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An approximate formula for the Riemann zeta function

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1. Introduction.

The approximate functional equation is one of the most powerful formulae for the Riemann zeta function $\zeta(s)$. Although the O-term is complicated, it can be replaced, due to C. L. Siegel [1], by an asymptotic series. We may write this as follows. Let $s=\sigma+it$ $(t>0)$, let $m=\left[\sqrt{\frac{t}{2\pi}}\right]$, and let a_n be the coefficients of Taylor's expansion of the function

$$\phi(z)=\exp\left((s-1)\log\left(1+\frac{z}{\sqrt{t}}\right)-i\sqrt{t}\,z+\frac{i}{2}z^2\right)$$

at the point $z=0$, i.e. $\phi(z)=\sum_{n=0}^{\infty} a_n z^n$. Put

$$\Psi(z)=\frac{\cos\pi\left(\frac{z^2}{2}-z-\frac{1}{8}\right)}{\cos\pi z},$$

and define, for every positive integer N,

$$S_N=\sum_{n=0}^{N-1}\sum_{k=0}^{[n/2]}\frac{n!\,i^{k-n}}{k!(n-2k)!2^n}\left(\frac{2}{\pi}\right)^{n/2-k}a_n\Psi^{(n-2k)}\left(\sqrt{\frac{2t}{\pi}}-2m\right).$$

If $0\leqq\sigma\leqq1$, and $N<At$, where A is a sufficiently small constant, then

$$\zeta(s)=\sum_{n=1}^{m}n^{-s}+\frac{(2\pi)^s}{\pi}\sin\frac{\pi s}{2}\Gamma(1-s)\sum_{n=1}^{m}n^{s-1}+(-1)^{m-1}e^{-i\pi(s-1)/2}(2\pi t)^{(s-1)/2}$$
$$\times e^{-it/2-\pi i/8}\Gamma(1-s)\left(S_N+O\left(\left(\frac{AN}{t}\right)^{N/6}\right)+O(e^{-At})\right).$$

The purpose of this paper is to construct some other asymptotic series of $\zeta(s)$. Our theorem is the same as Siegel's formula except for the sum S_N, the O-terms, and the conditions. In particular our sum S_N is defined as a function of $2g$ variables $\rho_1, \rho_2, \cdots, \rho_g, n_1, n_2, \cdots, n_g$, so that if we take suitable $2g$

variables, then we obtain some new asymptotic series of $\zeta(s)$ (see the corollaries). A key idea in our proof is the fact that $\phi(z)$ can be approximated by the sum of exponential functions $\sum_{k=1}^{g} P_k(z)\exp(\rho_k t^{-r}z)$.

Let r be a real number not less than $\frac{1}{2}$, let g be an integer not less than 2, let $\rho_1, \rho_2, \cdots, \rho_g$ be mutually distinct complex numbers, and let $n_1, n_2, \cdots, n_g$ be nonnegative integers. Put

$$N=\sum_{k=1}^{g}(n_k+1),$$

$$M=\max(1, |\rho_1|, |\rho_2|, \cdots, |\rho_g|),$$

$$B=\min_{1\leq k<h\leq g}|\rho_k-\rho_h|,$$

and

$$b_{kh}=\frac{1}{f_k(\rho_k)}\sum_{u=0}^{h}\sum_{j=0}^{N-n_k-1}\frac{(h-u+j)!}{u!(h-u)!}(-\rho_k)^u f_{kj}a_{h-u+j}t^{r(j-u)}$$
$$(1\leqq k\leqq g,\quad 0\leqq h\leqq n_k$$

where $f_k(z)$ and f_{kj} are defined by the relation

$$f_k(z)=\prod_{\substack{q=1\\ q\neq k}}^{g}(z-\rho_q)^{n_q+1}=\sum_{j=0}^{N-n_k-1}f_{kj}z^j \qquad (1\leqq k\leqq g).$$

Let S_N denote the sum

$$\sum_{k=1}^{g}\sum_{h=0}^{n_k}\sum_{q=0}^{[h/2]}\sum_{j=0}^{h-2q}\frac{h!\,i^{q-h}}{q!\,j!\,(h-j-2q)!\,2^h}\left(\frac{2}{\pi}\right)^{j/2}b_{kh}(\rho_k t^{-r})^{h-j-2q}$$
$$\times\exp\left(-\frac{i}{4}\rho_k^2 t^{-2r}\right)\Psi^{(j)}\left(\sqrt{\frac{2t}{\pi}}-2m-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}\right),$$

then we have the following theorem.

THEOREM. *If $0\leqq\sigma\leqq 1$, and $N+M^2<At$, where A is a sufficiently small constant, then*

$$\zeta(s)=\sum_{n=1}^{m}n^{-s}+\frac{(2\pi)^s}{\pi}\sin\frac{\pi s}{2}\Gamma(1-s)\sum_{n=1}^{m}n^{s-1}+(-1)^{m-1}e^{-i\pi(s-1)/2}(2\pi t)^{(s-1)/2}$$
$$\times e^{-it/2-\pi i/8}\Gamma(1-s)\left(S_N+O\left(e^M\left(\frac{N^{13}M^6}{B^6 t}\right)^{N/6}\right)+O\left(\left(\frac{M}{B}\right)^N(19rN)^{3rN}e^{-At}\right)\right).$$

If we take $g=N$, $n_k=0$ $(1\leqq k\leqq g)$, and $\rho_k=ik$, then we have $B=1$, $N=M$, and

$$f_{kj}=(-1)^{N-1}i^{N-1+j}c_{kj} \qquad (1\leqq k\leqq g,\quad 0\leqq j\leqq N-1),$$

where c_{kj} are the positive integers defined by

$$\prod_{\substack{q=1\\q\neq k}}^{N}(z+q)=\sum_{j=0}^{N-1}c_{kj}z^j \qquad (1\leqq k\leqq g).$$

We have further

$$f_k(\rho_k)=\prod_{\substack{q=1\\q\neq k}}^{N}(ik-iq)=(-1)^{N-k}i^{N-1}(k-1)!(N-k)!,$$

so that

$$b_{k0}=\sum_{j=0}^{n-1}\frac{j!(-1)^{k-1}i^j}{(N-k)!(k-1)!}c_{kj}a_jt^{rj}.$$

We thus obtain the following corollary.

