J. Math. Soc. Japan
Vol. 36, No. 4, 1984

The normality of Σ-products and the perfect κ-normality of Cartesian products

By Yukinobu YAJIMA

(Received March 3, 1983)
(Revised Nov. 29, 1983)

§ 0. Introduction.

Corson [3] introduced the concept of Σ-products, which are quite important subspaces of Cartesian products of topological spaces. He studied there the normality of Σ-products. On the other hand, Blair [2], Ščepin [16] and Terada [19] independently introduced the concept of perfect κ-normality (or Oz) which is analogous to that of normality. The former two studied there when Cartesian products of topological spaces are perfectly κ-normal. In these connections, the following two results (I) and (II) seem to be most remarkable:

(I) A Σ-product of metric spaces is (collectionwise) normal.

(II) A Cartesian product of metric spaces is perfectly κ-normal.

The former was proved by Gul'ko [4] and Rudin [9]. The latter was given by Ščepin [16]. Subsequently, Kombarov [8] obtained a nice extension of (I) as follows:

(III) For a Σ-product Σ of paracompact p-spaces, (a) Σ is normal, (b) Σ is collectionwise normal and (c) Σ has countable tightness are equivalent.

As another generalized metric spaces, Okuyama [13] introduced the concept of σ-spaces. Subsequently, Nagami [11] introduced the class of Σ-spaces which contains both ones of σ-spaces and paracompact p-spaces. These generalized metric spaces play important roles in this paper.

Recently, the author [21] has proved that for a Σ-product Σ of paracompact Σ-spaces the implication (c)$\Rightarrow$(b) in (III) is true. The first purpose of this paper is to prove that for such a Σ-product Σ the implication (a)$\Leftrightarrow$(b) is true. We also discuss the countable paracompactness of Σ-products. The second purpose of this paper is to obtain an extension of (II) for a Cartesian product of paracompact σ-spaces, the form of which is resemble to that of (c)$\Rightarrow$(a) in (III). In process of proving this result, we consider the union of $\aleph_0$-cubes in a Cartesian product of σ-spaces. This is closely related to a certain question of R. Pol and E. Pol [14] though it has been already solved by Klebanov [5].

All spaces considered here are assumed to be Hausdorff. The letters n, i, j,

k and r denote non-negative integers. The letter $\mathfrak{m}$ denotes an infinite cardinal number. For a set Λ, the cardinality of Λ is denoted by $|\Lambda|$. For a subset T of a space S, the closure of T in S is denoted by $\mathrm{Cl}\,T$.

§ 1. Main theorems.

Let $X=\prod_{\lambda\in\Lambda}X_\lambda$ be a Cartesian product of spaces. Take a point $s=\{s_\lambda\}\in X$. For each $x=\{x_\lambda\}\in X$, let $\mathrm{Supp}(x)=\{\lambda\in\Lambda : x_\lambda\neq s_\lambda\}$. The subspace $\Sigma=\{x\in X : |\mathrm{Supp}(x)|\leqq\aleph_0\}$ of X is called a *Σ-product* [3] of spaces X_λ, $\lambda\in\Lambda$. Such an $s\in\Sigma$ is called the *base point* of Σ, which is often omitted. For a finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of Λ, the finite product $\sum_{i=1}^{n}X_{\lambda_i}$ is called a *finite subproduct* of X or Σ.

A space S is called a *Σ-space* [**11**] if there exists a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of locally finite closed covers of S such that each sequence $\{x_n\}_{n=1}^{\infty}$ of S, with $x_n\in\bigcap\{F : x\in F\in\mathcal{F}_n\}$ for each $n\geqq 1$ and some $x\in S$, has a cluster point.

A space S is called a *σ-space* [**13**] if it has a σ-locally finite closed net.

A space S has *tightness*$\leqq\mathfrak{m}$ if for any $T\subset S$ and $x\in\mathrm{Cl}\,T$ there exists some $A\subset T$ such that $|A|\leqq\mathfrak{m}$ and $x\in\mathrm{Cl}\,A$. In particular, we say that the space S has *countable tightness* if $\mathfrak{m}=\aleph_0$.

Normal spaces and collectionwise normal spaces are quite well-known. A space S is said to be *perfectly κ-normal* [**16**] (or Oz [**2**], [**19**]) if for each disjoint open sets V_1 and V_2 in S there exist disjoint cozero-sets U_1 and U_2 in S such that $V_k\subset U_k$ ($k=1, 2$).

The author [**21**] has proved the following theorem, which causes the motivations for our main theorems.

THEOREM 0. *Let Σ be a Σ-product of paracompact Σ-spaces. If (each finite subproduct of) Σ has countable tightness, then it is collectionwise normal.*

The first main theorem is

THEOREM 1. *Let Σ be a Σ-product of paracompact Σ-spaces. Then Σ is collectionwise normal if and only if it is normal.*

The proof is performed in § 2.

