzero vector field which is inward normal on M and outward normal on N if and only if $\chi(L)=\chi(M)=\chi(N)$.*

For the proof see "Proof of Theorem (1)" in Reinhart [7]. By the similar way to Komiya [**3**] we may generalize this proposition to an equivariant version:

PROPOSITION 1.2. *Let L be a compact G-manifold with $\partial L=M+N$, the disjoint union of closed G-manifolds M and N, and G be a finite group. Then L admits a nonzero G-vector field which is inward normal on M and outward normal on N if and only if for any pair (H, V) of a subgroup H of G and a representation V of H, every component A of $L^{(H,V)}$ satisfies $\chi(A)=\chi(A\cap M)=\chi(A\cap N)$. (Here we make the convention $\chi(\emptyset)=0$.)*

This proposition characterizes a Reinhart G-cobordism in terms of the Euler characteristic.

§2. G-surgery and G-connected sum.

Let H be a subgroup of G, and V a representation of H containing no direct summand of trivial representation. Let L be a G-manifold with $\dim L^{(H,V)}>0$, and $\dim L^{(H,V)}+1=k_1+k_2$, where k_1 and k_2 are positive integers. Consider the G-manifold $G\times_H D(V\oplus R^{k_1})\times S(R^{k_2})$ where R^k is the k-dimensional trivial representation, and $D(\ )$ and $S(\ )$ denote the closed unit disc and the unit sphere, respectively. If there is a smooth G-embedding

$$\varphi: G\times_H D(V\oplus R^{k_1})\times S(R^{k_2}) \longrightarrow L,$$

then we obtain a G-manifold L_1 from the disjoint union of $L-\varphi(G\times_H D^o(V\oplus R^{k_1})\times S(R^{k_2}))$ and $G\times_H S(V\oplus R^{k_1})\times D(R^{k_2})$ by gluing the corresponding boundaries by φ, where $D^o(\ )$ denotes the open unit disc. L_1 is called a *G-manifold obtained from L by G-surgery of type (H, V, k_1, k_2).* We see the following:

(1) Let K be a subgroup of G, and U a representation of K. If K is not conjugate to a subgroup of H, or if $K=H$ and $GL^{(K,U)}\cap GL^{(H,V)}=\emptyset$, then the above G-surgery does not affect $L^{(K,U)}$, i.e., $L^{(K,U)}=L_1^{(K,U)}$.

(2) Restricting the G-surgery to the H-fixed point set, we see that $L_1^{(H,V)}$ is obtained from $L^{(H,V)}$ by deleting $\chi(G_{(H,V)}/H)$ copies of $D^o(R^{k_1})\times S(R^{k_2})$ and attaching as many copies of $S(R^{k_1})\times D(R^{k_2})$. Thus, if $\dim L^{(H,V)}$ is even,

$$\chi(L_1^{(H,V)})=\chi(L^{(H,V)})+(-1)^{k_1+1}2\chi(G_{(H,V)}/H).$$

(3) If $L^{(K,U)}$ is connected and $\dim V^K+k_1\geqq 2$, then $L_1^{(K,U)}$ is also connected.

Next let us introduce the G-connected sum. Let L and M be G-manifolds of dimension n, and H an isotropy subgroup occurring on both L and M. Let V be a representation of H containing no direct summand of trivial representation such that both $L^{(H,V)}$ and $M^{(H,V)}$ are not empty. Then there are smooth G-embeddings $\varphi: G\times_H D(V\oplus R^k)\to L$ and $\psi: G\times_H D(V\oplus R^k)\to M$, where $k=\dim L^{(H,V)}=\dim M^{(H,V)}$. We obtain a G-manifold from the disjoint union of $L-\varphi(G\times_H D^\circ(V\oplus R^k))$ and $M-\psi(G\times_H D^\circ(V\oplus R^k))$ by identifying $\varphi([g, x])$ with $\psi([g, x])$ for $g\in G$ and $x\in S(V\oplus R^k)$. The G-manifold is called a *G-connected sum of L and M of type (H, V)*.