COROLLARY 1.

$$S_N=\sum_{k=1}^{N}\sum_{j=0}^{N-1}\frac{j!(-1)^{k-1}i^j}{(N-k)!(k-1)!}c_{kj}a_jt^{rj}\exp\Big(\frac{i}{4}k^2t^{-2r}\Big)\Psi\Big(\sqrt{\frac{2t}{\pi}}-2m+\frac{k}{\sqrt{2\pi}}t^{-r}\Big),$$

and the sum of O-terms is

$$O\Big(\Big(\frac{N^{20}}{t}\Big)^{N/6}\Big)+O((19rN)^{5rN}e^{-At}).$$

We now take $g=N$, $n_k=0$ $(1\leqq k\leqq g)$, and $\rho_k=\omega^k$, where $\omega=\exp(2\pi i/N)$. Then we have $B\geqq\frac{1}{N}$, $M=1$, and

$$\sum_{j=0}^{N-1}f_{kj}z^j=\prod_{\substack{q=1\\q\neq k}}^{N}(z-\omega^q)=\omega^{k(N-1)}\prod_{\substack{q=1\\q\neq k}}^{N}(\omega^{-k}z-\omega^{q-k})=\omega^{-k}\sum_{j=0}^{N-1}\omega^{-kj}z^j,$$

so that

$$b_{k0}=\frac{1}{N}\sum_{j=0}^{N-1}j!\omega^{-kj}a_jt^{rj}.$$

Therefore we obtain the following corollary.

COROLLARY 2.

$$S_N=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}j!\omega^{-kj}a_jt^{rj}\exp\Big(-\frac{i}{4}\omega^{2k}t^{-2r}\Big)\Psi\Big(\sqrt{\frac{2t}{\pi}}-2m-\frac{i}{\sqrt{2\pi}}\omega^kt^{-r}\Big),$$

and the sum of O-terms is

$$O\Big(\Big(\frac{N^{19}}{t}\Big)^{N/6}\Big)+O((19rN)^{5rN}e^{-At}).$$

REMARK. If we take $g=1$, and $\rho_1=0$, ignoring the assumption of $g\geqq2$, then we get Siegel's asymptotic series with indeterminate O-terms.

2. Lemmas.

We need four lemmas to prove the theorem. In this paper we use Vinogradov's symbol "$\ll$".

LEMMA 1. $b_{kh}\ll\left(\frac{2NM}{eB}\right)^N N^{5/2}t^{-h/6+(r-1/6)(N-n_k-1)}$.

PROOF. We first prove that

$$a_n\ll(n+1)t^{-n/6}. \tag{1}$$

We use mathematical induction with respect to n. Suppose that

$$|a_j|\leqq(j+1)t^{-j/2+[j/3]}\qquad(1\leqq j\leqq n).$$

Then, from the fact ([3], p. 75) that $a_0=1$, $a_1=\frac{\sigma-1}{\sqrt{t}}$, and

$$(j+1)\sqrt{t}\,a_{j+1}=(\sigma-j-1)a_j+ia_{j-2}\qquad(j\geqq 2),$$

we get

$$\begin{aligned}|a_{n+1}|&\leqq\frac{|\sigma-n-1|}{(n+1)\sqrt{t}}|a_n|+\frac{1}{(n+1)\sqrt{t}}|a_{n-2}|\\&\leqq(n+1)t^{-(n+1)/2+[(n+1)/3]}+\left(1-\frac{2}{n+1}\right)t^{-(n+1)/2+[(n+1)/3]}\\&\leqq(n+2)t^{-(n+1)/2+[(n+1)/3]}.\end{aligned}$$

Therefore we have (1), so that

$$\begin{aligned}b_{kh}&=\prod_{\substack{q=1\\q\neq k}}^{g}(\rho_k-\rho_q)^{-n_q-1}\sum_{u=0}^{h}\sum_{j=0}^{N-n_k-1}\frac{(h-u+j)!}{u!(h-u)!}(-\rho_k)^u f_{kj}a_{h-u+j}t^{r(j-u)}\\&\ll B^{-(N-n_k-1)}\sum_{u=0}^{h}\sum_{j=0}^{N-n_k-1}(N-1)!M^u(2M)^{N-n_k-1}(h-u+j+1)t^{-(h-u+j)/6}t^{r(j-u)},\end{aligned}$$

since

$$|f_{kj}|\leqq(1+M)^{N-n_k-1}\leqq(2M)^{N-n_k-1}.$$

We thus have

$$\begin{aligned}b_{kh}&\ll B^{-N}N^2(N-1)!M^{n_k}(2M)^{N-n_k-1}Nt^{-h/6+(r-1/6)(N-n_k-1)}\\&\ll\left(\frac{2M}{B}\right)^N N^2N!\,t^{-h/6+(r-1/6)(N-n_k-1)}.\end{aligned}$$

By using Stirling's formula ([4], p. 253), we obtain

$$b_{kh}\ll\left(\frac{2NM}{eB}\right)^N N^{5/2}t^{-h/6+(r-1/6)(N-n_k-1)}.$$

LEMMA 2. *Define the function $R_N(z)$ by*

(2) $$R_N(z)=\phi(z)-\sum_{k=1}^{g}\sum_{h=0}^{n_k} b_{kh}z^h e^{\rho_k t^{-r}z}.$$

Then $R_N(z)$ vanishes at $z=0$ with an order not less than N.

