J. Math. Soc. Japan
Vol. 36, No. 4, 1984

On polarized manifolds of Δ-genus two; part I

By Takao FUJITA

(Received Dec. 23, 1983)

Introduction.

By a polarized manifold we mean a pair (M, L) of a projective manifold M and an ample line bundle L on M. Set $n=\dim M$, $d(M, L)=L^n$ and $\Delta(M, L)=n+d(M, L)-h^0(M, L)$. Then $\Delta(M, L)\geqq 0$ for any polarized manifold (M, L) (see [**F2**]). We have classified polarized manifolds with $\Delta=0$ in [**F2**] and those with $\Delta=1$ in [**F5**] (as for positive characteristic cases, see [**F6**]). In this series of papers we will study polarized manifolds with $\Delta=2$. However, because of various technical reasons, we assume here that things are defined over the complex number field $\boldsymbol{C}$, although some arguments work in positive characteristic cases too.

This series is an improved version of [**F1**], which contains most results here, but, unfortunately, is hardly readable. We remark that Ionescu [**I**] obtained independently the classification of (M, L) with $\Delta=2$ such that L is very ample.

§ 0. Outline of the classification.

In this section we give a brief account of the classification of polarized manifolds with $\Delta=2$. We freely use the notation in [**F2**], [**F5**], [**F6**], etc. The following result is used to reduce various problems to lower dimensional cases.

(0.1) THEOREM. *Let (M, L) be a polarized manifold with* $\dim M=n\geqq 3$, $d(M, L)=d\geqq 2$ *and* $\Delta(M, L)=2$. *Then any general member D of $|L|$ is non-singular. Moreover, the restriction homomorphism $r: H^0(M, L)\to H^0(D, L_D)$ is surjective and $\Delta(D, L_D)=2$.*

PROOF. [**F7**; (4.1)] shows that D is smooth. If r is not surjective, we have $H^1(M, \mathcal{O}_M)>0$ and $\Delta(D, L_D)<2$. The latter implies $H^1(D, L_D)=0$ by [**F2**] and [**F5**]. This is absurd because we have an exact sequence $H^1(M, -L)\to H^1(M, \mathcal{O}_M)\to H^1(D, \mathcal{O}_D)$ and $H^1(M, -L)=0$ by Kodaira's vanishing theorem. Thus r is surjective and hence $\Delta(D, L_D)=2$.

(0.2) THEOREM. *Let (M, L) be a polarized manifold with* $\dim M=n\geqq 2$, $\Delta(M, L)=2$ *and* $g(M, L)\leqq 1$, *where $g(M, L)$ is the sectional genus. Then* $M\cong\boldsymbol{P}(E)$

for an ample vector bundle E of rank two over an elliptic curve C and L is the tautological line bundle on it.

PROOF. We consider first the case $d(M, L)=d=1$. Then $h^0(M, L)=n+d-\Delta=n-1$, while $\dim \mathrm{Bs}|L|\leqq 1$ by [**F2**; Theorem 1.9]. Therefore, if $D_1, \cdots, D_{n-1}$ are general members of $|L|$ and if $C=D_1\cap\cdots\cap D_{n-1}$, then $\mathrm{Bs}|L|=\mathrm{Supp}(C)$ is a curve. Moreover $LC=L^n=1$. Hence the scheme theoretic intersection C is an irreducible reduced curve. By [**F2**; Proposition 1.3] we have $h^1(C, \mathcal{O}_C)=g(M, L)\leqq 1$.

Assume that $H^1(M, \mathcal{O}_M)=0$. Then we claim $H^i(V_j, (1-i)L)=0$ for each $j=1, \cdots, n$ and $i=1, \cdots, j-1$, where $V_j=D_j\cap D_{j+1}\cap\cdots\cap D_{n-1}$ (set $V_n=M$ and $V_1=C$). Indeed, this is true when $j=n$ by the assumption and Kodaira's vanishing theorem. In case $j<n$, we use the exact sequence $H^i(V_{j+1}, (1-i)L)\to H^i(V_j, (1-i)L)\to H^{i+1}(V_{j+1}, -iL)$ and the descending induction on j from above to prove the claim. Thus we have $H^1(V_j, \mathcal{O})=0$ for each $j\geqq 2$, which implies $\Delta(M, L)=\Delta(V_n, L)=\cdots=\Delta(V_1, L)=\Delta(C, L)$. However $\Delta(C, L)\leqq 1$ because $h^1(C, \mathcal{O}_C)\leqq 1$. This contradiction shows that $H^1(M, \mathcal{O}_M)\neq 0$.

On the other hand, by a similar argument as above, we get $H^i(V_j, -tL)=0$ for any $i<j$, $t>0$ by the descending induction on j and hence $H^1(V_{j+1}, \mathcal{O})\to H^1(V_j, \mathcal{O})$ is injective for each $j\geqq 1$. Therefore $h^1(M, \mathcal{O}_M)\leqq h^1(C, \mathcal{O}_C)\leqq 1$. So we conclude that $H^1(M, \mathcal{O}_M)\to H^1(C, \mathcal{O}_C)$ is bijective and $g(M, L)=h^1(C, \mathcal{O}_C)=1$.

Since $h^1(M, \mathcal{O}_M)=1$, the Albanese variety A of M is an elliptic curve. Let $\alpha: M\to A$ be the Albanese morphism. Then $\alpha(C)=A$ because $H^1(A, \mathcal{O}_A)\to H^1(M, \mathcal{O}_M)\to H^1(C, \mathcal{O}_C)$ is bijective. In view of $h^1(C, \mathcal{O}_C)=1$, we infer that C is a non-singular elliptic curve.

Now, when $n=2$, we apply [**F5**; (1.11)] to prove the theorem. So we will consider the case $n\geqq 3$ by induction on n. Let $\pi: M'\to M$ be the blowing-up with center C, let $E=\pi^{-1}(C)$ be the exceptional divisor, and let D_j' and V_j' be the proper transforms of D_j and V_j respectively. Since C is the ideal theoretical intersection of D_j's, we have $D_1'\cap\cdots\cap D_{n-1}'=\emptyset$. So $\mathrm{Bs}|\pi^*L-E|=\emptyset$ because $D_j'\in|\pi^*L-E|$. This linear system gives a morphism $\rho: M'\to \boldsymbol{P}^{n-2}$, whose restriction to each fiber of $E\to C$ is an isomorphism. From this we infer $E\cong C\times\boldsymbol{P}^{n-2}$, $D_j'\cap E\cong C\times\boldsymbol{P}^{n-3}$ and $V_j'\cap E\cong C\times\boldsymbol{P}^{j-2}$. This implies that V_j is smooth along C and V_j' is the blowing-up of V_j with center C. Thus, by Bertini's theorem, V_j is a submanifold of M. So, to prove the theorem, it suffices to derive a contradiction assuming $n=3$.

When $n=3$, any general member D of $|L|$ is a $\boldsymbol{P}^1$-bundle over $A=\mathrm{Alb}(M)\cong\mathrm{Alb}(D)$ by [**F5**; (1.11)]. Hence $\alpha: M\to A$ is a $\boldsymbol{P}^2$-bundle by [**F4**; (4.9)]. Moreover $M\cong\boldsymbol{P}_A(\mathcal{E})$ for some ample vector bundle $\mathcal{E}$ of rank 3 on A and L is the tautological line bundle on it. Then, as is well-known (cf., e.g., [**I**; Proposition 3.11]), we have $h^0(M, L)=h^0(A, \mathcal{E})=\deg(\det\mathcal{E})=L^3=d$, contradicting

$\Delta(M, L)=2$. Thus we complete the proof in case $d(M, L)=1$.

Next we consider the case $d(M, L)>1$. Using (0.1) and by similar arguments as above, we reduce the problem to the case $n=2$. The case in which $|L|$ has fixed components will be studied in the next section (cf. (1.13)). Here we assume that $|L|$ has at most finitely many base points. Then a general member C of $|L|$ is a smooth curve by [**F7**; (2.8)]. So, similarly as in the case $d(M, L)=1$, we infer $h^1(M, \mathcal{O}_M)>0$, $H^1(M, \mathcal{O}_M)\to H^1(C, \mathcal{O}_C)$ is bijective, $g(M, L)=h^1(C, \mathcal{O}_C)=1$ and hence [**F5**; (1.11)] applies.

(0.3) In case $g(M, L)>1$, since $\dim \mathrm{Bs}|L|<\Delta(M, L)=2$, we consider the following cases separately:

a) $d(M, L)=1$.

b) $d(M, L)>1$ and $\dim \mathrm{Bs}|L|=1$.

c) $d(M, L)>1$ and $\dim \mathrm{Bs}|L|\leqq 0$.

In case a), the precise structure of (M, L) is still a "mystery". Similarly as in (0.2), we can say that the scheme theoretic intersection C of general members $D_1, \cdots, D_{n-1}$ of $|L|$ is an irreducible reduced curve of arithmetic genus $g(M, L)$ with $LC=1$. But we do not know whether C is smooth or not.

The main purpose of this part I is the study of the case b).

(0.4) In view of $g(M, L)>1$ we divide the above case c) in the following subcases:

(c-i) $d(M, L)>4$.

(c-ii) $d(M, L)=4$.

(c-iii) $d(M, L)=2$ or 3.

(0.5) In case (c-i), we have $g(M, L)=2$ and L is simply generated (hence very ample) by [**F3**; Theorem 4.1, c)]. Moreover $H^i(M, tL)=0$ for any $0<i<n$ and any $t\in \boldsymbol{Z}$ by [**F6**; (3.8)]. Using this we infer $\mathrm{Bs}|K+(n-1)L|=\emptyset$ for the canonical bundle K of M by induction on n. This linear system gives the so-called adjunction mapping f. Since $g(M, L)=2$, f is a mapping onto $\boldsymbol{P}^1$. It turns out that $\mathcal{E}=f_*(\mathcal{O}_M[L])$ is a locally free sheaf of rank $n+1$ with $\deg(\det\mathcal{E})=d-3$, and M is a member of the linear system $|2H_\zeta+(6-d)H_\xi|$ on $P=\boldsymbol{P}(\mathcal{E})$, where H_ζ is the tautological line bundle of P and H_ξ is the pull-back of $\mathcal{O}_{\boldsymbol{P}^1}(1)$. Moreover L is the restriction of H_ζ to M. We list up below all such polarized manifolds (M, L). As for proofs of these facts, see [**F1**] or [**I**].

(I) The cases $n=2$.

(I-0) M is a blowing-up of $\boldsymbol{P}^1_\zeta\times\boldsymbol{P}^1_\xi$ with center being $12-d$ points. $L=2H_\zeta+3H_\xi-E$, where E is the sum of $12-d$ exceptional curves over these points.

(I-1) $M\cong\Sigma_1\cong\boldsymbol{P}(H_\beta\oplus\mathcal{O})$, a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1_\beta$, and $L=2H_\alpha+2H_\beta$, where H_α is the tautological line bundle. It is well-known that Σ_1 is a blowing-up of $\boldsymbol{P}^2$ with center being a point, and that the exceptional curve C is the unique section of $\Sigma_1\to\boldsymbol{P}^1_\beta$ with negative self-intersection number.