Let $RP(V\oplus R^{k+1})$ be the quotient space of $S(V\oplus R^{k+1})$ by the antipodal involution. $RP(V\oplus R^{k+1})$ inherits a structure of an n-dimensional H-manifold, and we see that $RP(V\oplus R^{k+1})^H=RP(R^{k+1})$ if H is of odd order, and that $RP(V\oplus R^{k+1})^H=RP(V)+RP(R^{k+1})$ if H is of order 2. Let L_2 be a G-connected sum of L and $G\times_H RP(V\oplus R^{k+1})$ of type (H, V). We then see the following:

(4) If G is of odd order and a pair (K, U) is as in (1), then the above G-connected sum does not affect $L^{(K,U)}$, i.e., $L^{(K,U)}=L_2^{(K,U)}$.

(5) Restricting the G-connected sum to the H-fixed point set, we see that if G is of odd order then $L_2^{(H,V)}$ is a (nonequivariant) connected sum of $L^{(H,V)}$ and each of $\chi(G_{(H,V)}/H)$ copies of $RP(R^{k+1})$. Thus, further if $k=\dim L^{(H,V)}$ is even,

$$\chi(L_2^{(H,V)})=\chi(L^{(H,V)})-\chi(G_{(H,V)}/H).$$

In case G is of order 2 we may also see the corresponding assertion.

§ 3. Construction of Reinhart G-cobordisms (1).

In this section G is always of odd order. Given a compact G-manifold L, denote by $I(L)$ a complete set of representatives of conjugacy classes of isotropy subgroups on L. For any $H\in I(L)$, denote by $R_H(L)$ the set of representations of H such that

(i) $L^{(H,V)}$ is not empty for any $V\in R_H(L)$,

(ii) if A is a component of $L^{=H}$, then A is contained in $GL^{(H,V)}$ for some $V\in R_H(L)$,

(iii) $GL^{(H,V)}$ for all $V\in R_H(L)$ are disjoint from each other.

Let $R(L)=\{(H, V)\mid H\in I(L), V\in R_H(L)\}$.

Theorem 3.1. *Suppose that G is of odd order. Let M and N be two G-cobordant closed G-manifolds of dimension n. Then there exists a Reinhart G-cobordism between them if and only if*

(i) $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ *for any* $(H, V)\in R(M+N)$, *and*

(ii) *when* $\dim V=n-1$, $\nu(M^{(H,V)})$ *and* $\nu(N^{(H,V)})$ *are Reinhart* $G_{(H,V)}$*-cobordant as* $G_{(H,V)}$*-vector bundles of type* (H, V).

For a proof of the theorem we need two lemmas.

LEMMA 3.2. *Suppose that G is of odd order. Let M and N be two G-cobordant closed G-manifolds of dimension n, and L a G-cobordism between them. For some $(H, V)\in R(L)$ with $\dim V\neq n+1$, n, $n-1$, suppose that*

(i) $\chi(M^{(H,V)})=\chi(N^{(H,V)})$, *and*

(ii) $\chi(L^{(H,V)})$ *is a multiple of* $\chi(G_{(H,V)}/H)$.

Then we may alter L to obtain a G-cobordism $\tilde{L}$ between M and N such that

(a) $R(\tilde{L})=R(L)$,

(b) $\tilde{L}^{(H,V)}$ *is connected, and* $\chi(\tilde{L}^{(H,V)})=\chi(M^{(H,V)})=\chi(N^{(H,V)})$, *and*

(c) *given $(K, U)\in R(L)$, if K is not conjugate to a subgroup of H, or if $H=K$ and $GL^{(K,U)}\cap GL^{(H,V)}=\emptyset$, then $\tilde{L}^{(K,U)}=L^{(K,U)}$.*

PROOF. Let $k=\dim L^{(H,V)}$. Then $k\geqq 3$ by the assumption. Doing G-surgery on L of type $(H, V, k, 1)$, we obtain a G-cobordism L_1 such that

(i) $R(L_1)=R(L)$,

(ii) $L_1^{(H,V)}$ is connected, and

(iii) $L_1^{(K,U)}=L^{(K,U)}$ for such $(K, U)\in R(L)$ as in (c).