PROOF. We are going to determine g polynomials $P_k(z)$ $(1\leqq k\leqq g)$ of degrees n_k $(1\leqq k\leqq g)$ such that the function

(3) $$f(z)=\phi(z)-\sum_{q=1}^{g}P_q(z)e^{\rho_q t^{-r}z}$$

vanishes at $z=0$ with an order not less than N. If we write the polynomials with indeterminate coefficients and equate the terms of order $0, 1, \cdots, N-1$ in the power series expansions of both sides of (3), then we obtain a system $\{L_j(\phi)=0\}$ composed of N linear equations for the N unknown coefficients in the polynomials. We recall our problem has a unique solution in the case of $\phi(z)=cz^{N-1}/(N-1)!$, where $c\neq 0$ ([2], p. 13, in which Siegel use the notation N instead of $N-1$). Thus, in this case, the coefficient matrix of the linear equations $\{L_j(\phi)=0\}$ with the N unknown variables has the rank N. On the other hand, the coefficient matrix of $\{L_j(\phi)=0\}$ is the same for every $\phi(z)$. It follows that our problem has a unique solution in the case of $\phi(z)=\exp\left((s-1)\log\left(1+\frac{z}{\sqrt{t}}\right)-i\sqrt{t}\,z-\frac{i}{2}z^2\right)$.

It is convenient to use the differential operator $D=d/dz$. Let j_1 be an integer satisfying $1\leqq j_1\leqq g$ and $j_1\neq k$. If we multiply both sides of (3) by $\exp(-\rho_{j_1}t^{-r}z)$, and differentiate $n_{j_1}+1$ times, then we have

$$f_1(z)=e^{-\rho_{j_1}t^{-r}z}(D-\rho_{j_1}t^{-r})^{n_{j_1}+1}\phi(z)$$
$$-\sum_{\substack{q=1\\ q\neq j_1}}^{g} e^{(\rho_q-\rho_{j_1})t^{-r}z}((\rho_q-\rho_{j_1})t^{-r}+D)^{n_{j_1}+1}P_q(z).$$

The function $f_1(z)$ vanishes at $z=0$ with an order not less than $N-n_{j_1}-1$. Let j_2 be an integer satisfying $1\leqq j_2\leqq g$, $j_2\neq k$, and $j_2\neq j_1$. If we multiply both sides of the above equation by $\exp((\rho_{j_1}-\rho_{j_2})t^{-r}z)$, and differentiate $n_{j_2}+1$ times, then we get

$$f_2(z)=e^{-\rho_{j_2}t^{-r}z}(D-\rho_{j_1}t^{-r})^{n_{j_1}+1}(D-\rho_{j_2}t^{-r})^{n_{j_2}+1}\phi(z)$$
$$-\sum_{\substack{q\neq j_1\\ q\neq j_2}} e^{(\rho_q-\rho_{j_2})t^{-r}z}((\rho_q-\rho_{j_1})t^{-r}+D)^{n_{j_1}+1}((\rho_q-\rho_{j_2})t^{-r}+D)^{n_{j_2}+1}P_q(z).$$

The function $f_2(z)$ vanishes at $z=0$ with an order not less than $N-n_{j_1}-n_{j_2}-2$. If we proceed in this way, then we have

$$f_{g-1}(z)=e^{-\rho_{j_{g-1}}t^{-r}z}\prod_{\substack{q=1\\q\neq k}}^{g}(D-\rho_q t^{-r})^{n_q+1}\phi(z)$$
$$-e^{(\rho_k-\rho_{j_{g-1}})t^{-r}z}\prod_{\substack{q=1\\q\neq k}}^{g}((\rho_k-\rho_q)t^{-r}+D)^{n_q+1}P_k(z).$$

The function $f_{g-1}(z)$ vanishes at $z=0$ with an order not less than n_k+1. We now multiply both sides of the above equation by $\exp((\rho_{j_{g-1}}-\rho_k)t^{-r}z)$. Then we obtain

$$F(z)=e^{-\rho_k t^{-r}z}\prod_{\substack{q=1\\q\neq k}}^{g}(D-\rho_q t^{-r})^{n_q+1}\phi(z)\tag{4}$$
$$-\prod_{\substack{q=1\\q\neq k}}^{g}((\rho_k-\rho_q)t^{-r}+D)^{n_q+1}P_k(z).$$

We notice that

$$Q_k(z)=\prod_{\substack{j=1\\j\neq k}}^{g}((\rho_k-\rho_j)t^{-r}+D)^{n_j+1}P_k(z)$$

is a polynomial having the same degree as $P_k(z)$, and the function $F(z)$ vanishes at $z=0$ with an order not less than n_k+1. Therefore $Q_k(z)$ is the sum of the first n_k+1 terms of Taylor's expansion of the function

$$e^{-\rho_k t^{-r}z}\prod_{\substack{j=1\\j\neq k}}^{g}(D-\rho_j t^{-r})^{n_j+1}\phi(z)$$

at the point $z=0$. Hence

$$Q_k(z)\equiv\sum_{u=0}^{n_k}\frac{(-\rho_k t^{-r}z)^u}{u!}t^{-r(N-n_k-1)}\sum_{j=0}^{N-n_k-1}f_{kj}(t^rD)^j\sum_{v=0}^{N}a_v z^v \qquad (\mathrm{mod}\ z^{n_k+1})$$
$$\equiv t^{-r(N-n_k-1)}\sum_{u=0}^{n_k}\frac{(-\rho_k t^{-r})^u}{u!}z^u\sum_{j=0}^{N-n_k-1}f_{kj}t^{rj}\sum_{v=0}^{n_k}\frac{(v+j)!}{v!}a_{v+j}z^v \qquad (\mathrm{mod}\ z^{n_k+1}).$$