REMARK 1. There exists a non-normal Σ-product of compact spaces (cf. [**6**]). The normality of Σ-products of paracompact Σ-spaces does not imply that they have countable tightness. Because there are two Lašnev spaces S and T such that $S\times T$ has not countable tightness (cf. [**1**, p. 68]).

The second main theorem is

THEOREM 2. *Let X be a Cartesian product of paracompact σ-spaces. If each finite subproduct of X has countable tightness, then X is perfectly κ-normal.*

The proof is obtained in the final part of § 4. It may be interesting to compare the forms of Theorems 0 and 2.

REMARK 2. Since perfect κ-normality is hereditary with respect to dense subspaces (cf. [2], [19]), the "Cartesian product" in Theorem 2 can be replaced by the "Σ-product".

REMARK 3. Ščepin [17] introduced the concept of κ-metrizability, in terms of which he obtained an extension of his result (II) in the introduction. Of course, the form of it is quite different from that of Theorem 2.

§ 2. Proof of Theorem 1.

LEMMA 1 ([11]). *Let S be a (strong) Σ-space. Then there exists a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of locally finite closed covers of S, satisfying the following conditions:*

(1) $\mathcal{F}_n=\{F(\alpha_1\cdots\alpha_n):\alpha_1,\cdots,\alpha_n\in\Omega\}$ *for each* $n\geqq 1$.

(2) *Each* $F(\alpha_1\cdots\alpha_n)$ *is the sum of all* $F(\alpha_1\cdots\alpha_n\alpha_{n+1})$, $\alpha_{n+1}\in\Omega$.

(3) *For each* $x\in S$ *there exists a sequence* $\alpha_1,\alpha_2,\cdots\in\Omega$, *satisfying*

(i) $\bigcap_{n=1}^{\infty}F(\alpha_1\cdots\alpha_n)$ *contains x (and is compact),*

(ii) *if* $\{K_n\}_{n=1}^{\infty}$ *is a decreasing sequence of non-empty closed sets in S such that* $K_n\subset F(\alpha_1\cdots\alpha_n)$ *for each* $n\geqq 1$, *then* $\bigcap_{n=1}^{\infty}K_n\neq\emptyset$.

The above sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is called a *spectral (strong) Σ-net* of S. Moreover, we say that the above sequence $\{F(\alpha_1\cdots\alpha_n)\}_{n=1}^{\infty}$ in (3) is a *local Σ-net* of x. Note that paracompact Σ-spaces and σ-spaces are strong Σ-spaces and that the classes of paracompact Σ-spaces and strong Σ-spaces are countably productive (cf. [11]).

The idea of the proof of Theorem 1 is essentially due to that of Theorem 0. So we use again the following notations which have been used in [21].

Notations for Σ: Let Σ be a Σ-product of spaces X_λ, $\lambda\in\Lambda$. For the set Λ, let Λ_ω be the set of all non-empty countable subsets of Λ. Let Ξ be an index set such that $R_\xi\in\Lambda_\omega$ is assigned for each $\xi\in\Xi$. Then a countable subproduct $\prod_{\lambda\in R_\xi}X_\lambda$ of Σ is abbreviated by X_ξ and the projection of Σ onto X_ξ is denoted by p_ξ for each $\xi\in\Xi$. For a collection $\mathcal{A}$ of subsets of Σ, $\bigcup\mathcal{A}$ denotes $\bigcup\{A:A\in\mathcal{A}\}$.

Notations for a $n\times n$ matrix $\xi=(\alpha_{ij})_{i,j\leq n}$: The $k\times k$ matrix $(\alpha_{ij})_{i,j\leq k}$ is denoted by ξ_k for $1\leqq k\leqq n$. In particular, ξ_{n-1} is often abbreviated by ξ_- and ξ_0 implies the 0×0 matrix which is the empty matrix ($\emptyset$).

PROOF OF THEOREM 1. Let Σ be a Σ-product of paracompact Σ-spaces X_λ, $\lambda\in\Lambda$, with a base point $s\in\Sigma$. Assume that Σ is normal. Let $\mathcal{D}$ be a discrete collection of closed sets in Σ.

Now, for each $n\geqq 0$ we construct a collection $\mathcal{U}_n$ of open sets in Σ and an index set Ξ_n of $n\times n$ matrices such that $R_\xi\in\Lambda_\omega$, $\Omega(\xi)$, $E(\xi)\subset\Sigma$, $G(\xi)\subset\Sigma$, $\mathcal{D}(\xi)\subset\mathcal{D}$ and $\{x(\xi, D): D\in\mathcal{D}(\xi)\}\subset\Sigma$ are given for each $\xi\in\Xi_n$, satisfying the following conditions (1)-(7):

(1) Each $\mathcal{V}_n$ is locally finite in Σ such that for each $U\in\mathcal{V}_n$ $\mathrm{Cl}U$ intersects at most one member of $\mathcal{D}$.

(2) For each $\xi\in\Xi_n$, $\{F(\alpha_1\cdots\alpha_k): \alpha_1, \cdots, \alpha_k\in\Omega(\xi)\}$, $k\geqq 1$, is a spectral Σ-net of X_ξ.