If k is odd, then $2\chi(L_1^{(H,V)})=\chi(\partial L_1^{(H,V)})$. Since $\partial L_1^{(H,V)}=M^{(H,V)}+N^{(H,V)}$, we see $\chi(L_1^{(H,V)})=\chi(M^{(H,V)})=\chi(N^{(H,V)})$. Thus L_1 is a desired G-cobordism.

If k is even, then $\chi(M^{(H,V)})=\chi(N^{(H,V)})=0$, and hence we must make $\chi(L_1^{(H,V)})$ zero. From the assumption of the lemma and (2) in §2, $\chi(L_1^{(H,V)})$ is also a multiple of $\chi(G_{(H,V)}/H)$, say $\chi(L_1^{(H,V)})=p\chi(G_{(H,V)}/H)$. If $p=0$, L_1 is a desired G-cobordism. Let $k+1=k_1+k_2$, where $k_1\geqq 2$ and $k_2\geqq 1$ are integers such that k_1 is even if $p>0$, or k_1 is odd if $p<0$. If $|p|$ is even, then the argument in §2 ensures that we obtain a desired G-cobordism by doing $|p|/2$ times G-sugeries on L_1 of type (H, V, k_1, k_2). If $|p|$ is odd, let L_2 be a G-connected sum of L_1 and $G\times_H RP(V\oplus R^{k+1})$. Then $\chi(L_2^{(H,V)})$ is an even multiple of $\chi(G_{(H,V)}/H)$, and we obtain a desired G-cobordism by doing G-surgeries on L_2 as above. Q.E.D.

LEMMA 3.3. *Suppose that G is of odd order. Let M and N be two G-cobordant closed G-manifolds, and L a G-cobordism between them. For some $(H, V)\in R(L)$, suppose that*

(i) *H is the only isotropy subgroup on $L^{(H,V)}$, and*

(ii) $\nu(M^{(H,V)})$ *and* $\nu(N^{(H,V)})$ *are Reinhart $G_{(H,V)}$-cobordant as $G_{(H,V)}$-vector bundles of type (H, V).*

Then we may alter L to obtain a G-cobordism $\tilde{L}$ between M and N such that

(a) $R(\tilde{L})=R(L)$,

(b) $\chi(A)=\chi(A\cap M)=\chi(A\cap N)$ *for any component A of $\tilde{L}^{(H,V)}$, and*

(c) *as* (c) *in Lemma* 3.2.

PROOF. Let $E\to X$ be a $G_{(H,V)}$-vector bundle of type (H, V) which is a Reinhart $G_{(H,V)}$-cobordism between $\nu(M^{(H,V)})$ and $\nu(N^{(H,V)})$. Here we may assume that X has no component without boundary. Since X admits a nonzero vector field which is inward normal on M and outward normal on N, then by Proposition 1.1 or Proposition 1.2 we see that $\chi(A)=\chi(A\cap M)=\chi(A\cap N)$ for any component A of X. Let $D\nu(M^{(H,V)})$, $D\nu(N^{(H,V)})$ and DE denote the associated disc bundles. There are smooth G-embeddings

$$\varphi: G\times_{G(H,V)}D\nu(M^{(H,V)})\longrightarrow M\subset L, \quad \text{and}$$