We thus obtain

$$Q_k(z)=t^{-r(N-n_k-1)}\sum_{h=0}^{n_k}z^h\sum_{u=0}^{h}\sum_{j=0}^{N-n_k-1}\frac{(h-u+j)!}{u!(h-u)!}(-\rho_k)^u f_{kj}a_{h-u+j}t^{r(j-u)}.\tag{5}$$

We next determine $P_k(z)$. From (4), we get

$$P_k(z)=\prod_{\substack{q=1\\q\neq k}}^{g}((\rho_k-\rho_q)t^{-r}+D)^{-n_q-1}Q_k(z)$$
$$=t^{r(N-n_k-1)}\prod_{\substack{q=1\\q\neq k}}^{g}(\rho_k-\rho_q)^{-n_q-1}\Big(\sum_{j=0}^{n_k}((\rho_k-\rho_q)^{-1}t^rD)^j\Big)^{n_q+1}Q_k(z).$$

If we differentiate h times, and put $z=0$, then we have

$$h!p_{kh}=h!q_{kh}t^{r(N-n_k-1)}\prod_{\substack{j=0\\j\neq k}}^{g}(\rho_k-\rho_j)^{-n_j-1} \quad (0\leqq h\leqq n_k),$$

where p_{kh} and q_{kh} are the coefficients of z^h in the polynomials $P_k(z)$ and $Q_k(z)$ respectively. Therefore we obtain from (5)

$$P_k(z)=\sum_{h=0}^{nk} b_{kh}z^h .$$

It turns out that $R_N(z)$ vanishes at $z=0$ with an order not less than N.

LEMMA 3. *The function* $R_N(z)$ *satisfies*

$$R_N(z)\ll\left(\frac{2NM}{eB}\right)^N N^{8/3}e^M t^{-N/6}|z|^N \qquad \left(\textit{if } N\leqq\frac{9261}{25600}t,\ |z|\leqq\frac{20}{21}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}\right),$$

and

$$R_N(z)\ll e^{(14/29)|z|^2}+\left(\frac{2^{2/3}N^2M}{e^2B}\right)^N N^{19/6}e^M e^{|z|} \qquad \left(\textit{if } |z|\leqq\frac{\sqrt{t}}{2}\right).$$

PROOF. We can write by Lemma 2

$$R_N(z)=\frac{1}{2\pi i}\int_\Gamma\frac{R_N(w)z^N}{w^N(w-z)}dw ,$$

where Γ is a contour including the points 0 and z. Therefore we have

$$R_N(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\phi(w)z^N}{w^N(w-z)}dw$$
$$-\frac{1}{2\pi i}\sum_{k=1}^{g}\sum_{h=0}^{nk} b_{kh}\int_\Gamma w^{h-N}e^{\rho_k t^{-r}w}\frac{z^N}{w-z}dw .$$

We know ([**3**], p. 76) that

$$\frac{1}{2\pi i}\int_\Gamma\frac{\phi(w)z^N}{w^N(w-z)}dw=O\left(|z|^N\left(\frac{5e}{2N\sqrt{t}}\right)^{N/3}\right) \tag{6}$$
$$\left(N\leqq\frac{27}{50}t,\ |z|\leqq\frac{20}{21}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}\right),$$

and

$$\frac{1}{2\pi i}\int_\Gamma\frac{\phi(w)z^N}{w^N(w-z)}dw=O(e^{(14/29)|z|^2}) \qquad \left(|z|\leqq\frac{\sqrt{t}}{2}\right). \tag{7}$$

We now choose Γ a circle with center 0 and radius t^r in order to evaluate the sum of the last N terms of the integrals for $|z|\leqq\frac{\sqrt{t}}{2}$. Let J denote the sum, then we have from Lemma 1

$$J\ll\sum_{k=1}^{g}\sum_{h=0}^{nk}\left(\frac{2NM}{eB}\right)^N N^{5/2}t^{-h/6+(r-1/6)(N-n_k-1)}t^{r(h-N)}\frac{e^M|z|^N}{t^r/2} \qquad \left(|z|\leqq\frac{\sqrt{t}}{2}\right).$$

We thus have

$$J \ll \left(\frac{2NM}{eB}\right)^N N^{5/2} e^M |z|^N \sum_{k=1}^{g} \sum_{h=0}^{n k} t^{-N/6+(r-1/6)(h-n_k\;1)-r}$$

$$\ll \left(\frac{2NM}{eB}\right)^N N^{5/2} e^M |z|^N N t^{-N/6-5/6}$$

$$\ll \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M |z|^N t^{-N/6} \qquad \left(|z| \leqq \frac{\sqrt{t}}{2}\right),$$

since $t^{-5/6} \leqq A^{5/6} N^{-5/6}$. It follows that

$$(8) \qquad J \ll \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M |z|^N t^{-N/6} \quad \left(N \leqq \frac{9261}{25600} t,\ |z| \leqq \frac{20}{21}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}\right),$$

because $\frac{20}{21}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3} \leqq \frac{\sqrt{t}}{2}$ implies $N \leqq \frac{9261}{25600} t$. If we make use of the fact that, for $1 \leqq |z| \leqq \frac{\sqrt{t}}{2}$,

$$t^{-N/6} |z|^N \leqq 2^{-N/3} |z|^{(2/3)N} \leqq N! 2^{-N/3} \exp(|z|^{2/3}) \leqq N! 2^{-N/3} \exp(|z|),$$

we get

$$J \ll \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M N! 2^{-N/3} \exp(|z|) \qquad \left(|z| \leqq \frac{\sqrt{t}}{2}\right).$$