(3) For each $\xi=(\alpha_{ij})_{i,j\leq n}\in\Xi_n$ and $1\leqq k\leqq n$, $\xi_{k-1}\in\Xi_{k-1}$ and $\alpha_{k1}, \cdots, \alpha_{kn}\in\Omega(\xi_{k-1})$.

(4) $\{G(\xi): \xi\in\Xi_n\}$ is a locally finite collection of open sets in Σ such that for each $\xi=(\alpha_{ij})_{i,j\leq n}\in\Xi_n$

$$E(\xi)=\bigcap_{i=1}^{n} p_{\xi_{i-1}}^{-1}(F(\alpha_{i1}\cdots\alpha_{in}))\subset G(\xi)$$

and $p_{\xi_-}^{-1}p_{\xi_-}(G(\xi))=G(\xi)$.

(5) Let $\mu=(\alpha_{ij})_{i,j\leq n-1}\in\Xi_{n-1}$, $\alpha_{in}\in\Omega(\mu_{i-1})$ and $\alpha_{nj}\in\Omega(\mu)$ for $1\leqq i, j\leqq n$. Then

$$\bigcap_{i=1}^{n} p_{\mu_{i-1}}^{-1}(F((\alpha_{i1}\cdots\alpha_{in}))\cap(\bigcup\mathcal{D})\not\subset\bigcup\mathcal{V}_n$$

implies $(\alpha_{ij})_{i,j\leq n}\in\Xi_n$.

(6) For each $\xi\in\Xi_n$, $n\geqq 1$, $\mathcal{D}(\xi)$ is an infinite countable subcollection of $\mathcal{D}$ with $x(\xi, D)\in E(\xi)\cap D$ for each $D\in\mathcal{D}(\xi)$.

(7) For each $\xi\in\Xi_n$, $n\geqq 1$,

$$R_\xi=R_{\xi_-}\cup\bigcup\{\mathrm{Supp}(x(\xi, D): D\in\mathcal{D}(\xi)\}\,.$$

Let $\Xi_0=\{\xi_0\}$ and $\mathcal{V}_0=\{\emptyset\}$. Let $E(\xi_0)=G(\xi_0)=\Sigma$. Take an arbitrary $R_{\xi_0}\in\Lambda_\omega$.

Assume that the above construction has been already performed for no greater than n. Take a $\xi\in\Xi_n$. Since $\xi_i\in\Xi_i$ and $\Omega(\xi_i)$ for $0\leqq i\leqq n$ have been already constructed, we set

$$\Xi(\xi)=\{\eta=(\alpha_{ij})_{i,j\leq n+1} : \eta_-=\xi,\ \alpha_{in+1}\in\Omega(\xi_{i-1}) \text{ and } \alpha_{n+1j}\in\Omega(\xi) \text{ for } 1\leqq i,\ j\leqq n+1\}.$$

Moreover, for each $\eta=(\alpha_{ij})_{i,j\leq n+1}\in\Xi(\xi)$ we set

$$E(\eta)=\bigcap_{i=1}^{n+1} p_{\xi_{i-1}}^{-1}(F(\alpha_{i1}\cdots\alpha_{in+1}))\,.$$

Then we have

$$p_\xi(E(\eta))=\bigcap_{i=1}^{n+1} p_\xi p_{\xi_{i-1}}^{-1}(F(\alpha_{i1}\cdots\alpha_{in+1}))\,.$$

By (2), $\{p_\xi(E(\eta)): \eta\in\Xi(\xi)\}$ is locally finite in X_ξ. Since X_ξ is paracompact and $E(\eta)\subset E(\xi)\subset G(\xi)$, there exists a locally finite collection $\{G(\eta): \eta\in\Xi(\xi)\}$ of open sets in Σ such that

$$E(\eta)\subset G(\eta)\subset G(\xi) \quad \text{and} \quad p_\xi^{-1}p_\xi(G(\eta))=G(\eta)$$

for each $\eta\in\Xi(\xi)$. We set

$$\Xi_+(\xi)=\{\eta\in\Xi(\xi): E(\eta) \text{ intersects at most finitely many members of } \mathcal{D}\}$$

and $\Xi_-(\xi)=\Xi(\xi)\setminus\Xi_+(\xi)$. Since Σ is normal, for each $\eta\in\Xi_+(\xi)$ there exists a finite collection $\mathcal{V}(\eta)$ of open sets in Σ such that

(i) for each $U\in\mathcal{V}(\eta)$, ClU intersects exactly one member of $\mathcal{D}$,

(ii) $E(\eta)\cap(\bigcup\mathcal{D})\subset\bigcup\mathcal{V}(\eta)$,

(iii) $\bigcup\mathcal{V}(\eta)\subset G(\eta)$.