(I-1′) M is a blowing-up of Σ_1 with center being a point p lying on C, and $L=2H_\alpha+2H_\beta-E_p$, where E_p is the exceptional curve over p.

(I-2) $M\cong\Sigma_2\cong\boldsymbol{P}(2H_\beta\oplus\mathcal{O})$ and $L=2H_\alpha+H_\beta$, where H_α is the tautological line bundle.

(II) The cases $n=3$.

(II-1) $\mathcal{E}=\mathcal{O}(1, 1, 0, 0)$. This means that $\mathcal{E}$ is the direct sum of four line bundles over $\boldsymbol{P}^1$ of degrees 1, 1, 0, 0. So $d=5$.

(II-2) $\mathcal{E}=\mathcal{O}(1, 1, 1, 0)$. So $d=6$. M is a double covering of $\boldsymbol{P}^1\times\boldsymbol{P}^2$ with branch locus being a divisor of bidegree (2,2).

(II-3) $\mathcal{E}=\mathcal{O}(1, 1, 1, 1)$. $d=7$. M is a blowing-up of $\boldsymbol{P}^3$ with center being a complete intersection curve of type (2,2).

(II-4) $\mathcal{E}=\mathcal{O}(2, 1, 1, 1)$. $d=8$. M is a blowing-up of a hyperquadric with center being a smooth conic curve.

(II-5) $\mathcal{E}=\mathcal{O}(2, 2, 1, 1)$. $d=9$. $M\cong\boldsymbol{P}^1\times\Sigma_1$.

(III) The cases $n>3$. In this case we have:

$\mathcal{E}=\mathcal{O}(1, 1, 1, 1, 1)$ and (M, L) is the Segre product of $(\boldsymbol{P}^1, \mathcal{O}(1))$ and $(\boldsymbol{Q}^3, \mathcal{O}(1))$.

Here, given polarized manifolds (M_1, L_1) and (M_2, L_2), by Segre product we mean the polarized manifold $(M_1\times M_2, p_1^*L_1+p_2^*L_2)$, where p_i denotes the projection onto M_i.

(0.6) In case (c-ii), we have $\mathrm{Bs}|L|=\emptyset$ by [**F3**; Theorem 4.1, b]. In view of [**F9**; (1.4)], we infer that there are three possibilities.

(1) $g(M, L)=2$ and (M, L) is the normalization of a singular hypersurface of degree four. It turns out that this is possible only when $n<4$.

(2) $g(M, L)=3$ and (M, L) is a smooth hypersurface of degree four.

(3) (M, L) is hyperelliptic in the sense of [**F9**]. Namely, $\rho_{|L|}$ makes M a double covering of a hyperquadric W. In view of Tables I and II in [**F9**; p. 24], we infer that (M, L) is of type (II_a^n), $(\Sigma(1, 1)_{a,b}^+)$, $(\Sigma(1, 1)_b^0)$ or $({}^*\mathrm{II}_a)$ in the notation of [**F9**]. In particular W is non-singular if $n\geqq3$.

(0.7) In case (c-iii), there are various types which do not appear in case (c-i) and (c-ii). For details, see [**F1**] or forthcoming parts of this series of papers.

§1. The rational mapping defined by $|L|$.

(1.1) From now on, throughout in this part I, let (M, L) be a polarized manifold with $n=\dim M\geqq2$, $d(M, L)=d\geqq2$, $\Delta(M, L)=2$ and $\dim\mathrm{Bs}|L|=1$.

(1.2) Set $\Lambda=|L|$ and take a Hironaka model (M', Λ') of (M, Λ) as in [**F7**; (1.4)]. We shall freely use the notation in [**F7**; (1.6)].

(1.3) By [**F7**; (4.2) & (4.13)] we have $\dim W=n-1$, where W is the image of the rational mapping $M'\to\boldsymbol{P}^{n+d-3}$ defined by Λ'. Moreover, applying [**F7**; (3.6)], we obtain $w=\deg W=d-1$, $LX=1$ and $\Delta(W, H)=0$ where X is a general fiber of $\rho: M' \to W$.

(1.4) By [**F7**; (4.5)], $Y = \mathrm{Bs}\Lambda$ is an irreducible rational normal curve. Therefore, the first blowing-up $\pi_1: M_1 \to M$ of the sequence $M' = M_r \to M_{r-1} \to \cdots \to M_1 \to M$ may be assumed to be the blowing-up of Y. We claim $\mathrm{Bs}|\pi_1^* L - E_1| = \emptyset$, where E_1 is the exceptional divisor lying over Y.

To see this we use the induction on n. When $n=2$, we have $M_1 = M$, $E_1 = Y$ and $LE_1 = 1$ by [**F7**; (3.7)]. We have also $E_1 X = 1$ and $E_1(L - E_1) = wE_1 X = d-1$ by [**F7**; (3.10)]. So $(L-E_1)^2 = 0$. On the other hand, using [**F7**; (3.9) & (3.7)], we infer that $|L - E_1|$ has no fixed component. Combining them we obtain $\mathrm{Bs}|L - E_1| = \emptyset$.

When $n \geqq 3$, take a general member D of $|L|$ and let D_1 be the proper transform of D in M_1. Then D is non-singular by (0.2) and hence D_1 is the blowing-up of D with center Y. The restriction of $\Lambda_1 = |\pi_1^* L - E_1|$ to D_1 is a complete linear system by (0.2). So this has no base point by the induction hypothesis. Hence $\mathrm{Bs}|\Lambda_1| = \emptyset$ because $D_1 \in \Lambda_1$. This completes the proof of the claim.

(1.5) Thus we see that $\pi: M' \to M$ is the blowing-up with center $Y = \mathrm{Bs}\Lambda \cong \boldsymbol{P}^1$, $E = E_1$ and $\Lambda' = |\pi^* L - E|$. Since $XE = \pi^* L \cdot X = 1$ for any general fiber X of $\rho: M' \to W$, the restriction ρ_E of ρ to E is a birational morphism onto W. Moreover, ρ_E is the rational mapping defined by $|\rho_E^* H|$, or equivalently, the natural mapping $H^0(W, H) \to H^0(E, \rho_E^* H)$ is bijective. Indeed, the injectivity is obvious, while we have $h^0(E, H_E) = \dim E + d(E, H_E) = n + d - 2 = h^0(W, H)$ since E is a $\boldsymbol{P}^{n-2}$-bundle over $\boldsymbol{P}^1$.

(1.6) CLAIM. *ρ_E is an isomorphism.*

By the above observation, this is equivalent to saying that $\rho_E^* H$ is ample. When $n=2$, the claim is obvious.

(1.7) Here we prove (1.6) in case $n=3$ and $d \geqq 3$. If ρ_E is not an isomorphism, then W is a cone over a Veronese curve of degree $d-1$ since $\Delta(W, H) = 0$. Since E is a $\boldsymbol{P}^1$-bundle over $Y \cong \boldsymbol{P}^1$, we have $E \cong \boldsymbol{P}_Y(\mathcal{O}(d-1) \oplus \mathcal{O})$, the Hirzebruch surface Σ_{d-1}. The morphism ρ_E contracts the unique section C_∞ of $E \to Y$ with $C_\infty^2 = 1-d$ to a normal point v on W, and v is the vertex of the cone W. We will derive a contradiction from this.

For any point w on W other than v, the fiber $X_w = \rho^{-1}(w)$ is an irreducible reduced curve. Indeed, $t\pi^* L - E$ is ample on M' for $t \gg 0$. The restriction of this to X_w is $(t-1)E_{X_w}$, because $L = E$ in $\mathrm{Pic}(X_w)$. So the restriction of E to X_w is an ample divisor. On the other hand, $E \cap X_w$ is a point and $EX = 1$. Hence X_w must be an irreducible reduced curve.

For $y \in Y$, let E_y be the fiber of $E \to Y$ over y. Then $(d-1)E_y + C_\infty$ is a member of $|\rho_E^* H|$. Let H_y be the corresponding hyperplane section of W and set $D_y = \rho^* H_y$. D_y is an effective Cartier divisor on M' such that the restriction to E is $(d-1)E_y+C_\infty$. The prime decomposition of D_y is of the form $(d-1)F_y+Z_y$, where $\rho(F_y)=H_y$ and Z_y is the sum of components contained in $\rho^{-1}(v)$. If $Z_y=0$, then D_y is divisible by $d-1$ and hence so is the restriction of D_y to E. This contradicts the above observation. So $Z_y\neq 0$. Moreover, we see easily that the restrictions of F_y and Z_y to E are E_y and C_∞ respectively.

Thus, the scheme theoretical intersection $Z_y\cap E$ is the non-singular rational curve C_∞. On the other hand, this is an ample divisor on Z_y because $Z_y\subset\rho^{-1}(v)$ and $[E]=L$ in $\mathrm{Pic}(Z_y)$. Since Z_y is smooth along C_∞, Z_y is irreducible and has at most finitely many singular points. Since every 2-dimensional component of $\rho^{-1}(v)$ is a component of Z_y, there is only one such component. In particular, Z_y is independent of the choice of $y\in Y$. Anyway, $Z=Z_y$ is normal by Serre's criterion. Now, [**F2**; Theorem 2.1, d)] applies since $[E]_{C_\infty}=L_{C_\infty}=\mathcal{O}(1)$. Thus we infer $Z\cong \boldsymbol{P}^2$.

Now we claim that $F_y\cap F_{y'}\neq\emptyset$ for any $y\neq y'$ on Y. Indeed, both $F_y\cap Z$ and $F_{y'}\cap Z$ are non-trivial effective divisors on $Z\cong\boldsymbol{P}^2$ because $F_y\cap C_\infty\neq\emptyset$ and $F_{y'}\cap C_\infty\neq\emptyset$. So $F_y\cap F_{y'}\cap Z\neq\emptyset$.

Thus we see $\dim(F_y\cap F_{y'})\geqq 1$. It is also clear that $F_y\cap F_{y'}\subset\rho^{-1}(v)$. Hence $[E]$ is ample on $F_y\cap F_{y'}$. So $F_y\cap F_{y'}\cap E\neq\emptyset$. On the other hand we have $F_y\cap E=E_y$, $F_{y'}\cap E=E_{y'}$ and $E_y\cap E_{y'}=\emptyset$. This gives a contradiction, as desired.

(1.8) Assuming $d\geqq 3$, we will prove (1.6) by induction on n. We should consider the case $n>3$ here.

Let T be a general hyperplane section of W and let N be the corresponding member of $|L|$. Namely $N=\pi(N')$ for $N'=\rho^*T$. Then (N, L_N) is a polarized manifold with $\Delta=2$ of the type under consideration. Therefore, by the induction hypothesis, the restriction of ρ to $E_T=E\cap N'$ is an isomorphism onto T. Taking π_* of the exact sequence $0\to\mathcal{O}_E\to\mathcal{O}_E[H]\to\mathcal{O}_{E_T}[H]\to 0$, we get an exact sequence $0\to\mathcal{O}_Y\to\mathcal{E}\to\mathcal{F}\to 0$ of locally free sheaves on Y. $\mathcal{F}$ is ample by assumption. So, if this sequence does not split, then $\mathcal{E}$ is ample (cf., e.g., [**F4**; (4.16)]) and hence H_E is very ample. Therefore we may assume that the above exact sequence splits. In this case E has a section C_∞ such that $\rho(C_\infty)$ is a point v on W. Moreover W is the cone over T with vertex v. We will derive a contradiction from this.