Therefore we obtain by Stirling's formula

$$(9) \qquad J \ll \left(\frac{2^{2/3} N^2 M}{e^2 B}\right)^N N^{19/6} e^M e^{|z|} \qquad \left(|z| \leqq \frac{\sqrt{t}}{2}\right).$$

Hence we have from (6) and (8)

$$R_N(z) \ll \left(\frac{5e}{2N\sqrt{t}}\right)^{N/3} |z|^N + \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M t^{-N/6} |z|^N$$

$$\ll \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M t^{-N/6} |z|^N \qquad \left(N \leqq \frac{9261}{25600} t,\ |z| \leqq \frac{20}{21}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}\right),$$

since $B \leqq 2M$. We have also from (7) and (9)

$$R_N(z) \ll e^{(14/29)|z|^2} + \left(\frac{2^{2/3} N^2 M}{e^2 B}\right)^N N^{19/6} e^M e^{|z|} \qquad \left(|z| \leqq \frac{\sqrt{t}}{2}\right).$$

This proves the lemma.

LEMMA 4. *Let*

$$(10) \qquad U_N = R_N\left(e^{-\pi i/4}\left(\frac{x}{\sqrt{2\pi}} + \sqrt{\pi}\left(\frac{\eta}{2\pi} - m\right)\right)\right),$$

where x is real, and $\eta=\sqrt{2\pi t}$; then we have

$$U_N \ll \left(\frac{4NM}{eB}\right)^N N^{11/3} e^M t^{-N/6}\left(\left|\frac{x}{\sqrt{2\pi}}\right|^N+2^N\right)$$

$$\left(if\ N\leqq\frac{9261}{25600}t,\ \left|\frac{x}{\sqrt{2\pi}}\right|\leqq\frac{40}{63}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}\right),$$

and

$$U_N \ll \exp\left(\frac{420}{841}\left|\frac{x}{\sqrt{2\pi}}\right|^2\right)+\left(\frac{2^{2/3}N^2M}{e^2B}\right)^N N^{19/6} e^M \exp\left(\left|\frac{x}{\sqrt{2\pi}}\right|\right)$$

$$\left(if\ \left|\frac{x}{\sqrt{2\pi}}\right|\leqq\frac{\sqrt{t}}{3}\right).$$

PROOF. If $\left|\frac{x}{\sqrt{2\pi}}\right|\leqq\frac{40}{63}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}$, and $\sqrt{\pi}\leqq\frac{20}{63}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}$ then we have

$$\left|\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\left(\frac{\eta}{2\pi}-m\right)\right|\leqq\frac{20}{21}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3},$$

so that we get from Lemma 3

$$U_N \ll \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M t^{-N/6}\left|\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\left(\frac{\eta}{2\pi}-m\right)\right|^N$$

$$\ll \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M t^{-N/6}\sum_{j=0}^{N}\binom{N}{j}\left|\frac{x}{\sqrt{2\pi}}\right|^j 2^{N-j}$$

$$\left(N\leqq\frac{9261}{25600}t,\quad \left|\frac{x}{\sqrt{2\pi}}\right|\leqq\frac{40}{63}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}\right),$$

since $\sqrt{\pi}\leqq 2$. If we use an inequality

$$\binom{N}{j}\left|\frac{x}{\sqrt{2\pi}}\right|^j 2^{N-j}\leqq 2^N\left(\left|\frac{x}{\sqrt{2\pi}}\right|^N+2^N\right) \qquad (0\leqq j\leqq N),$$

then we obtain

$$U_N \ll \left(\frac{2NM}{eB}\right)^N N^{8/3} e^M t^{-N/6}(N+1)2^N\left(\left|\frac{x}{\sqrt{2\pi}}\right|^N+2^N\right)$$

$$\ll \left(\frac{4NM}{eB}\right)^N N^{11/3} e^M t^{-N/6}\left(\left|\frac{x}{\sqrt{2\pi}}\right|^N+2^N\right)$$

$$\left(N\leqq\frac{9261}{25600}t,\ \left|\frac{x}{\sqrt{2\pi}}\right|\leqq\frac{40}{63}\left(\frac{2N\sqrt{t}}{5}\right)^{1/3}\right).$$

We next consider the case that $\left|\frac{x}{\sqrt{2\pi}}\right|\leqq\frac{\sqrt{t}}{3}$, and $\sqrt{\pi}\leqq\frac{\sqrt{t}}{6}$. We then have

$$\left|\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\left(\frac{\eta}{2\pi}-m\right)\right|\leqq\frac{\sqrt{t}}{2},$$

so that we get also from Lemma 3

$$U_N \ll \exp\Big(\frac{14}{29}\Big|\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)\Big|^2\Big)$$
$$+\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M \exp\Big(\Big|\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)\Big|\Big)$$
$$\Big(\Big|\frac{x}{\sqrt{2\pi}}\Big|\leqq\frac{\sqrt{t}}{3}\Big).$$