Here, running $\xi\in\Xi_n$, we set

$$\mathcal{V}_{n+1}=\bigcup\{\mathcal{V}(\eta) : \eta\in\Xi_+(\xi) \text{ and } \xi\in\Xi_n\}$$

and $\Xi_{n+1}=\bigcup\{\Xi_-(\xi):\xi\in\Xi_n\}$. Then (1), (3), (4) and (5) are satisfied. By the choices of $\Xi_-(\xi)$ and Ξ_{n+1}, for each $\eta\in\Xi_{n+1}$ we can take some $\mathcal{D}(\eta)\subset\mathcal{D}$ and $\{x(\eta, D): D\in\mathcal{D}(\eta)\}$, satisfying (6). Moreover, we define $R_\eta\in\Lambda_\omega$ as it satisfies (7). Since X_η is a Σ-space, it follows from Lemma 1 that there exists a spectral Σ-net of X_η with an index set $\Omega(\eta)$, which satisfies (2). Thus, we have inductively accomplished the desired construction.

Set $\mathcal{V}=\bigcup_{n=1}^{\infty}\mathcal{V}_n$. Then, by (1), $\mathcal{V}$ is a σ-locally finite collection of open sets in Σ such that the closure of each member of $\mathcal{V}$ intersects at most one member of $\mathcal{D}$. In order to prove that Σ is collectionwise normal, it suffices to prove that $\mathcal{V}$ covers $\bigcup\mathcal{D}$. Assume the contrary and pick some $y\in\bigcup\mathcal{D}\setminus\bigcup\mathcal{V}$. By (2) and (5), we can inductively choose a sequence $(\alpha_{ij})_{i,j=1,2,\dots}$ such that for each $n\geqq1$ $\xi^n=(\alpha_{ij})_{i,j\leq n}\in\Xi_n$ and $\{F(\alpha_{n1}\cdots\alpha_{nk})\}_{k=1}^{\infty}$ is a local Σ-net of $p_{\xi^{n-1}}(y)$ in $X_{\xi^{n-1}}$, where $\alpha_{nk}\in\Omega(\xi^{n-1})$ and $\xi^0=(\emptyset)$. By (6), we can also choose a sequence $\{D_n\}_{n=1}^{\infty}$ of distinct members of $\mathcal{D}$ such that $D_n\in\mathcal{D}(\xi^n)$ for each $n\geqq1$. Let $x_n=x(\xi^n, D_n)$ for each $n\geqq1$. Moreover, for each n, k with $1\leqq n\leqq k$, we set $L_{nk}=\{p_{\xi^n}(x_i): i\geqq k\}$. Then we have

$$\mathrm{Cl}\,L_{nk}\subset F(\alpha_{n1}\cdots\alpha_{nk}) \quad \text{and} \quad \mathrm{Cl}\,L_{nk+1}\subset\mathrm{Cl}\,L_{nk}\,.$$

In the same way as the both proofs of [**6**, Theorem 1] and [**21**, Theorem 1], one can find a point x_∞ of Σ such that each basic open neighborhood of x_∞ in Σ contains infinitely many x_n's. This verification is a standard one. So the detail of it is left to the reader. Thus the infinite subcollection $\{D_n: n\geqq1\}$ of $\mathcal{D}$ is not discrete at x_∞ in Σ. This is a contradiction. The proof of Theorem 1 is complete.

Recall that a space S is said to be *collectionwise Hausdorff* if for each closed discrete set D in S there exists a disjoint collection $\{V_x: x\in D\}$ of open sets such that each V_x contains x.

THEOREM 3. *A Σ-product of paracompact Σ-spaces is collectionwise Hausdorff.*

In the proof of Theorem 1, we consider a discrete closed set D and the regularity of Σ instead of the above $\mathscr{D}$ and the normality of it, respectively. Then the proof of Theorem 3 is quite parallel to that of Theorem 1.

For a Cartesian product $X=\prod_{\lambda\in\Lambda}X_\lambda$ of spaces, the subspace $\Sigma_{\mathfrak{m}}=\{x\in X: |\mathrm{Supp}(x)|\leqq\mathfrak{m}\}$ is called a *$\Sigma_{\mathfrak{m}}$-product* [7] (with a base point $s\in\Sigma_{\mathfrak{m}}$). We can also obtain the following result which is more general than Theorem 1.

THEOREM 4. *Let $\Sigma_{\mathfrak{m}}$ be a $\Sigma_{\mathfrak{m}}$-product of paracompact Σ-spaces. Then $\Sigma_{\mathfrak{m}}$ is collectionwise normal if and only if it is normal.*

Using [12, Theorem 2.7], one will notice that the proof is also quite parallel to that of Theorem 1.

§3. The countable paracompactness of Σ-products.

Until now, the countable paracompactness of Σ-products has been hardly discussed. Because, as in [3], the normality of Σ-products often yields the countable paracompactness of them as a corollary. Here, for a Σ-product which may be non-normal, we consider when it is a P-space (in the sense of Morita [10]). In the sequel, such a Σ-product is countably paracompact if it is normal.