Set $Z=\rho^{-1}(v)$. Then $E\cap Z=C_\infty$ is an ample divisor on Z. Hence $\dim Z\leqq 2$.

For any point y on Y, let T_y and E_y be the fibers of $E_T\to Y$ and $E\to Y$ respectively. Then $\rho(T_y)$ and $\rho(E_y)$ are linear subspaces in $\boldsymbol{P}^{n+d-3}\supset W$ and $\rho(E_y)$ is the linear span of $\rho(T_y)$ and v. Let F_y be the $(n-1)$-dimensional component of $\rho^{-1}(\rho(E_y))$. Clearly $F_y\cap E=E_y$ and $\rho^{-1}(\rho(E_y))=F_y\cup Z$. Moreover $Z_y=F_y\cap Z$ is a curve in Z.

Take another point y' on Y. Then $\rho(F_y\cap F_{y'})\subset\rho(E_y)\cap\rho(E_{y'})=v$. So

$F_y \cap F_{y'} \subset Z$. If $F_y \cap F_{y'} \neq \emptyset$, then $\dim(F_y \cap F_{y'} \cap E) \geqq n-3$ because E is ample on Z. But $F_y \cap F_{y'} \cap E = E_y \cap E_{y'} = \emptyset$. Hence $F_y \cap F_{y'} = \emptyset$, so $Z_y \cap Z_{y'} = \emptyset$. Thus Z contains a one-dimensional family of curves. So $\dim Z = 2$. Moreover, since $C_\infty \cong \boldsymbol{P}^1$ and $[E]_{C_\infty} = \mathcal{O}(1)$, the normalization $\tilde{Z}$ of Z is isomorphic to $\boldsymbol{P}^2$ by [**F2**; Theorem 2.1, d)]. But Z_y and $Z_{y'}$ are curves on Z disjoint with each other. This is impossible. Thus we get a contradiction.

(1.9) Now we consider the remaining case $d=2$. When $n=2$, we have $E(L-E)=1=LE$ and so $E^2=0$. Since $E=Y$ is a component of $\mathrm{Bs}|L|$, we have $h^0(M, E)=1$. Therefore E is a fiber of a ruling $\alpha: M \to A$ over an irrational curve A. $LF=LE=1$ for every fiber F of α. Hence α is a $\boldsymbol{P}^1$-bundle. Moreover $\rho: M \to W \cong \boldsymbol{P}^1$ is an isomorphism restricted to each fiber F. So $M \cong A \times W$ with α and ρ being the first and second projections respectively.

Next we consider the case $n=3$. For any general member S of $|L|$, (S, L_S) is a polarized surface of the above type. So $S \cong A \times \boldsymbol{P}^1$ for an irrational curve A. Using the Albanese mapping we can extend the morphism $S \to A$ to a morphism $\mu: M \to A$. Moreover, by [**F4**; (4.9)], μ is a $\boldsymbol{P}^2$-bundle. Y is a line in a fiber $F \cong \boldsymbol{P}^2$ of μ. Combining ρ and μ we get a birational morphism $M_1' \to A \times \boldsymbol{P}^2$. One easily sees that this is nothing but the contraction of the proper transform F' of F to a point p. Thus, from the converse view-point, M' is the blowing-up of $A \times \boldsymbol{P}^2$ at a point p. Now, let Z be the proper transform on M' of the fiber of $A \times \boldsymbol{P}^2 \to \boldsymbol{P}^2$ passing p. Then $E \cap Z = \emptyset$ and $H_Z = 0 = L_Z$. This is impossible because $tL-E$ is ample on M' for $t \gg 0$. Thus the case $n=3$ is ruled out.

In view of (0.1), we conclude $n=2$ if $d=2$. In particular, the claim (1.6) is true in this case too. Thus we have completed the proof of (1.6).

(1.10) For every fiber X of ρ, E_X is an ample divisor on X and E_X is a simple point. Hence X is an irreducible reduced curve. So ρ is a flat morphism. In particular every fiber is of the same arithmetic genus g.

(1.11) The sectional genus $g(M, L)$ of (M, L) is equal to $(d-1)g$. In order to see this, we take general members of $|L|$, use (0.1) and reduce the problem to the case $n=2$. When $n=2$, we have $E^2=E(L-H)=LE-(d-1)XE=2-d$ and $KE=-2-E^2=d-4$ for the canonical bundle K of $M=M'$. Using $KX=2g-2$ we get $KL=K((d-1)X+E)=2(d-1)(g-1)+d-4$ and $2g(M, L)-2=(K+L)L$ $=2(d-1)(g-1)+2d-4$. This gives $g(M, L)=(d-1)g$.

(1.12) We claim $g \geqq 1$. To prove this, we may assume $n=2$ as in (1.11). If $g=0$, $\rho: M \to W \cong \boldsymbol{P}^1$ is a $\boldsymbol{P}^1$-bundle. Then $H^1(M, L-E)=H^1(M, (d-1)X)=0$ and $H^0(M, L) \to H^0(E, L_E)$ is surjective. This contradicts $E \subset \mathrm{Bs}|L|$.

(1.13) Now we complete the proof of (0.2). We should consider the case $\dim \mathrm{Bs}|L|=1$ here. By (1.11), we infer $d=2$ from $g(M, L)=1$ and $d \geqq 2$. So the argument (1.9) proves (0.2).

(1.14) Summarizing the preceding arguments we obtain the following

THEOREM. *Let* (M, L) *be a polarized manifold with* $\dim M=n\geqq 2$, $d(M, L)=d\geqq 2$, $\Delta(M, L)=2$ *and* $\dim \mathrm{Bs}|L|=1$. *Then*

1) $Y=\mathrm{Bs}|L|$ *is an irreducible rational normal curve.*

2) *Let* $\pi: M'\to M$ *be the blowing-up of* Y *and let* E *be the exceptional divisor over* Y. *Then* $\mathrm{Bs}|\pi^*L-E|=\emptyset$.

3) *Let* W *be the image of the morphism* $M'\to \boldsymbol{P}^{n+d-3}$ *defined by* $|\pi^*L-E|$. *Then* $\dim W=n-1$, $\deg W=d-1$ *and* $\Delta(W, \mathcal{O}_W(1))=0$.

4) E *is a section of the morphism* $\rho: M'\to W$. *So* $E\cong W$ *and this is a* $\boldsymbol{P}^{n-2}$-*bundle over* Y.

5) ρ *is flat and every fiber of* ρ *is an irreducible reduced curve of arithmetic genus* $g\geqq 1$. *This number* g *is determined by the relation* $g(M, L)=(d-1)g$.

6) *If* $n\geqq 3$, (D, L_D) *is a polarized manifold of the above type for any general member* D *of* $|L|$.

7) *If* $d=2$, *then* $n=2$ *and* $M\cong A\times \boldsymbol{P}^1$ *for some curve* A *of genus* $g\geqq 1$. *Moreover* $L=E+X$ *where* E *(resp.* X*) is a fiber of the projection onto* A *(resp.* $\boldsymbol{P}^1$*).*

(1.15) COROLLARY. *There exists a morphism* $\psi: M\to Y\cong \boldsymbol{P}^1$ *such that* ψ_Y *is the identity and that* (M_y, L_y) *is a polarized manifold with* $d(M_y, L_y)=\Delta(M_y, L_y)=1$ *for any smooth fiber* $M_y=\psi^{-1}(y)$ *over* $y\in Y$. *Here* L_y *denotes the restriction of* L *to* M_y.

To see this, consider the morphism $M'\to W\cong E\to Y$. It is easy to see that this factors through M. So we have a morphism $\psi: M\to Y$. Comparing (1.14) and [**F5**; (13.7)], we infer $d(M_y, L_y)=\Delta(M_y, L_y)=1$ for any smooth fiber M_y.

(1.16) Here we consider the converse of (1.14).

Let W be a rational scroll in $\boldsymbol{P}^{n+d-3}$ with $\dim W=n-1$, $\deg W=d-1$ and $\Delta(W, H)=0$. So W is a $\boldsymbol{P}^{n-2}$-bundle over $Y\cong \boldsymbol{P}^1_\xi$. Suppose that we have a flat morphism $f: N'\to W$ such that every fiber of f is an irreducible reduced curve of arithmetic genus $g\geqq 1$. Suppose further that there is a section E of f with its normal bundle $[E]_E$ being $H_\xi-H$, where H_ξ is the pull-back of $\mathcal{O}_Y(1)$. Then, the restriction of $[E]$ to a fiber of $E\cong W\to Y\cong \boldsymbol{P}^1_\xi$ is $\mathcal{O}(-1)$ and hence E can be blown-down smoothly to Y. Let $\pi: N'\to N$ be the blowing-down morphism. From the converse view-point, N' is the blowing-up of N with center $Y\subset N$ and E is the exceptional divisor. We have a line bundle L on N such that $\pi^*L=f^*H+E$, because the restriction of f^*H+E to each fiber of $E\to Y$ is trivial. Then (N, L) is a polarized manifold with $d(N, L)=d$, $\Delta(N, L)=2$ and $\mathrm{Bs}|L|=Y$.

Indeed, the ampleness of L is proved similarly as in [**F5**; (13.7)]. Here the irreducibility of *every* fiber of f is essential. We have $L^n=L^{n-1}(E+H)=L^{n-1}H=\cdots=L^2H^{n-2}=LEH^{n-2}+EH^{n-1}=1+(d-1)=d$ in the Chow ring of N'. So $d(N, L)=d$. Since $g\geqq 1$, E is in the fixed part of $|f^*H+E|$ and we have

$h^0(N, L)=h^0(N', f^*H+E)=h^0(N', f^*H)=h^0(W, H)=n+d-2$. Hence $\Delta(N, L)=2$. Moreover $\mathrm{Bs}|f^*H+E|=E$ implies that $\mathrm{Bs}|L|=Y$.

(1.17) THEOREM. *Let things be as in* (1.14). *Then* $d>n$. *Moreover, if* $d=n$, *then the fibration* $\phi: M\to Y\cong \boldsymbol{P}_\eta^1$ *in* (1.15) *is trivial and* $(M_y, L_y)\cong(N, A)$ *for some fixed polarized manifold* (N, A) *with* $d(N, A)=\Delta(N, A)=1$, $g(N, A)=g$. *Thus* (M, L) *is the Segre product of* (N, A) *and* $(\boldsymbol{P}_\eta^1, H_\eta)$.

PROOF. $W\cong E$ is a $\boldsymbol{P}^{n-2}$-bundle over Y and $\mathcal{F}=\pi_*\mathcal{O}_E[H]$ is an ample locally free sheaf on Y. So $d-1=\deg W=\deg(\det\mathcal{F})\geqq\operatorname{rank}\mathcal{F}=n-1$, proving the inequality.