If we suppose $\Big|\frac{x}{\sqrt{2\pi}}\Big|\geqq 116$, then we have $\Big|\frac{x}{\sqrt{2\pi}}\Big|^2+4\Big|\frac{x}{\sqrt{2\pi}}\Big|\leqq\frac{30}{29}\Big|\frac{x}{\sqrt{2\pi}}\Big|^2$. Hence we obtain

$$U_N \ll \exp\Big(\frac{14}{29}\Big(\Big|\frac{x}{\sqrt{2\pi}}\Big|+2\Big)^2\Big)+\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M \exp\Big(\Big|\frac{x}{\sqrt{2\pi}}\Big|+2\Big)$$
$$\ll \exp\Big(\frac{420}{841}\Big|\frac{x}{\sqrt{2\pi}}\Big|^2\Big)+\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M \exp\Big(\Big|\frac{x}{\sqrt{2\pi}}\Big|\Big)$$
$$\Big(116\leqq\Big|\frac{x}{\sqrt{2\pi}}\Big|\leqq\frac{\sqrt{t}}{3}\Big).$$

This proves the lemma.

3. Proof of Theorem.

We recall an inequality for $\zeta(s)$ ([**3**], p. 72). Let η be $\sqrt{2\pi t}$, let c be an absolute constant, $0<c\leqq\frac{1}{2}$, and let C_2 be a line segment joining $c\eta+i\eta(1+c)$ and $-c\eta+i\eta(1-c)$, then we have

$$\zeta(s)=\sum_{n=1}^{m} n^{-s}+\frac{(2\pi)^s}{\pi}\sin\frac{\pi s}{2}\Gamma(1-s)\sum_{n=1}^{m} n^{s-1} \tag{11}$$
$$+\frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s)\Big(\int_{C_2}\frac{w^{s-1}e^{-mw}}{e^w-1}dw+O(e^{-(\pi/2)t-At})\Big),$$

where A is a sufficiently small constant. Here we consider the case where $|e^w-1|>A$ on C_2, and $c=2^{-5/2}$. Let

$$I=\int_{C_2}\frac{w^{s-1}e^{-mw}}{e^w-1}dw,$$

then we have ([**3**], p. 75)

$$I=(i\eta)^{s-1}\int_{C_2}\exp\Big((s-1)\log\Big(1+\frac{w-i\eta}{i\eta}\Big)-mw\Big)\frac{dw}{e^w-1}$$
$$=(i\eta)^{s-1}\int_{C_2}\exp\Big(\frac{i}{4\pi}(w-i\eta)^2+\frac{\eta}{2\pi}(w-i\eta)-mw\Big)\phi\Big(\frac{w-i\eta}{i\sqrt{2\pi}}\Big)\frac{dw}{e^w-1}.$$

It follows from (2) that

$$(12)\quad I=(i\eta)^{s-1}\sum_{k=1}^{g}\sum_{h=0}^{nk}b_{kh}\int_{C_2}\exp\Big(\frac{i}{4\pi}(w-i\eta)^2+\Big(\frac{\eta}{2\pi}-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}\Big)(w-i\eta)$$

$$-mw\Big)\Big(\frac{w-i\eta}{i\sqrt{2\pi}}\Big)^h\frac{dw}{e^w-1}$$

$$+(i\eta)^{s-1}\int_{C_2}\exp\Big(\frac{i}{4\pi}(w-i\eta)^2+\frac{\eta}{2\pi}(w-i\eta)-mw\Big)R_N\Big(\frac{w-i\eta}{i\sqrt{2\pi}}\Big)\frac{dw}{e^w-1}.$$

We first estimate the last term of the above series. Let I_1 denote the term, and put $w-i\eta=\lambda\exp\Big(\frac{\pi i}{4}\Big)$, where λ is real, then we have

$$I_1\ll\eta^{\sigma-1}e^{-\pi t/2}\int_{-\eta/4}^{\eta/4}\exp\Big(-\frac{\lambda^2}{4\pi}+\frac{1}{\sqrt{2}}\Big(\frac{\eta}{2\pi}-m\Big)\lambda\Big)\Big|R_N\Big(e^{-\pi i/4}\frac{\lambda}{\sqrt{2\pi}}\Big)\Big|d\lambda$$

$$\ll\eta^{\sigma-1}e^{-\pi t/2}\exp\Big(\frac{1}{2}\Big(\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)\Big)^2\Big)$$

$$\times\int_{-\eta/4}^{\eta/4}\exp\Big(-\frac{1}{2}\Big(\frac{\lambda}{\sqrt{2\pi}}-\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)\Big)^2\Big)\Big|R_N\Big(e^{-\pi i/4}\frac{\lambda}{\sqrt{2\pi}}\Big)\Big|d\lambda.$$

If we substitute $\frac{x}{\sqrt{2\pi}}$ for $\frac{\lambda}{\sqrt{2\pi}}-\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)$ in the integral, then we obtain

$$I_1\ll\eta^{\sigma-1}e^{-\pi t/2}\int_{-\eta/3}^{\eta/3}\exp\Big(-\frac{x^2}{4\pi}\Big)\Big|R_N\Big(e^{-\pi i/4}\Big(\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)\Big)\Big)\Big|dx,$$

for $t>120$, since $0<\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)<2$. We thus have by (10)

$$I_1\ll\eta^{\sigma-1}e^{-\pi t/2}\int_{-\eta/3}^{\eta/3}\exp\Big(-\frac{x^2}{4\pi}\Big)\big|U_N\big|dx.$$