We use a certain characterization of P-spaces in [18]: A space S is called a *P-space* if for each finite decreasing sequence $\{K_1, \cdots, K_r\}$ of closed sets in S one can assign a closed set $\Phi(K_1, \cdots, K_r)$ in S, satisfying

(i) $\Phi(K_1, \cdots, K_r)\cap K_r=\emptyset$,

(ii) for each decreasing sequence $\{K_r\}_{r=1}^{\infty}$ of closed sets in S with $\bigcap_{r=1}^{\infty}K_r=\emptyset$, $\{\Phi(K_1, \cdots, K_r): r\geqq 1\}$ covers S.

THEOREM 5. *A Σ-product of strong Σ-spaces is a P-space.*

PROOF. Let Σ be a Σ-product of strong Σ-spaces X_λ, $\lambda\in\Lambda$, with a base point $s\in\Sigma$. We also use the notations in §2.

Let $\{K_1, \cdots, K_r\}$ be a finite decreasing sequence of closed sets in Σ. For each $0\leqq n\leqq r$, we construct two index sets Ξ_n and Ξ_n^* of $n\times n$ matrices with $\Xi_n^*\subset\Xi_n$ such that for each $\xi\in\Xi_n$ $E(\xi)\subset\Sigma$ is given and for each $\xi\in\Xi_n^*$ $R_\xi\doteqdot\Lambda_\omega$, $\Omega(\xi)$ and $x_\xi\in\Sigma$ are given, satisfying the following conditions (1)-(6):

(1) For each $\xi\in\Xi_n^*$, $\{F(\alpha_1\cdots\alpha_k): \alpha_1, \cdots, \alpha_k\in\Omega(\xi)\}$, $k\geqq 1$, is a spectral strong Σ-net of X_ξ.

(2) For each $\xi=(\alpha_{ij})_{i,j\leqq n}\in\Xi_n$ and $1\leqq k\leqq n$, $\xi_{k-1}\in\Xi_{k-1}^*$ and $\alpha_{k1}, \cdots, \alpha_{kn}\in\Omega(\xi_{k-1})$.

(3) For each $\xi=(\alpha_{ij})_{i,j\leqq n}\in\Xi_n$, $E(\xi)=\bigcap_{i=1}^{n}p_{\xi_{i-1}}^{-1}(F(\alpha_{i1}\cdots\alpha_{in}))$.

(4) If $\mu=(\alpha_{ij})_{i,j\leq n-1}\in\Xi^*_{n-1}$, $\alpha_{in}\in\Omega(\mu_{i-1})$ and $\alpha_{nj}\in\Omega(\mu)$ for $1\leqq i, j\leqq n$, then $\xi=(\alpha_{ij})_{i,j\leq n}\in\Xi_n$. If $\xi\in\Xi_n$ and $E(\xi)\cap K_n\neq\emptyset$, then $\xi\in\Xi^*_n$.

(5) For each $\xi\in\Xi^*_n$, $n\geqq 1$, $x_\xi\in E(\xi)\cap K_n$.

(6) For each $\xi\in\Xi^*_n$, $n\geqq 1$, $R_\xi=R_{\xi\_}\cup\mathrm{Supp}(x_\xi)$.

The above construction is rather easier than that of the proof of Theorem 1. So the detail is left to the reader.

Now, we set

$$\Phi(K_1, \cdots, K_r)=\bigcup\{E(\xi):\xi\in\Xi_n\setminus\Xi^*_n \text{ and } n\leqq r\}\ .$$

Since it follows from (1) and (3) that $\{E(\xi):\xi\in\Xi_n\}$ is locally finite in Σ, $\Phi(K_1, \cdots, K_r)$ is closed in Σ. Moreover, by (4), $\Phi(K_1, \cdots, K_r)$ is disjoint from K_r.

Let $\{K_r\}_{r=1}^\infty$ be a decreasing sequence of closed sets in Σ with the empty intersection. It suffices to show that $\{\Phi(K_1, \cdots, K_r)\}_{r=1}^\infty$ covers Σ. Assuming the contrary, pick some $y\in\Sigma\setminus\bigcup_{r=1}^\infty\Phi(K_1, \cdots, K_r)$. By (1) and (4), we can inductively choose a sequence $(\alpha_{ij})_{i,j=1,2,\dots}$ such that for each $n\geqq 1$ $\xi^n=(\alpha_{ij})_{i,j\leq n}\in\Xi^*_n$ and $\{F(\alpha_{n1}\cdots\alpha_{nk})\}_{k=1}^\infty$ is a local strong Σ-net of $p_{\xi^{n-1}}(y)$ in $X_{\xi^{n-1}}$, where $\alpha_{nk}\in\Omega(\xi^{n-1})$ and $\xi^0=(\emptyset)$. For each n, k with $1\leqq n\leqq k$, we set $L_{nk}=\{p_{\xi^n_-}(x_{\xi^i}):i\geqq k\}$. In the same way as the both proofs of [**6**, Theorem 1] and [**21**, Theorem 1], one can find a point $x_\infty\in\Sigma$ such that each basic open neighborhood of x_∞ in Σ intersects all K_r's. This implies $x_\infty\in\bigcap_{r=1}^\infty K_r$, which is a contradiction. The proof is complete.