We prove the assertion for the case $d=n$ by induction on n. When $n=2$, (1.9) shows our assertion. So we consider the case in which $n\geqq 3$.

Since $\deg(\det\mathcal{F})=\operatorname{rank}\mathcal{F}$, we infer that $\mathcal{F}$ is a direct sum of H_η's. So W is a Segre variety $\cong\boldsymbol{P}_\eta^1\times\boldsymbol{P}_\xi^{n-2}$ and $H=H_\eta+H_\xi$. Let Z be a general member of $\rho^*|H_\xi|$. Then we have $E\cap Z\cong\boldsymbol{P}_\eta^1\times\boldsymbol{P}_\xi^{n-3}$, $\pi(E\cap Z)=Y$, $\pi(Z)$ (denoted by T in the sequel) is a non-singular member of $|L-\phi^*H_\eta|$ and $\pi_Z: Z\to T$ can be viewed as the blowing-up of the manifold T with center Y. Furthermore, in view of (1.16), we see that (T, L) is a polarized manifold of the type (1.14) such that $d(T, L)=d-1$. The rational scroll associated to (T, L) is identified with the member of $|H_\xi|$ on W corresponding to Z. Applying the induction hypothesis to (T, L), we see that the restriction of ϕ to T is a trivial fibration and $T\cong Y\times F$ for the fiber F. Note also that $[T]_T=[L-H_\eta]_T$ is the pull-back of an ample line bundle on F.

Now it follows that $H^1(T, [mT])=0$ and $\mathrm{Bs}|[mT]_T|=\emptyset$ for any $m\gg 0$. So the mapping $H^1(M, (m-1)T)\to H^1(M, mT)$ is surjective and $h^1(M, mT)$ is a non-increasing function in m. Hence we have an integer $m_0\gg 0$ such that $h^1(M, mT)=h^1(M, m_0T)$ for every $m\geqq m_0$. Then $H^0(M, mT)\to H^0(T, [mT]_T)$ is surjective for any $m>m_0$. This implies $\mathrm{Bs}|mT|=\emptyset$ for every $m\gg 0$.

Now, applying (A1) in the Appendix, we obtain a fibration $f: M\to N$ over a normal variety N together with an ample line bundle A on N such that $f^*A=[T]$. Define a morphism $\Psi: M\to Y\times N$ by $\Psi(x)=(\phi(x), f(x))$. Since $L=\Psi^*(H_\eta+A)$ is ample, Ψ is a finite morphism. Clearly $Y\times N$ is normal. We have $d=L^n=(\deg\Psi)\cdot(H_\eta+A)^n\{Y\times N\}=(\deg\Psi)\cdot n\cdot A^{n-1}\{N\}$. So the assumption $d=n$ implies $A^{n-1}\{N\}=\deg\Psi=1$. Thus Ψ is birational. Hence Ψ is an isomorphism by Zariski's Main Theorem.

The rest of our assertion is now obvious.

§2. The case of elliptic fibration.

(2.1) Let things be as in (1.14) and we assume $g=1$ in this section. By the method in [**F5**; §14], we study the structure of (M, L) in the following way.

(2.2) Set $\mathcal{D}=\mathcal{O}_{M'}[\pi^*2L]$ and $\mathcal{F}=\rho_*\mathcal{D}$. Then $\mathcal{F}$ is a locally free sheaf of rank two on W and the natural homomorphism $\rho^*\mathcal{F}\to\mathcal{D}$ is surjective. So we have a morphism $\beta: M'\to \boldsymbol{P}_W(\mathcal{F})=V$ such that $\beta^*\mathcal{O}_V(1)=\mathcal{D}$. Of course V is a $\boldsymbol{P}^1$-bundle over W and $S=\beta(E)$ is a section of $p: V\to W$. β is a finite double covering and hence $M'\cong R_B(V)$ in the notation in [**F9**] etc., where the branch locus B is a smooth divisor on V. Furthermore, S is a component of B and E is a component of the ramification locus of β.

(2.3) Let H_η denote the pull-back of $\mathcal{O}_Y(1)$ (recall that W is a $\boldsymbol{P}^{n-2}$-bundle over $Y\cong\boldsymbol{P}^1_\eta$) and set $H_\xi=H-H_\eta$. Then $\mathrm{Pic}(W)\cong\mathrm{Pic}(S)\cong\mathrm{Pic}(E)$ is generated by H_η and H_ξ. The normal bundle of E in M' is $[L-H]_E=-H_\xi$. Since $\beta^*S=2E$, the normal bundle of S in V is $-2H_\xi$. Taking p_* of the exact sequence $0\to\mathcal{O}_V[2H_\xi]\to\mathcal{O}_V[S+2H_\xi]\to\mathcal{O}_S\to 0$, we get an exact sequence $0\to\mathcal{O}_W[2H_\xi]\to\mathcal{E}\to\mathcal{O}_W\to 0$, where $\mathcal{E}$ is a locally free sheaf such that $V\cong\boldsymbol{P}(\mathcal{E})$.

If H_ζ is the tautological line bundle of $\boldsymbol{P}(\mathcal{E})$, we see $S\in|H_\zeta-2H_\xi|$ and $[H_\zeta]_S=\mathcal{O}_S$. Now, we have $H^1(W, 2H_\xi)=0$ since $H=H_\xi+H_\eta$ is ample on the rational scroll W. Hence the above exact sequence splits and $\mathcal{E}\cong[2H_\xi]\oplus\mathcal{O}_W$.

Write $B=S+B^*$. Since B is non-singular, we have $S\cap B^*=\emptyset$. We may set $[B^*]=zH_\zeta+xH_\xi+yH_\eta$, because $\mathrm{Pic}(V)$ is generated by H_ζ, H_ξ and H_η. Then $x=y=0$ because $[B^*]_S=0$. Moreover $z=3$ since the restriction of β over $w\in W$ is the rational mapping $X_w\to V_w\cong\boldsymbol{P}^1$ defined by $|2E|_{X_w}$, which is ramified over four points. Thus $B^*\in|3H_\zeta|$.

It is easy to see $\mathrm{Bs}|H_\zeta|=\emptyset$ on V, since $\mathcal{E}$ is generated by global sections. On the other hand, we have $H_\zeta^2H_\xi^{n-3}H_\eta\{V\}=c_1(\mathcal{E})H_\xi^{n-3}H_\eta\{W\}=2H_\xi^{n-2}H_\eta\{W\}=2$. Hence $\dim\rho_{|H_\zeta|}(V)\geqq 2$ and $H^1(V, -3H_\zeta)=0$ by Kodaira-Ramanujam's vanishing theorem. So B^* is connected.

(2.4) Summarizing we obtain the following

THEOREM. *Let (M, L) be a polarized manifold of the type* (1.14) *and suppose that $g=1$. Then M' is a finite double covering of a $\boldsymbol{P}^1$-bundle $V=\boldsymbol{P}_E(\mathcal{O}_E\oplus[2H_\xi])$ over $E\cong W$, where $H_\xi=H_E-L_E$. The image S of E by the morphism $\beta: M'\to V$ is the unique member of $|H_\zeta-2p^*H_\xi|$, where H_ζ is the tautological line bundle on V and p is the morphism $V\to E$. The branch locus B of β is of the form B^*+S, where B^* is a smooth connected member of $|3H_\zeta|$ and $B^*\cap S=\emptyset$.*

(2.5) For further study of such polarized manifolds, see §4.

§3. The case of hyperelliptic fibration.

(3.1) Let things be as in (1.14) and we assume $g\geqq 2$ in this section. Let ω be the dualizing sheaf of M' and set $\mathcal{F}_t=\rho_*(\omega^{\otimes t})$ for each positive integer t. Similarly as in [**F5**; §15], $\mathcal{F}_t$ is a locally free sheaf for each $t\geqq 1$ and the natural morphism $\rho^*\mathcal{F}_1\to\omega$ is surjective. So we have a morphism $\beta: M'\to\boldsymbol{P}(\mathcal{F}_1)$

such that the restriction β_w of β to each fiber $X_w=\rho^{-1}(w)$ over $w\in W$ is the canonical mapping of the curve X_w. Let V be the image of β.

(3.2) DEFINITION. We say that the fibration $\rho: M'\to W$ is *hyperelliptic* if any general fiber X_w of ρ is a hyperelliptic curve.

From now on, throughout in this part I, we assume that ρ is hyperelliptic. Then, by a similar reasoning as in [**F5**; §15], we infer that V is a $\boldsymbol{P}^1$-bundle over W and $\beta: M'\to V$ is a double branched covering. The branch locus B of β is a smooth divisor on V.

(3.3) Let i be the involution of M' such that $M'/i\cong V$. Then we have the following three possibilities:

a) $i(E)=E$.

b) $i(E)\cap E=\emptyset$.

c) $i(E)\neq E$ and $i(E)\cap E\neq\emptyset$.

In case a) (resp. b), c)), (M, L) is said to be of type $(-)$ (resp. (∞), $(+)$).

(3.4) REMARK. Let $\psi: M\to Y\cong\boldsymbol{P}^1$ be as in (1.15). Then ρ is hyperelliptic if and only if (M_y, L_y) is sectionally hyperelliptic in the sense of [**F5**; III] for any general point y on Y. In this case we will see that (M, L) is of type $(-)$ (resp. (∞), $(+)$) if and only if (M_y, L_y) is of type $(-)$ (resp. (∞), $(+)$).

This is almost clear by the definition of ψ. But we should prove that (M_y, L_y) is of type $(+)$ if (M, L) is of type $(+)$. See §6.

§4. Type $(-)$.

In this section we assume that $\rho: M'\to W$ is hyperelliptic and that (M, L) is of type $(-)$.

(4.1) Since $i(E)=E$, the restriction of i to E is the identity. So $S=\beta(E)$ is a component of the branch locus B of $\beta: M'\to V$. By a quite similar method as in (2.3), we obtain the following

THEOREM. *Let things be as in* (1.14) *and assume that* $\rho: M'\to W$ *is hyperelliptic and of type* $(-)$. *Then* M' *is a double branched covering of a* $\boldsymbol{P}^1$*-bundle* $V=\boldsymbol{P}(\mathcal{O}_W\oplus[2H_\xi]_W)$ *over* W, *where* H_ξ *denotes* $[\rho^*H-\pi^*L]_E\in\mathrm{Pic}(E)\cong\mathrm{Pic}(W)$. *The image* S *of* E *by* $\beta: M'\to V$ *is a section of* $p: V\to W$ *and is the unique member of* $|H_\zeta-2p^*H_\xi|$, *where* H_ζ *is the tautological line bundle on* V. *The branch locus* B *of* β *is of the form* $S+B^*$, *where* B^* *is a smooth connected member of* $|(2g+1)H_\zeta|$ *such that* $S\cap B^*=\emptyset$.

(4.2) Because of the similarity of this theorem and (2.4), the case $g=1$ may be regarded as a special case of type $(-)$. In particular, the following results in this section are valid in case $g=1$ too.