From Lemma 4, we get

$$I_1\ll\eta^{\sigma-1}e^{-\pi t/2}\Big(\int_0^{\alpha}\exp\Big(-\frac{x^2}{4\pi}\Big)\Big(\frac{4NM}{eB}\Big)^N N^{11/3}e^M t^{-N/6}\Big(\Big|\frac{x}{\sqrt{2\pi}}\Big|^N+2^N\Big)dx$$

$$+\int_{\alpha}^{\eta/3}\exp\Big(-\frac{x^2}{4\pi}\Big)\Big(\exp\Big(-\frac{420}{841}\Big|\frac{x}{\sqrt{2\pi}}\Big|^2\Big)+\Big(\frac{2^{2/3}N^2M}{e^2M}\Big)^N N^{19/6}e^M\exp\Big(\Big|\frac{x}{\sqrt{2\pi}}\Big|\Big)\Big)dx\Big)$$

$$\ll\eta^{\sigma-1}e^{-\pi t/2}\Big(\Big(\frac{4NM}{eB}\Big)^N N^{11/3}e^M t^{-N/6}\int_0^{\infty}\exp\Big(-\frac{x^2}{4\pi}\Big)\Big(\Big(\frac{x}{\sqrt{2\pi}}\Big)^N+2^N\Big)dx$$

$$+\int_{\alpha}^{\infty}\exp\Big(-\frac{x^2}{3364\pi}\Big)dx+\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M\int_{\alpha}^{\infty}\exp\Big(-\frac{x^2}{4\pi}+\frac{x}{\sqrt{2\pi}}\Big)dx\Big),$$

where α is $\frac{40}{63}\Big(\frac{2N\sqrt{t}}{5}\Big)^{1/3}\sqrt{2\pi}$. It follows that

$$I_1 \ll \eta^{\sigma-1}e^{-\pi t/2}\Big(\Big(\frac{4NM}{eB}\Big)^N N^{11/3}e^M t^{-N/6}2^N I\Big(\frac{N}{2}+\frac{1}{2}\Big)$$
$$+\Big(\frac{8NM}{eB}\Big)^N N^{11/3}e^M t^{-N/6}+\exp\Big(-\frac{1}{3364\pi}\Big(\frac{40}{63}\Big(\frac{2N\sqrt{t}}{5}\Big)^{1/3}\sqrt{2\pi}\Big)^2\Big)$$
$$+\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M\exp\Big(-\frac{1}{4\pi}\Big(\frac{20}{63}\Big(\frac{2N\sqrt{t}}{5}\Big)^{1/3}\sqrt{2\pi}\Big)^2\Big)\Big),$$

because $\frac{20}{63}\Big(\frac{2N\sqrt{t}}{5}\Big)^{1/3}>1$. By using Stirling's formula, we have

$$I_1 \ll \eta^{\sigma-1}e^{-\pi t/2}\Big(\Big(\Big(\frac{N}{e}\Big)^{3/2}\frac{4M}{B}\Big)^N N^{11/3}e^M t^{-N/6}$$
$$+\Big(\frac{8NM}{eB}\Big)^N N^{11/3}e^M t^{-N/6}+\exp\Big(-\frac{800}{3337929}\Big(\frac{2N\sqrt{t}}{5}\Big)^{2/3}\Big)$$
$$+\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M\exp\Big(-\frac{200}{3969}\Big(\frac{2N\sqrt{t}}{5}\Big)^{2/3}\Big)\Big).$$

The last term can be written as

$$\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M t^{-N/6}\cdot t^{N/6}\exp\Big(-\frac{200}{3969}\Big(\frac{2N\sqrt{t}}{5}\Big)^{2/3}\Big).$$

The second factor $t^{N/6}\exp\Big(-\frac{200}{3969}\Big(\frac{2N\sqrt{t}}{5}\Big)^{2/3}\Big)$ has the maximum $\Big(\frac{7^6 3^{12}N}{2^{14}5^4}\Big)^{N/6}$ $\times\exp\Big(-\frac{N}{2}\Big)$ for $t=\frac{7^6 3^{12}N}{2^{14}5^4}$, so that the last term is

$$\ll\Big(\frac{2^{2/3}N^2M}{e^2B}\Big)^N N^{19/6}e^M t^{-N/6}\Big(\frac{7^6 3^{12}N}{2^{14}5^4}\Big)^{N/6}\exp\Big(-\frac{N}{2}\Big)$$
$$\ll\Big(\frac{N^{13}M^6}{B^6}\Big)^{N/6}e^M t^{-N/6},$$

since $\frac{7^6 3^{12}}{2^{10}5^4 e^{15}}<1$. Therefore we finally obtain

$$I_1 \ll \eta^{\sigma-1}e^{-\pi t/2}e^M\Big(\frac{N^{13}M^6}{B^6 t}\Big)^{N/6}. \tag{13}$$

In the first N terms of (12) we now replace C_2 by the infinite straight line of which it is a part. We denote this new straight line by C_2', and put

$$I_2=(i\eta)^{s-1}\sum_{k=1}^{g}\sum_{h=0}^{n_k} b_{kh}\int_{C_2'-C_2}\exp\Big(\frac{i}{4\pi}(w-i\eta)^2+\Big(\frac{\eta}{2\pi}-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}\Big)(w-i\eta)$$
$$-mw\Big)\Big(\frac{w-i\eta}{i\sqrt{2\pi}}\Big)^h\frac{dw}{e^w-1}.$$