$$\phi: G\times_{G(H,V)}D\nu(N^{(H,V)})\longrightarrow N\subset L$$

onto G-invariant tubular neighborhoods of $GM^{(H,V)}$ and $GN^{(H,V)}$ in M and N, respectively. Let L_1 be a G-manifold obtained from the disjoint union of L and $G\times_{G(H,V)}DE$ by identifying $\mathrm{Im}\,\varphi$ with $G\times_{G(H,V)}(DE|M^{(H,V)})$ and $\mathrm{Im}\,\phi$ with $G\times_{G(H,V)}(DE|N^{(H,V)})$. Let L_2 be a G-manifold obtained from the disjoint union of $M\times[0, 1]$, $N\times[0, 1]$ and $G\times_{G(H,V)}DE$ by identifying $\mathrm{Im}\,\varphi\times\{1\}$ with $G\times_{G(H,V)}(DE|M^{(H,V)})$ and $\mathrm{Im}\,\phi\times\{1\}$ with $G\times_{G(H,V)}(DE|N^{(H,V)})$. We then see that $\partial L_2=M+N+Y$ and $Y\approx\partial L_1$. So let L_3 be a G-manifold obtained from the disjoint union of L_1 and L_2 by identifying ∂L_1 with Y. For L_3 we see that

(i) $R(L_3)=R(L)$,

(ii) $\partial L_3=M+N$,

(iii) $L_3^{(H,V)}=X+Z$, where Z is a closed manifold, and

(iv) $L_3^{(K,U)}\approx L^{(K,U)}$ for such $(K, U)\in R(L)$ as in (c).

Indeed Z is diffeomorphic to a manifold obtained from the disjoint union of $L^{(H,V)}$ and X by identifying $\partial L^{(H,V)}$ with ∂X. From the assumption of the lemma H is the only isotropy subgroup on Z. Thus GZ is a G-invariant submanifold of L_3. Let T be a G-invariant open tubular neighborhood of GZ in L_3, and let $L_4=L_3-T$. Then $\partial L_4=M+N+S$, where S is a sphere bundle over GZ. Let L_5 be a G-manifold obtained by sewing L_4 along S by antipodal involution on every fibre. This sewing yields no new fixed point since G is of odd order, and we see $L_5^{(H,V)}=X$. L_5 is a desired G-cobordism. Q.E.D.

We are now in a position to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let L be a G-cobordism between closed G-manifolds M and N of dimension n. First suppose that L is Reinhart. Then (i) holds from Proposition 1.2. When $\dim V=n-1$, i.e., $\dim L^{(H,V)}=2$, all the isotropy subgroups on $L^{(H,V)}$ are H, since G is of odd order and L has no isolated fixed point. Thus $\nu(L^{(H,V)})$ gives a Reinhart $G_{(H,V)}$-cobordism between $\nu(M^{(H,V)})$ and $\nu(N^{(H,V)})$. Hence (ii) holds.

Conversely, suppose that (i) and (ii) hold. We alter L to a Reinhart G-cobordism. This is done separately in the cases in which n is even or odd.

[I] *The case when n is even.* Since G is of odd order, $\dim L^{(H,V)}$ is odd for any $(H, V)\in R(L)$. For (H, V) with $\dim L^{(H,V)}=1$ we do G-surgery on L of type $(H, V, 1, 1)$ to obtain a G-cobordism L_1 such that any component A of $L_1^{(H,V)}$ is either a circle or a curve joining points of M and N, and hence $\chi(A)=\chi(A\cap M)=\chi(A\cap N)$. For (H, V) with $k=\dim L_1^{(H,V)}\geqq 3$ we do G-surgery on L_1 of type $(H, V, k, 1)$ to obtain a G-cobordism L_2 such that $L_2^{(H,V)}$ is connected and that if $\dim L_2^{(K,U)}=1$ then $L_2^{(K,U)}=L_1^{(K,U)}$. Since $2\chi(L_2^{(H,V)})=\chi(\partial L_2^{(H,V)})$ and $\partial L_2^{(H,V)}=M^{(H,V)}+N^{(H,V)}$, $\chi(L_2^{(H,V)})=\chi(M^{(H,V)})=\chi(N^{(H,V)})$. Thus Proposition 1.2 implies that L_2 is Reinhart.