Immediately, we have

COROLLARY 1. *A (normal) Σ-product of strong Σ-spaces is countably metacompact (paracompact).*

§4. Subsets of Cartesian products of σ-spaces.

Let $X=\prod_{\lambda\in\Lambda}X_\lambda$ be a Cartesian product of spaces. For a subset R of Λ, the subproduct $\prod_{\lambda\in R}X_\lambda$ of X is denoted by X_R and the projection of X onto X_R is denoted by p_R. A subset of the form $\prod_{\lambda\in\Lambda}K_\lambda$, where $K_\lambda\subset X_\lambda$ for each $\lambda\in\Lambda$, is called an $\mathfrak{m}$-*cube* in X if $|\{\lambda\in\Lambda:K_\lambda\neq X_\lambda\}|\leqq\mathfrak{m}$. In particular, we call it an $\aleph_0$-*cube* if $\mathfrak{m}=\aleph_0$.

R. Pol and E. Pol [**14**] raised the question of whether, for a Cartesian product of completely metric spaces, a closed union of $\aleph_0$-cubes in it is a G_δ-set. Recently, Klebanov [**5**] gave an affirmative answer to this question, showing that, for a Cartesian product of metric spaces, the closure of union of $\aleph_0$-cubes in it is a zero-set. Here, we prove the following result, which yields an extension of his one in the sequel (see our Theorem 2′ below).

THEOREM 6. *Let X be a Cartesian product of σ-spaces. Then a closed set in X is a G_δ-set if and only if it is a union of $\aleph_0$-cubes.*

PROOF. Let X be a Cartesian product of σ-spaces X_λ, $\lambda\in\Lambda$. Let K be a closed set in X which is a union of $\aleph_0$-cubes.

As before, let Λ_ω be the set of all non-empty countable subset of Λ. In the below, for an $R(F)\in\Lambda_\omega$, the subproduct of $X_{R(F)}$ of X and the projection $p_{R(F)}$ are abbreviated by X_F and p_F, respectively.

For each $n\geqq 0$, we construct two collections $\mathcal{F}_n$ and $\mathcal{F}_n^*$ of closed sets in X, a function ϕ of $\mathcal{F}_{n+1}$ into $\mathcal{F}_n^*$ and two functions x and R of $\mathcal{F}_n^*$ into X and Λ_ω, respectively, satisfying the following conditions (1)-(5) for each $n\geqq 0$:

(1) $\mathcal{F}_n$ is σ-locally finite in X, where $\mathcal{F}_0=\{X\}$.

(2) $\mathcal{F}_n^*=\{F\in\mathcal{F}_n : F\cap K\neq\emptyset\}$.

(3) For each $F\in\mathcal{F}_n$, $p_{F_-}(F)$ is a closed set in X_{F_-} and $p_{F_-}^{-1}p_{F_-}(F)=F$, where $F_-=\phi(F)$.

(4) For each $F\in\mathcal{F}_n^*$, $\{p_F(H) : H\in\mathcal{F}_{n+1}$ with $\phi(H)=F\}$ forms a closed net of the closed set $p_F(F)$ in X_F.

(5) For each $F\in\mathcal{F}_n^*$, $x(F)\in F\cap K$, $R(\phi(F))\subset R(F)$ and $p_F^{-1}p_F(x(F))\subset K$.

For the case of $n=0$, the construction is easily performed. Assume that the construction has been already performed for no greater than n. Fix an $F\in\mathcal{F}_n^*$ with $\phi(F)=E$. It should be noted by (3) that $p_E(F)$ is closed in X_E and $p_F(F)=p_Fp_E^{-1}p_E(F)$. Since X_F is a σ-space (cf. [**13**, Theorem 2.2]), so is $p_F(F)$. There exists a σ-locally finite closed net $\mathcal{N}_{n+1}(F)$ of $p_F(F)$. We set $\mathcal{F}_{n+1}(F)=\{p_F^{-1}(N) : N\in\mathcal{N}_{n+1}(F)\}$. Here, running $F\in\mathcal{F}_n^*$, we set $\mathcal{F}_{n+1}=\bigcup\{\mathcal{F}_{n+1}(F) : F\in\mathcal{F}_n^*\}$ and define the function ϕ of $\mathcal{F}_{n+1}$ into $\mathcal{F}_n^*$ as $\phi(\mathcal{F}_{n+1}(F))=\{F\}$ for each $F\in\mathcal{F}_n^*$. Moreover, $\mathcal{F}_{n+1}^*$ is defined as in (2). Then $\mathcal{F}_{n+1}$, $\mathcal{F}_{n+1}^*$ and ϕ satisfy (1)-(4). For each $H\in\mathcal{F}_{n+1}^*$, pick any point $x(H)$ of $H\cap K$. Since $x(H)$ is a point of some $\aleph_0$-cube contained in K, we can take some $R(H)\in\Lambda_\omega$ satisfying (5). Thus we have inductively accomplished the desired construction.