(4.3) Conversely, let $W\subset\boldsymbol{P}^{n+d-3}$ be a rational scroll with $\deg W=d-1$, $\dim W=n-1$, let $\pi: W\to Y\cong\boldsymbol{P}^1_Y$ be the $\boldsymbol{P}^{n-2}$-bundle morphism, let $H_\xi=H-\pi^*\mathcal{O}_Y(1)$, let V be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(\mathcal{O}_W \oplus [2H_\xi])$ over W with the tautological bundle H_ζ, let S be the unique member of $|H_\zeta - 2H_\xi|$ and let B^* be a smooth member of $|(2g+1)H_\zeta|$ with $g \geqq 1$. Then, taking a double covering $\beta: N' \to V$ with branch locus $B = S + B^*$, we obtain $\rho: N' \to W$ as in (1.16). So, by blowing-down $E = \beta^{-1}(S)$ to a smooth rational curve $\cong Y$, we get a polarized manifold (M, L) of the type (4.1).

Note that the isomorphism class of (M, L) depends only on the type of the rational scroll W and on the choice of B^*.

(4.4) For any fixed (n, d, g), all the polarized manifolds of the type (4.1) with $n = \dim M$, $d = d(M, L)$ and with g being the genus of general fibers of ρ (or equivalently, with $g(M, L) = (d-1)g$) are deformations of each other.

This is clear if the rational scroll W is the same. In general, we prove the assertion similarly as in [**F9**; (8.12)]. We sketch the outline of the proof.

Suppose that we have a family $\{\mathcal{E}_t\}$ of vector bundles on $Y \cong \boldsymbol{P}^1_\eta$ with $\mathrm{rank}(\mathcal{E}_t) = n-1$, $\deg(\det(\mathcal{E}_t)) = d-n$ parametrized by $t \in \boldsymbol{A}^1$. Assume that the tautological line bundle $(H_\xi)_t$ on $\boldsymbol{P}(\mathcal{E}_t) = W_t$ is semipositive (or equivalently, $H_\xi + H_\eta$ is ample on W_t) for every t. Set $V_t = \boldsymbol{P}(2H_\xi \oplus \mathcal{O}_{W_t})$. Then $\{V_t\}$ is a family of manifolds and $h^0(V_t, (2g+1)[H_\zeta]_t)$ does not depend on t, where $[H_\zeta]_t$ is the tautological line bundle on V_t. So, all the pairs consisting of V_t and a smooth member of $|(2g+1)[H_\zeta]_t|$ are parametrized by a connected (non-compact) manifold, which is fibered over $\boldsymbol{A}^1$. Performing the construction (4.3) simultaneously we get a family of polarized manifolds of the type (4.1). Thus we see that the deformation type of (M, L) depends only on the deformation type of W. On the other hand, rational scrolls of the same (n, d) are deformations of each other. Putting things together, we complete the proof.

(4.5) LEMMA. *Let (M, L) be as in* (4.1) *and suppose that $d > n$. Then there is a polarized manifold $(M^\#, L^\#)$ with* $\dim M^\# = n+1$ *of the type* (4.1) *such that, for any smooth member D of $|L^\#|$, $(D, L^\#_D)$ is a polarized deformation of (M, L).*

PROOF. Obvious by (4.3) and (4.4).

(4.6) PROPOSITION. *Let (M, L) be as in* (4.1) *and let $\phi: M \to Y \cong \boldsymbol{P}^1_\eta$ be as in* (1.15). *Then*

1) *$H^q(M, \mathcal{O}_M) = 0$ for any $0 < q < n$ unless $q+1 = n = d$.*
2) *M is simply connected if $d > 2$.*
3) *The canonical bundle K^M of M is $(2g-n)L + (d-2-2g)H_\eta$.*
4) Pic(M) *is generated by L and H_η if $d > 3$ and $n \geqq 3$.*

PROOF. 1). Similarly as in [**F9**], we have $h^q(M, \mathcal{O}_M) = h^q(M', \mathcal{O}) = h^q(V, -F)$, where $F = B/2 = (g+1)H_\zeta - H_\xi$. By Serre duality we have $h^q(V, -F) = h^{n-q}(V, K^V + F) = h^{n-q}(V, (g-1)H_\zeta - (n-2)H_\xi + (d-n-2)H_\eta) = h^{n-q}(W, S^{g-1}(2H_\xi \oplus \mathcal{O}_W) \otimes [-(n-2)H_\xi + (d-n-2)H_\eta])$. If this does not vanish, then $h^{n-q}(W, jH_\xi + (d-n-2)H_\eta) > 0$ for some $j \geqq 0$, because W is a $\boldsymbol{P}^{n-2}$-bundle over $\boldsymbol{P}^1_\eta$. This is possible only when $n-q=1$ and $d-n-2\leqq -2$ since H_ξ is semipositive. From this observation we deduce the assertion 1).

2). By virtue of (4.5) and Lefschetz Theorem, we may assume $n\geqq 3$. Let Σ be the singular locus of $\phi: M\to Y$ and set $U=Y-\Sigma$. Then $M_y=\phi^{-1}(y)$ is simply connected by [**F5**; (16.6; 6)] for every $y\in U$. Since $\phi_U: \phi^{-1}(U)\to U$ is topologically locally trivial, we infer $\pi_1(\phi^{-1}(U))\cong\pi_1(U)$. Then, by the technique in [**F8**; (4.19)], we obtain $\pi_1(M)=\{1\}$ because $L^{n-1}\{M_y\}=1$ implies that every fiber of ϕ is irreducible and reduced.

3). In general, for any locally free sheaf $\mathcal{F}$ of rank r over a manifold X, the canonical bundle K^P of $\boldsymbol{P}(\mathcal{F})=P$ is $K^X-H+\det\mathcal{F}$, where H is the tautological line bundle $\mathcal{O}_P(1)$. So we infer $K^W=-2H_\eta-(n-1)(H_\xi+H_\eta)+(d-1)H_\eta=-(n-1)H_\xi+(d-n-2)H_\eta$ and $K^V=-2H_\zeta-(n-3)H_\xi+(d-n-2)H_\eta$. Hence $K^{M'}=K^V+[B]/2=(g-1)H_\zeta-(n-2)H_\xi+(d-n-2)H_\eta$. On the other hand, we have $K^{M'}=K^M+(n-2)E$ while $L=E+H_\xi+H_\eta$ and $2E=[S]=H_\zeta-2H_\xi$ in $\mathrm{Pic}(M')$. So $K^M+(n-2)L=K^{M'}+(n-2)(H_\xi+H_\eta)=(g-1)H_\zeta+(d-4)H_\eta=(2g-2)L+(d-2-2g)H_\eta$. From this we get 3).

4). We have $h^1(M, \mathcal{O})=h^2(M, \mathcal{O})=0$ by 1). So $\mathrm{Pic}(M)\cong H^2(M;\boldsymbol{Z})$. Hence, by virtue of (4.5), we may assume $n\geqq 4$. Then, for any $F\in\mathrm{Pic}(M)$, the restriction of F to $M_y=\phi^{-1}(y)$ is mL_y for some integer m by [**F5**; (16.6, 5)]. Then $\mathcal{F}=\phi_*(\mathcal{O}_M[F-mL])$ is an invertible sheaf on Y and the natural homomorphism $\phi^*\mathcal{F}\to\mathcal{O}_M[F-mL]$ is an isomorphism. Therefore F is an integral combination of L and H_η.

REMARK. The conditions in 2) and 4) are best possible. Indeed, M is not simply connected if $d=n=2$. If $d=n=3$, M is isomorphic to $Y\times N$ for a surface N by (1.17). So 4) is not true in this case unless $\mathrm{Pic}(N)$ is generated by L_N.

(4.7) THEOREM. *Let (M, L) be a polarized manifold as in* (1.14). *Then the following conditions are equivalent to each other.*

a) *The fibration $\rho: M'\to W$ is hyperelliptic of type* $(-)$.

b) $\mathrm{Bs}|2L|=\emptyset$.

c) $h^0(M, 2L)\geqq n(n-1)/2+3d$.

PROOF. Note first that (W, H) is a rational scroll and hence $(W, H)\cong(\boldsymbol{P}(F), \mathcal{O}(1))$ for some ample vector bundle F on $\boldsymbol{P}^1_\eta$. So $h^0(W, 2H)=h^0(\boldsymbol{P}^1, \mathrm{S}^2F)=\mathrm{rank}(\mathrm{S}^2F)+c_1(\mathrm{S}^2F)=n(n-1)/2+3(d-1)$ since $\mathrm{rank}(F)=\dim W=n-1$ and $c_1(F)=\deg W=d-1$.

a)→c): By (4.1), we have $h^0(M, 2L)=h^0(M', 2L)=h^0(M', H_\zeta+2H_\eta)\geqq h^0(V, H_\zeta+2H_\eta)=h^0(W, 2H_\xi+2H_\eta)+h^0(W, 2H_\eta)=h^0(W, 2H)+3=n(n-1)/2+3d$.

c)→b): Since E is a section of $\rho': M'\to W$ and $g>0$, E must be a fixed component of $|2H+E|=|L+H|$ on M'. So $h^0(M', L+H)=h^0(M', 2H)=n(n-1)/2+3d-3$. In view of the exact sequence $0\to H^0(M', L+H)\to H^0(M', 2L)\to H^0(E, 2L_E)$ and the fact $L_E=H_\eta$, we infer that $H^0(M', 2L)\to H^0(E, 2L_E)$ is surjective. So $\mathrm{Bs}|2L|=\mathrm{Bs}|2L_E|=\emptyset$.

b)→a): For any general fiber X of ρ', we have $\mathrm{Bs}|2L_X|=\emptyset$. So X is a hyperelliptic curve. Moreover, since $L_X=E_X$, $E\cap X$ is a ramification point of the canonical mapping of X. So (M, L) is of type $(-)$ by the reasoning as in §2.

§5. Type (∞).

(5.1) Suppose that $\rho: M'\to W$ is hyperelliptic and that (M, L) is of type (∞). Since $E\cap i(E)=\emptyset$, both E and $i(E)$ do not meet the ramification locus of $\beta: M'\to V$. Therefore $S=\beta(E)=\beta(i(E))$ is isomorphic to E and gives a section of $p: V\to W$. Moreover the normal bundle of S is $[E]_E=L_E-H_E$. Set $H_\xi=[-S]_S \in \mathrm{Pic}(S)\cong\mathrm{Pic}(W)\cong\mathrm{Pic}(E)$.

Taking p_* of the exact sequence $0\to\mathcal{O}_V[p^*H_\xi]\to\mathcal{O}_V[S+p^*H_\xi]\to\mathcal{O}_S[S+H_\xi]\to 0$, we obtain $0\to\mathcal{O}_W[H_\xi]\to\mathcal{E}\to\mathcal{O}_W\to 0$, where $\mathcal{E}$ is a locally free sheaf on W such that $V\cong\boldsymbol{P}(\mathcal{E})$. Let H_ζ be the tautological line bundle on V. Then S is a member of $|H_\zeta-p^*H_\xi|$ and $[H_\zeta]_S=0$. Furthermore, since H_ξ is semipositive on the rational scroll W, we have $H^1(W, H_\xi)=0$. This implies $\mathcal{E}\cong\mathcal{O}_W[H_\xi]\oplus\mathcal{O}_W$.