[II] *The case when n is odd.* In this case $\dim L^{(H,V)}$ is even for any $(H, V)\in R(L)$. If $\dim L^{(H,V)}=0$, we cut off from L an open disc about each point of $L^{(H,V)}$, and sew L by antipodal involution along each sphere resulting as boundary. By this way we may remove all the isolated fixed points from L, and obtain a G-cobordism L_0 such that $\dim L_0^{(H,V)}\geqq 2$ for any $(H, V)\in R(L_0)$. For any component A of $L_0^{(H,V)}$, $A\cap M$ and $A\cap N$ have Euler characteristic zero since they are odd dimensional and closed. Thus we must make the Euler characteristic of A zero. Let

$$R(L_0)=\{(H_1, V_1), (H_2, V_2), \cdots, (H_a, V_a)\}$$

be ordered in such a way that if H_i is conjugate to a subgroup of H_j then $j\leqq i$. In virtue of Lemma 3.2 and Lemma 3.3 we may inductively alter L_0 to obtain G-cobordisms L_i $(1\leqq i\leqq a)$ such that

(i) $R(L_i)=R(L_0)$,

(ii) $\chi(A)=0$ for any component A of $L_i^{(H_j,V_j)}$, $1\leqq j\leqq i$, and

(iii) $L_i^{(H_j,V_j)}=L_{i-1}^{(H_j,V_j)}$ for any j with $1\leqq j<i$.

Then L_a is a desired Reinhart G-cobordism.

The induction step proceeds as follows. Suppose that we have obtained L_{i-1}. Consider $L_{i-1}^{(H_i,V_i)}$ as a $G_{(H_i,V_i)}$-manifold. Then H_i is a principal isotropy subgroup on it. For any subgroup K with $H_i<K\leq G_{(H_i,V_i)}$ we see $\chi((L_{i-1}^{(H_i,V_i)})^K)=0$. This implies that $\chi(L_{i-1}^{(H_i,V_i)})$ is a multiple of $\chi(G_{(H_i,V_i)}/H_i)$. Thus, if $\dim L_{i-1}^{(H_i,V_i)}\geqq 4$, i.e., $\dim V_i\neq n+1, n, n-1$, then L_i is obtained by Lemma 3.2, and if $\dim L_{i-1}^{(H_i,V_i)}=2$, i.e., $\dim V_i=n-1$, then L_i is obtained by Lemma 3.3. Q.E.D.

§ 4. Construction of Reinhart G-cobordisms (2).

In this section we consider the case when G is of order 2. In this case a representation V of G containing no direct summand of trivial representation is determined by its dimension. So, for a G-manifold M we denote $M^{(G,V)}$ by $M^{(G,k)}$ where $k=\dim V$.

PROPOSITION 4.1. *Let M and N be two G-cobordant closed G-manifolds of dimension n. Suppose that G is of order 2 and that n is odd and greater than 1. Then there exists a Reinhart G-cobordism between M and N, if and only if $\chi(M^{(G,k)})=\chi(N^{(G,k)})$ for any k $(0\leqq k\leqq n)$.*

This proposition will be proved by a similar way to the proof of Theorem 3.1. The following two lemmas are needed.

LEMMA 4.2. *Let M and N be two G-cobordant closed G-manifolds of dimension n, and L a G-cobordism between them. Suppose that G is of order 2 and that n is odd. For some integer k $(0\leqq k\leqq n-2)$, suppose that $\chi(M^{(G,k)})=\chi(N^{(G,k)})$. Then L is altered to a G-cobordism $\tilde{L}$ such that*

(a) *$\tilde{L}^{(G,k)}$ is connected, and $\chi(\tilde{L}^{(G,k)})=\chi(M^{(G,k)})=\chi(N^{(G,k)})$, and*

(b) *for any $m\neq k$, $\tilde{L}^{(G,m)}=L^{(G,m)}+A$, where A is a closed manifold with $\chi(A)=0$, in fact, A is the empty set or an odd dimensional real projective space.*

PROOF. Similar to the proof of Lemma 3.2 except the point that $RP(V\oplus R^{n-k+2})^G=RP(V)+RP(R^{n-k+2})$ since G is of order 2. If $\dim L^{(G,k)}=n-k+1$ is even and $\chi(L^{(G,k)})$ is odd, one needs to do connected sum of L and $RP(V\oplus R^{n-k+1})$. This yields the new component $RP(V)$ of the fixed point set. $RP(V)$ is odd dimensional, since n is odd. Q.E.D.