Now, we set $G=\bigcup\{F\in\mathcal{F}_n : F\cap K=\emptyset$ and $n\geqq 0\}$. It follows from (1) that G is an F_σ-set disjoint from K. Assume $G\neq X\setminus K$. Pick a point y of $X\setminus(G\cup K)$ and take a basic open neighborhood U of y in X, disjoint from K. Then we can inductively choose a sequence $\{F_n\}_{n=0}^\infty$ such that for each $n\geqq 0$

(i) $F_n\in\mathcal{F}_n^*$ with $\phi(F_n)=F_{n-1}$,

(ii) $p_{F_n}(y)\in p_{F_n}(F_{n+1})\subset p_{F_n}(U)$.

Indeed, assume that F_i, $i\leqq n$, have been already chosen. By (ii) and (3), we have $p_{F_n}(y)\in p_{F_n}(F_n)$. By (4), we can choose some $F_{n+1}\in\mathcal{F}_{n+1}$, satisfying (i) and (ii). Again by (3), we have $y\in F_{n+1}$. So $F_{n+1}\notin\mathcal{F}_{n+1}^*$ implies $y\in F_{n+1}\subset G$, which is a contradiction. Hence $F_{n+1}\in\mathcal{F}_{n+1}^*$.

We set $R_\infty=\bigcup_{n=0}^{\infty} R(F_n)$. Since $\{R(F_n)\}_{n=0}^\infty$ is non-decreasing, we can take some $k\geqq 1$ such that

$$p_{F_{k-1}}(U)\times\prod\{X_\lambda:\lambda\in R_\infty\setminus R(F_{k-1})\}\times p_{\Lambda\setminus R_\infty}(U)=U.$$

We take the point z of X defined by $p_{R_\infty}(z)=p_{R_\infty}(x_k)$, where $x_k=x(F_k)$, and $p_{\Lambda\setminus R_\infty}(z)=p_{\Lambda\setminus R_\infty}(y)$. Then we have $z\in U\subset X\setminus K$. On the other hand, by (5), we have

$$z\in p_{R_\infty}^{-1}p_{R_\infty}(x_k)\subset p_{F_k}^{-1}p_{F_k}(x_k)\subset K.$$

This is a contradiction. Hence K is a G_δ-set in X. Since the converse is obvious, the proof is complete.

Recall that a space S is said to be *perfect* if each closed set in S is a G_δ-set.

COROLLARY 2. *Let Y be a space which is a closed continuous image of a Cartesian product of σ-spaces. Then Y is perfect if and only if each point of Y is a G_δ-set.*

PROOF. Let X be a Cartesian product of σ-spaces and f a closed continuous map of X onto Y. Assume that each $y\in Y$ is a G_δ-set. Let F be a closed set in Y. Since $f^{-1}(F)$ is a closed set in X which is a union of G_δ-sets, it is a union of $\aleph_0$-cubes. It follows from Theorem 6 that $f^{-1}(F)$ is a G_δ-set. Since f is a closed map, F is also a G_δ-set.

Next, we show a generalization of [**14**, Corollary 2].

THEOREM 7. *Let X be a Cartesian product of spaces, each finite subproduct of which has tightness $\leqq\mathfrak{m}$. If Y is a union of $\mathfrak{m}$-cubes in X, then* ClY *is also a union of $\mathfrak{m}$-cubes.*

PROOF. Let $X=\prod_{\lambda\in\Lambda}X_\lambda$. Pick a point y of ClY. We construct two sequences $\{A_n\}_{n=0}^\infty$ and $\{R_n\}_{n=0}^\infty$ of subsets of Y and Λ, respectively, satisfying for each $n\geqq 0$

(1) $|A_n|\leqq\mathfrak{m}$, $|R_n|\leqq\mathfrak{m}$,

(2) $p_{R_{n-1}}(y)\in \mathrm{Cl}\, p_{R_{n-1}}(A_n)$,

(3) $p_{R_n}^{-1}p_{R_n}(A_n)\subset Y$ and $R_n\subset R_{n+1}$.

Assume that the construction has been already performed for no greater than n. It follows from [**9**, Remark 3] and (1) that X_{R_n} has tightness $\leqq\mathfrak{m}$. Since $p_{R_n}(y)\in\mathrm{Cl}\,p_{R_n}(Y)$, we can take some $A_{n+1}\subset Y$ satisfying (1) and (2). For each $a\in A_{n+1}$, there exists some $R_a\subset\Lambda$ such that $|R_a|\leqq\mathfrak{m}$ and $p_{R_a}^{-1}p_{R_a}(a)\subset Y$. Here, we set $R_{n+1}=\bigcup\{R_a:a\in A_{n+1}\}\cup R_n$. Then R_{n+1} and A_{n+1} satisfy (1) and (3). The construction has been accomplished.