Let B be the branch locus of β. We may set $[B]=zH_\zeta+xH_\xi+yH_\eta$ because $\mathrm{Pic}(V)$ is generated by H_ξ, H_η and H_ζ. Since $[B]_S=0$, we have $x=y=0$. Similarly as before, we have $z=2g+2$. Hence B is a non-singular member of $|(2g+2)H_\zeta|$. Moreover, similarly as in (2.3), we obtain $H^1(V, [-B])=0$ from $H_\zeta^2H_\xi^{n-3}H_\eta\{V\}=H_\xi^{n-2}H_\eta\{W\}=1$. So B is connected.

Thus we obtain the following

(5.2) THEOREM. *Let (M, L) be a polarized manifold as in* (1.14) *and suppose that $\rho: M'\to W$ is hyperelliptic and that (M, L) is of type (∞). Then M' is a double covering of a $\boldsymbol{P}^1$-bundle $V=\boldsymbol{P}(H_\xi\oplus\mathcal{O}_W)$ over W, where $H_\xi=H-H_\eta$. The image S of E via $\beta: M'\to V$ is a section of $p: V\to W$ and is the unique member of $|H_\zeta-p^*H_\xi|$, where H_ζ is the tautological line bundle on V. The branch locus B of β is a smooth connected member of $|(2g+2)H_\zeta|$ such that $S\cap B=\emptyset$.*

(5.3) COROLLARY. *Let (M, L) be as in* (5.2). *Then, any smooth fiber (M_y, L_y) of $\phi: M\to Y$ in* (1.15) *is a polarized manifold with $\Delta(M_y, L_y)=d(M_y, L_y)=1$, $g(M_y, L_y)=g$ which is sectionally hyperelliptic of type (∞) in the sense of* [**F5**; §17].

(5.4) REMARK. Let $\mathcal{F}$ be the locally free sheaf on $Y\cong\boldsymbol{P}_\eta^1$ such that $(\boldsymbol{P}(\mathcal{F}), \mathcal{O}(1))\cong(W, H_\xi)$. Then, V is the blowing-up of $V''=\boldsymbol{P}(\mathcal{F}\oplus\mathcal{O}_Y)$ with center C being the section corresponding to the quotient bundle $\mathcal{O}_Y$ of $\mathcal{F}\oplus\mathcal{O}_Y$. Moreover, the exceptional divisor of this blowing-up is S. The pull-back of $\mathcal{O}_{V''}(1)$ to V is H_ζ. So, by abuse of notation, $\mathcal{O}_{V''}(1)$ will be denoted by H_ζ. Note that B is mapped isomorphically onto a divisor B'' on V''. It is now easy to see that M is a blowing-up of the double covering M'' of V'' with branch locus B'', and the exceptional divisor of this blowing-up is $\pi(i(E))$. The structure of such a double covering $M''\to V''$ is studied in [**F9**; §5]. From these observations, we obtain, for example:

(5.5) COROLLARY. *M is simply connected* (cf. [**F9**; (5.17)]).

(5.6) Applying [**F5**; (17.14)] to (M_y, L_y) in (5.3), we infer $n-1\leqq g+1$. So $g\geqq n-2$ in case (5.2).

We will further analyse the case $g=n-2$ using the technique in [**F5**; §17]. B gives a section b of the bundle $\boldsymbol{P}((\mathrm{S}^{2g+2}\mathcal{E})^{\vee})$ over W. On the other hand, we have a natural morphism $\mu:\boldsymbol{P}((\mathrm{S}^{g+1}\mathcal{E})^{\vee})=G\to\boldsymbol{P}(\mathrm{S}^{2g+2}\mathcal{E}^{\vee})$ defined by square. Then we should have $b(W)\cap\mu(G)=\emptyset$ (compare [**F5**; (17.7)]).

By a similar calculation as in [**F5**; (17.9)], we infer $0=(2H_{\tau}+H_{\xi})\cdots(2H_{\tau}+(2g+2)H_{\xi})\{G\}$ for the tautological line bundle H_{τ} on G. This intersection number is equal to $d(W, H_{\xi})\cdot 2^{g+1}\cdot\prod_{t=0}^{g}(2t+1)$ as in [**F5**; (17.11)]. Hence $0=H_{\xi}^{n-1}\{W\}=d-n$. So (1.17) applies. Thus we obtain:

(5.7) COROLLARY. *Let things be as in* (5.2). *Then $n\leqq g+2$. Moreover, if the equality holds, then $d=n$ and M is a product of $\boldsymbol{P}^1_{\eta}$ and a polarized manifold of the type* [**F5**; §17].

(5.8) Conversely, suppose that we are given a rational scroll $W\subset\boldsymbol{P}^N$ with $n-1=\dim W$, $d-1=\deg W$. Set $H_{\xi}=H-H_{\eta}$, $V=\boldsymbol{P}(H_{\xi}\oplus\mathcal{O}_W)$ and let H_{ζ} be the tautological line bundle on V. Then a general member B of $|(2g+2)H_{\zeta}|$ is non-singular because $\mathrm{Bs}|H_{\zeta}|=\emptyset$. Moreover, if $g\geqq n-1$, we easily see $b(W)\cap\mu(G)=\emptyset$, where b, μ and G are as in (5.7). This implies that, on every fiber of $V\to W$, the restriction of B is not divisible by two as a divisor. So, if $\beta: M'\to V$ is the double covering with branch locus B, every fiber of $\rho: M'\to W$ is an irreducible reduced curve.

Let S be the unique member of $|H_{\zeta}-H_{\xi}|$ on V. Then S is a section of $p: V\to W$ and S can be blown-down with respect to the mapping $S\cong W\to\boldsymbol{P}^1_{\eta}$. Since $B\cap S=\emptyset$, $\beta^{-1}(S)$ consists of two connected components, each of which is isomorphic to S and can be blown-down to $\boldsymbol{P}^1$. So (1.16) applies and we get a polarized manifold (M, L) of the type (5.2) by blowing-down one of these two components of $\beta^{-1}(S)$.

(5.9) Similarly as in (4.4), we now see that polarized manifolds of the type (5.2) form a single deformation family for any fixed triple (n, d, g). Using this fact one can get an alternate proof of (5.5). Compare (4.7; 1).

§ 6. Type (+).

(6.1) Suppose that $\rho: M'\to W$ is hyperelliptic and that (M, L) is of type (+). Let $\beta: M'\to V$ and $p: V\to W$ be as in (3.2), and let B be the branch locus of the double covering β. The image $\beta(E)=S$ is a section of p. We have $S\cap B \neq \emptyset$ since $E\cap i(E)\neq\emptyset$. But $E\neq i(E)$. This is possible only when the restriction of the Cartier divisor B to S is divisible by two. So we set $B_S=2Z$. Then $[Z]_E=[i(E)]_E$. Hence the pull-back of the normal bundle $[S]_S$ of S in V to Pic(E) is equal to $[Z]+[E]_E$.

(6.2) When $n=2$, we have $W\cong \boldsymbol{P}^1_\eta$ and $M'\cong M$. Therefore, replacing the polarization suitably, M can be viewed as a hyperelliptic polarized surface in the sense of [**F9**]. Moreover, one easily sees that it is of type (Σ^+) or (Σ^-).

In fact, we actually find various polarized surfaces of this type.

(6.3) From now on, we consider the case $n\geqq 3$. First, by a similar argument as in [**F5**; (18.3)], we have $[S]_{Z'}=[B]_{Z'}$ for each prime component Z' of Z.

Suppose that Pic$(S)\cong$Pic$(W)\cong$Pic(E) is generated (after tensored by $\boldsymbol{Q}$) by the classes of components of Z. Then, by the above observation we infer $[S]=[B]=2[Z]$. Hence $[Z]=[E]$ by (6.1). But $0\leqq ZF=EF=-1$ for any general fiber F of $E\to Y$. This contradiction shows that Pic(S) is not generated by components of Z.

Suppose that Z has a component Z' which is a fiber of $S\to Y$. By the above observation we infer that Z has no horizontal component. Hence $[S]_{Z'}=[B]_{Z'}=[2Z]_{Z'}=0$. So the restriction of $Z+E$ to a fiber of $E\to Y$ is trivial by (6.1). This is impossible because E is exceptional.

Thus we see that Z has no vertical component with respect to $S\to Y$. So Z has a horizontal component. From this we infer that any general fiber of $\phi: M\to Y$ in (1.15) is a polarized manifold with $\Delta=d=1$, which is sectionally hyperelliptic of type $(+)$ in the sense of [**F5**; § 15]. In particular we have $n=3$ by [**F5**; (18.3)].

(6.4) Since $n=3$, $W\cong S\cong E$ is a $\boldsymbol{P}^1$-bundle over $Y\cong \boldsymbol{P}^1_\eta$. So we set $W\cong \boldsymbol{P}([kH_\eta]\oplus\mathcal{O})$ for some $k\geqq 0$, and let H_ξ be the tautological line bundle on it. Note that, if $k>0$, W has a unique section Y_∞ such that $Y_\infty^2=-k$ and $[H_\xi]_{Y_\infty}=0$. If $k=0$, then $W\cong \boldsymbol{P}^1_\xi\times\boldsymbol{P}^1_\eta$.

Set $[Z]_S=xH_\xi+yH_\eta$ and $[E]_E=-H_\xi+\alpha H_\eta$. Then $[B]_S=2xH_\xi+2yH_\eta$. Moreover, in view of the results in [**F5**; § 18], we infer $[S]_S=\sigma H_\eta$ for some σ. Then $[E+i^*(E)]=\beta^*[S]$ implies $x=1$ and $y+\alpha=\sigma$. From $x=1$ we infer that Z is a section of $S\to Y$ because Z has no vertical component. Furthermore, the relation $[S]_Z=[B]_Z$ gives $\sigma=2(k+2y)$. Hence $y+\alpha=2(k+2y)$, or equivalently, $2k+3y=\alpha$.

Recall that $H+E=L_E=H_\eta$. So $H_W=H_\xi-(\alpha-1)H_\eta$. As we have seen before, $H_W-H_\eta=H_\xi-\alpha H_\eta$ is semipositive. Hence $0\leqq(H_\xi-\alpha H_\eta)\{Z\}=k-\alpha+y=-k-2y$. When $k=0$, we obtain $y=0$ from this. When $k>0$, we obtain $y<0$, which implies $Z=Y_\infty$ because $ZY_\infty=y<0$. Therefore $y=-k$. In either case we have $y=-k$, and hence $\alpha=-k$, $\sigma=-2k$. So $d-1=H_W^2=k-2(\alpha-1)=3k+2$.