LEMMA 4.3. *Let M and N be two G-cobordant closed G-manifolds of dimension $n\geqq 3$, and G be of order 2. Any G-cobordism L between M and N is altered to a G-cobordism $\tilde{L}$ such that*

(a) *each component of $\tilde{L}^{(G,n-1)}$ has Euler characteristic zero, and*

(b) *$\tilde{L}^{(G,m)}\approx L^{(G,m)}$ except for $m=1, n-1$.*

PROOF. By Stong [8; Theorem 3.1 and Corollary] we see that two normal bundles $\nu(M^{(G,n-1)})$ and $\nu(N^{(G,n-1)})$ are Reinhart G-cobordant. Thus the proof proceeds as the proof of Lemma 3.3 except that new fixed point set arises when one sews L_4 along S by antipodal involution. This new fixed point set is of dimension n, and does not affect $\tilde{L}^{(G,n-1)}$ since $n\geqq 3$. Q.E.D.

PROOF OF PROPOSITION 4.1. The "only if" part is easy from Proposition 1.2. To prove the "if" part, let L be a G-cobordism between M and N. As in the proof of Theorem 3.1, we may remove the isolated fixed points from L and alter $L^{(G,n)}$ to a disjoint union of circles and curves joining points of M and N. Then we obtain a G-cobordism L_1 such that

$$(*)\qquad \chi(A)=\chi(A\cap M)=\chi(A\cap N)$$

for any component A of $L_1^{(G,k)}$ with $k=n+1, n$. Applying Lemma 4.3 to L_1, we obtain L_2 such that the equation $(*)$ holds for any component A of $L_2^{(G,k)}$

with $k=n+1, n, n-1$. Applying also Lemma 4.2 repeatedly to L_2, we obtain L_3 such that the equation (*) holds for any component A of L_3^G. By G-surgery of type $(\{1\}, \{0\}, n+1, 1)$ we make $L_4=L_3^{(\{1\},0)}$ connected, where $\{1\}$ is the identity subgroup of G and $\{0\}$ is the 0-dimensional representation. We want to make the Euler characteristic of L_4 vanish, since $\chi(L_4\cap M)=\chi(L_4\cap N)=0$. Since $0=\chi(L_4\cap M)\equiv\chi(L_4^G\cap M) \mod 2$, $\chi(L_4^G)=\chi(L_4^G\cap M)$ is even. Also since $\chi(L_4)\equiv\chi(L_4^G) \mod 2$, $\chi(L_4)$ is even. Thus, as in the proof of Lemma 3.2, we may make the Euler characteristic of L_4 vanish, and the resulting manifold is a Reinhart G-cobordism between M and N. Q.E.D.

§5. Proofs of Theorem 0.1, Theorem 0.2, and Corollary 0.3.

Let M and N be two G-cobordant closed G-manifolds of dimension n. If a representation of an odd order group does not contain a direct summand of trivial representation, it is even dimensional. Thus, if n is even, (ii) of Theorem 3.1 is vacuously valid. So, if n is even and G is of odd order, Theorem 0.1 follows from Theorem 3.1. If G is of order 2, Theorem 0.1 follows from Proposition 4.1 and Stong [8; Theorem 4.4].

If n is odd, then for any $(H, V)\in R(M+N)$, $M^{(H,V)}$ and $N^{(H,V)}$ are odd dimensional and closed. Thus the Euler characteristics of them are zero, and hence (i) of Theorem 3.1 holds. So Theorem 0.2 also follows from Theorem 3.1.