Now, we set $R=\bigcup_{n=0}^\infty R_n$. Then $|R|\leqq\mathfrak{m}$ is clear. We show that the $\mathfrak{m}$-cube $p_R^{-1}p_R(y)$ is contained in ClY. Pick any point x of $p_R^{-1}p_R(y)$ and take any basic open neighborhood U of x in X. We can take some $k\geqq 1$ such that

$$p_{R_k}(U)\times\prod\{X_\lambda:\lambda\in R\setminus R_k\}\times p_{\Lambda\setminus R}(U)=U.$$

Since $p_{R_k}(y)=p_{R_k}(x)\in p_{R_k}(U)$, by (2), there exists some $a\in R_{k+1}$ such that $p_{R_k}(a)\in p_{R_k}(U)$. So we take the point z of X defined by $p_R(z)=p_R(a)$ and $p_{\Lambda\setminus R}(z)=p_{\Lambda\setminus R}(x)$. Then we have $z\in U$. On the other hand, by (3), we have $z\in Y$. Hence U intersects Y, which implies $x\in \mathrm{Cl}\,Y$. The proof is complete.

In the case of $\mathfrak{m}=\aleph_0$, we have

COROLLARY 3. *Let X be a Cartesian product of spaces, each finite subproduct of which has countable tightness. Then the closure of a union of $\aleph_0$-cubes in X is also a union of $\aleph_0$-cubes.*

Let's complete the proof of Theorem 2 in § 1. Note that a perfectly κ-normal space is equivalently a space whose regular closed sets are always zero-sets (cf. [**2**], [**16**]). So, in order to prove Theorem 2, it suffices to show the following result, which is barely more general than it.

THEOREM 2′. *Let X be a Cartesian product of paracompact σ-spaces. If each finite subproduct of X has countable tightness, then the closure of a union of $\aleph_0$-cubes in X is a zero-set.*

PROOF. Let $X=\prod_{\lambda\in\Lambda}X_\lambda$. Let Σ be a Σ-product of the spaces X_λ, $\lambda\in\Lambda$. It follows from Theorem 0 and [**20**, Theorem 1] that Σ is normal and C-embedded. Let F be the closure of a union of $\aleph_0$-cubes in X. By Corollary 3, F is a closed union of $\aleph_0$-cubes. Moreover, by Theorem 6, F is a G_δ-set. Hence it follows from [**14**, Proposition 2] that F is a zero-set. The proof is complete.

References

[1] A. V. Arhangel'skiĭ, Structure and classification of topological spaces and cardinal invariants, Russian Math. Surveys, **33** (1978), 33-96.

[2] R. L. Blair, Spaces in which special sets are z-embedded, Canad. J. Math., **28** (1976), 673-690.

[3] H. H. Corson, Normality in subsets of product spaces, Amer. J. Math., **81** (1959), 785-796.

[4] S. P. Gul'ko, On the properties of subsets of Σ-products, Soviet Math. Dokl., **18** (1977), 1438-1442.

[5] B. S. Klebanov, Remarks on subsets of Cartesian products of metric spaces, Comment. Math. Univ. Carolinae, **23** (1982), 767-784.

[6] A. P. Kombarov, On Σ-products of topological spaces, Soviet Math. Dokl., **12** (1971), 1101-1104.

[7] A. P. Kombarov, On the normality of $\Sigma_{\mathfrak{m}}$-products, Soviet Math. Dokl., **14** (1973), 1050-1054.

[8] A. P. Kombarov, On tightness and normality of Σ-products, Soviet Math. Dokl., **19** (1978), 403-407.

[9] V. I. Malyhin, On tightness and Souslin number in $\exp X$ and in a product of spaces, Soviet Math. Dokl., **13** (1972), 496-499.

[10] K. Morita, Products of normal spaces with metric spaces, Math. Ann., **154** (1964),

365-382.
[11] K. Nagami, Σ-spaces, Fund. Math., **65** (1969), 169-192.
[12] K. Nagami, Countable paracompactness of inverse limits and products, Fund. Math., **73** (1972), 261-270.
[13] A. Okuyama, Some generalizations of metric spaces, their metrization theorems and product spaces, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A, **9** (1967), 236-254.
[14] R. Pol and E. Pol, Remarks on Cartesian products, Fund. Math., **93** (1976), 57-69.
[15] M. E. Rudin, Book Review, Bull. Amer. Math. Soc., **84** (1978), 271-272.
[16] V. E. Ščepin, On topological products, groups and a new class of spaces more general than metric spaces, Soviet Math. Dokl., **17** (1976), 152-155.
[17] V. E. Ščepin, Topology of limit spaces of uncountable inverse spectra, Russian Math. Surveys, **31** (1976), 155-191.
[18] R. Telgársky, A characterization of P-spaces, Proc. Japan Acad., **51** (1975), 802-807.
[19] T. Terada, Note on z-, C^*-, and C-embedding, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A, **13** (1975), 129-132.
[20] M. G. Tkačenko, The notion of o-tightness and C-embedded subspaces of products, Topology and Appl., **15** (1983), 93-98.
[21] Y. Yajima, On Σ-products of Σ-spaces, Fund. Math., to appear.

Yukinobu YAJIMA
Department of Mathematics
Kanagawa University
Yokohama 221, Japan