(6.5) Since $[S]_S=-2kH_\eta$, the exact sequence $0\to\mathcal{O}_V[2kH_\eta]\to\mathcal{O}_V[S+2kH_\eta]\to\mathcal{O}_S\to 0$ gives an exact sequence $0\to\mathcal{O}_W[2kH_\eta]\to\mathcal{E}\to\mathcal{O}_W\to 0$ on W, which splits because $H^1(W, 2kH_\eta)=0$. Hence $V\cong\boldsymbol{P}_W([2kH_\eta]\oplus\mathcal{O}_W)$. Moreover, letting H_ζ denote the tautological line bundle on it, we have $S\in|H_\zeta-2kH_\eta|$ and $[H_\zeta]_S=0$. Since $[B]_S=2Z$, it is now easy to see $[B]=(2g+2)H_\zeta+2H_\xi-2kH_\eta$ in $\mathrm{Pic}(V)$.

Combining these observations (6.3), (6.4) and (6.5), we obtain the following

(6.6) THEOREM. *Let (M, L) be a polarized manifold as in* (1.14) *and suppose that $\rho: M'\to W$ is hyperelliptic and that (M, L) is of type $(+)$ in the sense* (3.3). *Then $n=\dim M\leqq 3$. If $n=3$, one has $d=3(k+1)$ for some non-negative integer k. Moreover, in this case, we have $W\cong\boldsymbol{P}([kH_\eta]\oplus\mathcal{O})$, $V\cong\boldsymbol{P}_W([2kH_\eta]\oplus\mathcal{O}_W)$, $S=\beta(E)\in|H_\zeta-2kH_\eta|$ and $B\in|(2g+2)H_\zeta+2H_\xi-2kH_\eta|$, where H_ξ and H_ζ are tautological line bundles on W and V respectively.*

REMARK. V is isomorphic to a fiber product of W and $\boldsymbol{P}([2kH_\eta]\oplus\mathcal{O})$ over $\boldsymbol{P}^1_\eta$.

(6.7) COROLLARY. *In the above case $n=3$, M is simply connected, uniruled and $H^q(M, \mathcal{O}_M)=0$ for any $q>0$. Moreover $H^1(M, L)=0$ if $k>0$.*

PROOF. Any general fiber of $\psi: M\to Y$ is a rational surface by [**F5**; (18.12)]. So M is uniruled. Similarly as in (4.6; 2), we infer that M is simply connected. Moreover, using [**FR**; Proposition 6.7], we obtain $H^q(M, \mathcal{O}_M)=0$ for $q>0$. In order to show $H^1(M, L)=0$, we recall $H_W=H_\xi+(k+1)H_\eta$. So $h^1(M', H)=h^1(V, H)+h^1(V, H-(g+1)H_\zeta-H_\xi+kH_\eta)=h^1(V, -(g+1)H_\zeta+(2k+1)H_\eta)=h^1(\Sigma_{2k}, -(g+1)H_\zeta+(2k+1)H_\eta)=h^1(\Sigma_{2k}, (g-1)H_\zeta-3H_\eta)$, where $\Sigma_{2k}=\boldsymbol{P}([2kH_\eta]\oplus\mathcal{O}_Y)$ and H_ζ is the tautological line bundle on it. This is equal to 2 unless $k=0$. Now, using the exact sequence $0\to H^0(M', H)\to H^0(M', L)\to H^0(E, L_E)\to H^1(M', H)\to H^1(M', L)\to H^1(E, L_E)$ and $L_E=H_\eta$, we infer that $h^0(E, L_E)=2$, $h^1(E, L_E)=0$ and $h^1(M', L)=h^1(M', H)-h^0(E, L_E)=0$. This implies $h^1(M, L)=h^1(M', L)=0$.

REMARK. When $k=0$, (M, L) is the Segre product of $(\boldsymbol{P}^1, \mathcal{O}(1))$ and a polarized manifold (N, A) with $\Delta=d=1$ of the type [**F5**; §18]. See (1.17).

(6.8) Let things be as in (6.6). Then every fiber V_x of V over $x\in W$ meets B at some point with odd multiplicity. Indeed, otherwise, the fiber of $\rho: M'\to W$ over x would not be irreducible.

Conversely, given (g, k) with $g\geqq 2$ and $k\geqq 1$, let Y, W, V, S, H_η, H_ξ, H_ζ be as in (6.6). Then any general member B of $|(2g+2)H_\zeta+2H_\xi-2kH_\eta|$ is non-singular and satisfies the above condition. So, via the process (1.16), we can construct a polarized manifold (M, L) of the type (6.6).

Indeed, since $\mathrm{Bs}|B-S|=\emptyset$, the singular locus of B is contained in $B\cap S$. Next let $T=p^{-1}(Y_\infty)$, where Y_∞ is the unique member of $|H_\xi-kH_\eta|$ on W. Then $T\cong\Sigma_{2k}$ and $[B]_T=(2g+2)H_\zeta-2kH_\eta$. It is easy to see that $H^0(V, [B])\to H^0(T, [B]_T)$ is surjective. Therefore B_T is of the form S_T+B', B' being a member of $|(2g+1)H_\zeta|$. In particular B is non-singular along $\mathrm{Supp}(S_T)=S\cap T$. Since $\mathrm{Supp}(B\cap S)=S\cap T$, we conclude that B is non-singular.

Other assertions are easy to verify.

(6.9) COROLLARY. *For any fixed* (g, k), *polarized threefolds* (M, L) *of the type* (6.6) *form a single deformation family.*

§ 7. Deformations.

(7.1) By a deformation family of polarized manifolds over a complex manifold T we mean a proper smooth morphism $f: \mathcal{M}\to T$ together with an f-ample line bundle $\mathcal{L}$ on $\mathcal{M}$. Then (M_t, L_t) is a polarized manifold for every $t\in T$, where $M_t=f^{-1}(t)$ and L_t is the restriction of $\mathcal{L}$ to M_t. Each (M_t, L_t) is said to be a member of this family.

From now on, we usually consider the case in which T is the disk $\{z\in \boldsymbol{C}\,|\,|z|<\varepsilon\}$ with radius ε being a small positive number. (M_0, L_0) is called a special fiber of this family. We say that any general fiber has a property (♯) if there exists a positive number δ such that (M_t, L_t) has the property (♯) for every t with $0<|t|<\delta$. If so, we say that (M_0, L_0) is a specialization of polarized manifolds having the property (♯).

Given a polarized manifold (M, L), we say that any small deformation of (M, L) has the property (♯) if, for every deformation family of polarized manifolds over the disk T with special fiber being isomorphic to (M, L), any general fiber of this family has the property (♯).

(7.2) For any deformation family of polarized manifolds over the disk T as above, $d=d(M_t, L_t)$ is independent of t. So we have $\Delta(M_t, L_t)\geqq\Delta(M_0, L_0)$ for any general t by the upper-semicontinuity theorem. Moreover we have the following

(7.3) LEMMA. *If* $H^1(M_0, L_0)=0$, *then* $h^0(M_t, L_t)=h^0(M_0, L_0)$ *and* $\Delta(M_t, L_t)=\Delta(M_0, L_0)$ *for any general* t.

(7.4) LEMMA. *If* $\Delta(M_t, L_t)=\Delta(M_0, L_0)$ *for any general* t, *then* $\dim \mathrm{Bs}|L_t|\leqq\dim \mathrm{Bs}|L_0|$ *for any general* t.

PROOF. Since $h^0(M_t, L_t)$ is a constant function in t, $f_*\mathcal{L}$ is locally free at 0. Moreover, we have $\mathrm{Bs}|L_t|=M_t\cap\mathrm{Supp}(\mathrm{Coker}(f^*f_*\mathcal{L}\to\mathcal{L}))$. From this we obtain the inequality.

(7.5) THEOREM. *Suppose that there is a deformation family of polarized manifolds over the disk* T *and that* $\Delta(M_t, L_t)=2$ *for any general* t. *Then* $\Delta(M_0, L_0)=2$ *unless* $d(M_0, L_0)=1$.

PROOF. By (7.2) we have $\Delta(M_0, L_0)\leqq\Delta(M_t, L_t)=2$. If $\Delta(M_0, L_0)\leqq1$ and if $d(M_0, L_0)>1$, then $H^1(M_0, L_0)=0$ by [**F6**; (3.8)] and [**F9**; (3.1)]. This is impossible by (7.3).

(7.6) COROLLARY. *Suppose that* (M_t, L_t) *is of the type* (1.14) *for any general* t. *Then* (M_0, L_0) *is also of the type* (1.14).

For a proof, use (7.4).

REMARK. In this case, as a consequence, we see that $\{\mathrm{Bs}|L_t|\}$, $\{M'_t\}$, $\{E_t\}$ and $\{W_t\}$ become (smooth) deformation families of manifolds.

(7.7) THEOREM. *Suppose that* (M_t, L_t) *is of the type* (1.14) *and that* $\rho_t: M'_t \to W_t$ *is hyperelliptic in the sense* (3.2) *for any general* t. *Then* $\rho_0: M'_0 \to W_0$ *is also hyperelliptic.*

PROOF. Let M' and W be the total spaces of the deformation families $\{M'_t\}$ and $\{W_t\}$ respectively. Then the natural morphism $\rho: M' \to W$ is a fibration, whose general fibers are hyperelliptic curves. So every fiber of ρ is hyperelliptic. Hence ρ_0 is also hyperelliptic.

(7.8) THEOREM. *Let things be as in* (7.7). *Suppose that* (M_t, L_t) *is of the type* $(-)$ (*resp.* (∞), $(+)$) *for any general* t *and that* $n=\dim M_t \geqq 3$. *Then* (M_0, L_0) *is of the same type* $(-)$ (*resp.* (∞), $(+)$).

PROOF. V_t is a $\boldsymbol{P}^1$-bundle over W_t. So $\{V_t\}$ is a smooth family of manifolds. Moreover, $\{S_t\}$ gives a family of sections of $\{V_t \to W_t\}$. Comparing (4.1), (5.2) and (6.6), we infer that V_0 must be a $\boldsymbol{P}^1$-bundle of the same type as V_t. Hence (M_0, L_0) must be of the same type as (M_t, L_t).

(7.9) Thus, under certain mild conditions, we have seen that these types $(-)$, (∞), $(+)$ studied in this article are stable under smooth polarized specializations. We will next study small deformations.

(7.10) THEOREM. *Let* (M, L) *be a polarized manifold of the type* (4.1) *and suppose that* $d=d(M, L) \geqq 5$ *or* $n=\dim M \geqq 3$ *and* $d \geqq 4$. *Then any small deformation of* (M, L) *is of the same type* (4.1).

To prove this, we use the following

(7.11) LEMMA. *Let* (M, L) *be of the type* (4.1). *Then*

1) $H^1(M, L)=0$ *if* $d \geqq 3$.
2) $H^1(M, 2L)=0$ *either if* $d \geqq 5$ *or if* $n \geqq 3$.