Corollary 0.3 follows from Theorem 0.2 and the following lemma:

LEMMA 5.1. *Let G be an abelian group of odd order, and $E\to X$ a G-vector bundle of type (H, V) over a closed G-manifold X of dimension 1. Then $E\to X$ is Reinhart G-cobordant to zero as a G-vector bundle of type (H, V).*

For a proof of the lemma we first give some remarks. Let G be an abelian group of odd order, V a representation of a subgroup H of G containing no direct summand of trivial representation. Since H is of odd order, V has a structure of a complex representation. Since G is abelian, the H-action on V extends to a G-action on V. So we may consider V as a complex representation of G. Let $E\to X$ be a G-vector bundle of type (H, V). From the above remark we may consider $E\to X$ as a complex G-vector bundle. From now on we fix complex structures of V and $E\to X$. Let $\{V_j \mid j\in J(H)\}$ be a complete set of nontrivial, nonisomorphic complex irreducible representations of H, and let $V=\bigoplus_{j\in J(H)}V_j^{n_j}$. The H-equivariant complex linear homomorphisms from V_j to every fibre of E form a complex n_j-dimensional G-vector bundle $\mathrm{Hom}^H(V_j, E)$ over X. Let $X\times V_j$ be a complex G-vector bundle (with diagonal G-action) over X. We obtain a canonical homomorphism

$$\phi_j: (X\times V_j)\otimes_C \mathrm{Hom}^H(V_j, E) \longrightarrow E$$

such that

$$\oplus_{j\in J(H)}\phi_j : \oplus_{j\in J(H)}((X\times V_j)\otimes_C \mathrm{Hom}^H(V_j, E)) \longrightarrow E$$

is an isomorphism of complex G-vector bundles. Since H acts trivially on $\mathrm{Hom}^H(V_j, E)$, we obtain a complex vector bundle $\mathrm{Hom}^H(V_j, E)/G$ over X/G. If X is 1-dimensional and closed, then so is X/G, and $\mathrm{Hom}^H(V_j, E)/G$ is isomorphic to a product bundle $X/G\times C^{n_j}$ since $BU(n_j)$ is 1-connected. Thus $\mathrm{Hom}^H(V_j, E)$ is isomorphic to $X\times C^{n_j}$, and we see that E is isomorphic to a product bundle $X\times V$.

PROOF OF LEMMA 5.1. X admits a free G/H-action. Let $f: X/G\rightarrow B(G/H)$ be a classifying map for its G/H-action. X/G is a disjoint union of circles: $X/G=C_1+C_2+\cdots+C_k$. Denote by f_i the restriction of f to each C_i. Since the fundamental group $\pi_1(B(G/H))\cong G/H$ is of odd order, then there is $[g_i]\in\pi_1(B(G/H))$ such that $2[g_i]=[f_i]$ in $\pi_1(B(G/H))$. Let M_i be a Möbius band with $\partial M_i=C_i$ and the axis $D_i\approx$circle. Since $2[g_i]=[f_i]$, there is a map $F_i: M_i\rightarrow B(G/H)$ such that $F_i|C_i=f_i$ and $F_i|D_i=g_i$. Pulling back a universal G/H-space by the map $F_1+\cdots+F_k: M_1+\cdots+M_k\rightarrow B(G/H)$, we obtain a 2-dimensional compact G-manifold Y such that (1) $\partial Y=X$, (2) H is the only isotropy subgroup on Y, (3) Y admits a nonzero G-vector field pointing inward on $\partial Y=X$ (since M_i admits a nonzero vector field pointing inward on $\partial M_i=C_i$). Then $Y\times V$ is a Reinhart G-cobordism between E and zero, since E is isomorphic to $X\times V$. Q.E.D.

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Katsuhiro KOMIYA
Department of Mathematics
Yamaguchi University
Yamaguchi 753
Japan