PROOF. 1). The involution i of M' acts on the sheaf $\beta_*(\mathcal{O}_{M'}[-E])$. Considering the decomposition with respect to eigenvalues ± 1 of i, we see $\beta_*(\mathcal{O}_{M'}[-E]) \cong \mathcal{O}_V[-S] \oplus \mathcal{O}_V[-B/2] \cong \mathcal{O}_V[2H_\xi - H_\zeta] \oplus \mathcal{O}_V[-(g+1)H_\zeta + H_\xi]$. Since $L=2E+H_\xi+H_\eta-E=H_\zeta-H_\xi+H_\eta-E$, we have $h^1(M, L)=h^1(M', L)=h^1(V, H_\xi+H_\eta)+h^1(V, -gH_\zeta+H_\eta)$. Moreover $h^1(V, H_\xi+H_\eta)=h^1(W, H_\xi+H_\eta)=0$ and $h^1(V, -gH_\zeta+H_\eta)=h^{n-1}(V, (g-2)H_\zeta-(n-3)H_\xi+(d-n-3)H_\eta)=\sum_{j=0}^{g-2} h^{n-1}(W, (2j-n+3)H_\xi+(d-n-3)H_\eta)$. This is zero unless $n=2$. When $n=2$, we have $W \cong \boldsymbol{P}^1_\eta$ and $[H_\xi]_W=(d-2)H_\eta$. Then $\deg((2j-n+3)H_\xi+(d-n-3)H_\eta)=2j(d-2)+2d-7 \geqq -1$.

Thus in any case we have $h^1(M, L)=0$.

Next we prove 2). Similarly as above, we have $h^1(M, 2L)=h^1(M', 2L)$ $=h^1(V, H_\zeta+2H_\eta)+h^1(V, -gH_\zeta+H_\xi+2H_\eta)$. Clearly $h^1(V, H_\zeta+2H_\eta)=h^1(W, 2H_\xi+2H_\eta)$ $+h^1(W, 2H_\eta)=0$. By duality we have $h^1(V, -gH_\zeta+H_\xi+2H_\eta)=h^{n-1}(V, (g+2)H_\zeta$ $-(n-2)H_\xi+(d-n-4)H_\eta)$. If this is not zero, we have $n=2$ and $d-n-4\leqq-2$. This is impossible if $d\geqq 5$.

(7.12) PROOF OF (7.10). By (7.11; 1), we can apply (7.3) to infer $\Delta(M_t, L_t)$ $=2$ for any small deformation (M_t, L_t) of (M, L). Moreover, by (7.4), we have $\dim \mathrm{Bs}|L_t|\leqq 1$.

Assume that $\mathrm{Bs}|L_t|$ is a finite set. Then, if $d>4=2\Delta$, we have $g(M_t, L_t)$ $=2$ by [**F3**; Theorem 4.1, c)]. But we have $g(M, L)=(d-1)g\geqq d-1\geqq 4$ by (1.14; 5). This contradicts the deformation invariance of the sectional genus $g(M, L)$. We will derive a contradiction in case $d=4$, $n\geqq 3$ too. Indeed, we have $g(M_t, L_t)=g(M, L)\geqq d-1\geqq 3$ similarly as above. By (0.6), (M_t, L_t) is a smooth hypersurface of degree four or a double covering of a non-singular hyperquadric. Then $b_2(M_t)=1$ by Lefschetz theorem (cf. [**F9**; (3.11)]). On the other hand we have $b_2(M)\geqq 2$ by (4.1).

Thus, from this contradiction, we infer $\dim \mathrm{Bs}|L_t|=1$. So (M_t, L_t) is of the type (1.14). Moreover, by virtue of (7.11; 2), we infer $h^0(M_t, 2L_t)=h^0(M, 2L)$. So, by the criterion (4.7), (M_t, L_t) is of the type (4.1).

(7.13) THEOREM. *Suppose that (M, L) is a polarized manifold of the type* (5.2) *and that $n=\dim M\geqq 3$. Then any small deformation of (M, L) is of the same type* (5.2) *unless $n=d=3$.*

REMARK. When $n=d=3$, we have $M\cong N\times \boldsymbol{P}^1$ for a certain $K3$-surface N (cf. (1.17)).

PROOF OF (7.13). As we saw in (5.4), M is a blowing-up of M'', which is a double covering of a $\boldsymbol{P}^{n-1}$-bundle V'' over $\boldsymbol{P}^1_\eta$. By virtue of the theory of Kodaira [**K**; Theorem 5], any small deformation of M is a blowing-up of a small deformation of M''. Furthermore, by [**F9**; (7.12) & (7.13; 3)], the double covering structure of M'' is stable under small deformation except when $V''\cong \boldsymbol{P}^1_\eta\times \boldsymbol{P}^2_\xi$ and the branch locus of the mapping $M''\to V''$ is the pull-back of a hypersurface of degree 6 on $\boldsymbol{P}^2_\xi$. In this exceptional case M has the structure described above. Moreover $g=2$.

(7.14) THEOREM. *Suppose that (M, L) is a polarized manifold of the type* (6.6) *and that $n=3$, $k\geqq 1$. Then any small deformation of (M, L) is of the same type* (6.6).

PROOF. (7.3) applies by (6.7). We have $g(M, L)=(d-1)g=(3k+2)g\geqq 10$. Recalling (0.5), we infer $\dim \mathrm{Bs}|L_t|=1$ for any small deformation (M_t, L_t) of (M, L). So, by (1.14), we obtain a famiy $\{M'_t\}$ of deformations of M'. We should show that the double covering structure $M' \to V$ is stable under small deformation. Similarly as in [**F9**; (7.12)], it suffices to show $H^1(V, \Theta_V[-(g+1)H_\zeta - H_\xi + kH_\eta]) = 0$ where the notations are as in (6.6) and Θ_V denotes the sheaf of vector fields on V.

Using the exact sequence $0 \to [2H_\zeta - 2kH_\eta] \to \Theta_V \to p^*\Theta_W \to 0$, we get $h^1(\Theta_V[-(g+1)H_\zeta - H_\xi + kH_\eta]) \leqq h^1(V, p^*\Theta_W[-(g+1)H_\zeta - H_\xi + kH_\eta])$ $= h^0(W, R^1p_*(\mathcal{O}_V[-(g+1)H_\zeta]) \otimes \Theta_W[-H_\xi + kH_\eta])$ because $(g-1)H_\zeta + H_\xi + kH_\eta$ is very ample on V and hence $h^1(V, -(g-1)H_\zeta - H_\xi - kH_\eta) = 0$. By duality $R^1p_*(\mathcal{O}_V[-(g+1)H_\zeta])$ is the dual of $p_*(\omega_{V/W}[(g+1)H_\zeta]) = p_*(\mathcal{O}_V[(g-1)H_\zeta + 2kH_\eta])$ $\cong \bigoplus_{j=1}^{g} \mathcal{O}_W[2kjH_\eta]$. Hence it suffices to show $h^0(W, \Theta_W[-H_\xi - k(2j-1)H_\eta]) = 0$ for each $j = 1, \cdots, g$. We have an exact sequence $0 \to [2H_\xi - kH_\eta] \to \Theta_W \to [2H_\eta]$ $\to 0$ on W. Therefore $h^0(\Theta_W[-H_\xi - k(2j-1)H_\eta]) \leqq h^0(W, H_\xi - 2kH_\eta) = 0$. This completes the proof.

Appendix.

THEOREM (A1). *Let L be a line bundle on a variety V. Then the following conditions are equivalent to each other.*

a) *There is an integer m such that* $\mathrm{Bs}|tL| = \emptyset$ *for every* $t \geqq m$.

b) *There is a morphism $f: V \to W$ and an ample line bundle A on W such that $L = f^*A$.*

PROOF. Clearly b) implies a). So we show that a) implies b). For each t, let W_t be the image of the rational mapping ρ_t defined by $|tL|$. Let X be the image of the mapping $g: V \to W_m \times W_{m+1}$ given by ρ_m and ρ_{m+1}. Let $V \to W \to X$ be the Stein factorization of g. So, $f_*\mathcal{O}_V = \mathcal{O}_W$ for $f: V \to W$ and $\pi: W \to X$ is finite. Let H_m and H_{m+1} be pull-backs of hyperplane sections on W_m and W_{m+1} respectively and set $A = H_{m+1} - H_m$. We claim that (W, A) has the desired property b).

In fact, $f^*A = f^*H_{m+1} - f^*H_m = (m+1)L - mL = L$. Furthermore, by Lemma (A2) below, we have $mA = H_m$ and $H_{m+1} = (m+1)A$. Since π is finite, $H_m + H_{m+1}$ is ample on W. Hence so is A. Thus we prove the claim.

LEMMA (A2). *Let $f: V \to W$ be a morphism of schemes such that $f_*\mathcal{O}_V = \mathcal{O}_W$. Then $f^*: \mathrm{Pic}(W) \to \mathrm{Pic}(V)$ is injective.*

PROOF. Suppose that $f^*\mathcal{F} = \mathcal{O}_V$ for some $\mathcal{F} \in \mathrm{Pic}(W)$. Then the natural homomorphism $\mathcal{F} \to f_*f^*\mathcal{F}$ is an isomorphism. So $\mathcal{F} = \mathcal{O}_W$.

REMARK (A3). In case (A1), W can be taken to be normal if V is normal.

References

[F1] T. Fujita, On Δ-genera of polarized manifolds (in Japanese), Master Thesis, Univ. of Tokyo, 1974.

[F2] T. Fujita, On the structure of polarized varieties with Δ-genera zero, J. Fac. Sci. Univ. Tokyo, **22** (1975), 103-115.

[F3] T. Fujita, Defining equations for certain types of polarized varieties, in Complex Analysis and Algebraic Geometry (edited by Baily and Shioda), Iwanami, 1977, 165-173.

[F4] T. Fujita, On the hyperplane section principle of Lefschetz, J. Math. Soc. Japan, **32** (1980), 153-169.

[F5] T. Fujita, On the structure of polarized manifolds with total deficiency one, part I, II and III, J. Math. Soc. Japan, **32** (1980), 709-725 & **33** (1981), 415-434 & **36** (1984), 75-89.

[F6] T. Fujita, On polarized varieties of small Δ-genera, Tôhoku Math. J., **34** (1982), 319-341.

[F7] T. Fujita, Theorems of Bertini type for certain types of polarized manifolds, J. Math. Soc. Japan, **34** (1982), 709-718.

[F8] T. Fujita, On the topology of non-complete algebraic surfaces, J. Fac. Sci. Univ. Tokyo, **29** (1982), 503-566.

[F9] T. Fujita, On hyperelliptic polarized varieties, Tôhoku Math. J., **35** (1983), 1-44.

[FR] T. Fujita and J. Roberts, Varieties with small secant varieties: the extremal case, Amer. J. Math., **103** (1981), 953-976.

[I] P. Ionescu, Varieties with sectional genus 2, Preprint Series in Math., No. 110/1981, INCREST Bucharest.

[KO] S. Kobayashi and T. Ochiai, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ., **13** (1973), 31-47.

[K] K. Kodaira, On stability of compact submanifolds of complex manifolds, Amer. J. Math., **85** (1963), 79-94.

[R] C. P. Ramanujam, Remarks on the Kodaira vanishing theorem, J. Indian Math. Soc. (N. S.), **36** (1972), 41-51.

Takao FUJITA
Department of Mathematics
College of Arts and Sciences
University of Tokyo
Komaba, Meguro, Tokyo 153
Japan