We then get from Lemma 1

$$I_2 \ll \eta^{\sigma-1}e^{-\pi t/2}\sum_{k=1}^{g}\sum_{h=0}^{n_k}\Big(\frac{2NM}{eB}\Big)^N N^{5/2}t^{-h/6+(r-1/6)(N-n_k-1)}$$

$$\times\int_{\eta/4}^{\infty}\exp\Big(-\frac{\lambda^2}{4\pi}+\Big(\frac{1}{\sqrt{2}}\Big(\frac{\eta}{\sqrt{2\pi}}-m\Big)+\frac{Mt^{-r}}{\sqrt{2\pi}}\Big)\lambda\Big)\Big(\frac{\lambda}{\sqrt{2\pi}}\Big)^h d\lambda$$

$$\ll \eta^{\sigma-1}e^{-\pi t/2}\Big(\frac{2NM}{eB}\Big)^N N^{5/2}\sum_{k=1}^{g}\sum_{h=0}^{n_k}t^{-h/6+(r-1/6)N}$$

$$\times\int_{\eta/4}^{\infty}\exp\Big(-\frac{1}{2}\Big(\frac{\lambda}{\sqrt{2\pi}}-\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)-Mt^{-r}\Big)^2\Big)\Big(\frac{\lambda}{\sqrt{2\pi}}\Big)^h d\lambda,$$

since $0<\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)-Mt^{-r}<\sqrt{\pi}+A^{1/2}t^{1/2-r}<2$. If we substitute $\frac{x}{\sqrt{2\pi}}$ for $\frac{\lambda}{\sqrt{2\pi}}-\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)-Mt^{-r}$ in the integral, then we have

$$I_2 \ll \eta^{\sigma-1}e^{-\pi t/2}\Big(\frac{2NM}{eB}\Big)^N N^{5/2}\sum_{k=1}^{g}\sum_{h=0}^{n_k}t^{-h/6+(r-1/6)N}\int_{\eta/5}^{\infty}\exp\Big(-\frac{x^2}{4\pi}\Big)$$

$$\times\Big(\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)+Mt^{-r}\Big)^h dx,$$

if $\frac{\sqrt{t}}{20}>2$. It follows that

$$I_2 \ll \eta^{\sigma-1}e^{-\pi t/2}\Big(\frac{4NM}{eB}\Big)^N N^{7/2}\sum_{k=1}^{g}\sum_{h=0}^{n_k}t^{-h/6-(r-1/6)N}\int_{\eta/5}^{\infty}\exp\Big(-\frac{x^2}{4\pi}\Big)\Big(\Big(\frac{x}{\sqrt{2\pi}}\Big)^h+2^h\Big)dx,$$

because $\Big(\frac{x}{\sqrt{2\pi}}+\sqrt{\pi}\Big(\frac{\eta}{2\pi}-m\Big)+Mt^{-r}\Big)^h \leqq (N+1)2^N\Big(\Big(\frac{x}{\sqrt{2\pi}}\Big)^h+2^h\Big)$. Here we estimate the term

$$t^{-h/6+(r-1/6)N}\int_{\eta/5}^{\infty}\exp\Big(-\frac{x^2}{4\pi}\Big)\Big(\frac{x}{\sqrt{2\pi}}\Big)^h dx. \tag{14}$$

We can write the integrand as

$$\exp\Big(-\frac{50A}{4\pi}x^2\Big)\cdot\exp\Big(-\frac{1-50A}{4\pi}x^2\Big)\Big(\frac{x}{\sqrt{2\pi}}\Big)^h.$$

The second factor is steadily decreasing for $x \geqq 2\sqrt{h\pi}$, and so throughout the interval of integration if A is sufficiently small. Therefore the term (14) is

$$\ll t^{-h/6+(r-1/6)N}\exp\Big(-\frac{1-50A}{4\pi}\Big(\frac{\sqrt{2\pi t}}{5}\Big)^2\Big)\Big(\frac{\sqrt{t}}{5}\Big)^h$$

$$\ll t^{(r-1/6)N}\exp\Big(-\Big(\frac{1}{50}-2A\Big)t\Big)\exp(-At).$$

The factor $t^{(r-1/6)N}\exp\left(-\left(\frac{1}{50}-2A\right)t\right)$ has the maximum

$$\left(\frac{\left(r-\frac{1}{6}\right)N}{\frac{1}{50}-2A}\right)^{(r-1/6)N} e^{-(r-1/6)N} \qquad \text{for } t=\frac{\left(r-\frac{1}{6}\right)N}{\frac{1}{50}-2A},$$

so that the term is

$$\ll\left(\frac{51}{e}\left(r-\frac{1}{6}\right)N\right)^{(r-1/6)N} e^{-At},$$

if $A<\frac{1}{5100}$. It is easily verified that

$$t^{-h/6+(r-1/6)N}\int_{\eta/6}^{\infty}\exp\left(-\frac{x^2}{4\pi}\right)2^h dx\ll\left(\frac{51}{e}\left(r-\frac{1}{6}\right)N\right)^{(r-1/6)N} e^{-At},$$

because $2^h t^{-h/6}\leqq 2^h A^{h/6}N^{-h/6}<1$, for sufficiently small A. Therefore we obtain

$$I_2\ll\eta^{\sigma-1}e^{-\pi t/2}\left(\frac{4NM}{eB}\right)^N N^{9/2}\left(\frac{51}{e}\left(r-\frac{1}{6}\right)N\right)^{(r-1/6)N} e^{-At}$$

(15) $$\ll\eta^{\sigma-1}e^{-\pi t/2}\left(\frac{M}{B}\right)^N(19rN)^{3rN}e^{-At},$$

since $\frac{51}{e}<19$, and $1+\left(r-\frac{1}{6}\right)\leqq 3r$.

Let

$$I_3=(i\eta)^{s-1}\sum_{k=1}^{g}\sum_{h=0}^{nk}b_{kh}(i\sqrt{2\pi})^{-h}\int_{C_2'}\exp\left(\frac{i}{4\pi}(w-i\eta)^2+\left(\frac{\eta}{2\pi}-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}\right)(w-i\eta)-mw\right)(w-i\eta)^h\frac{dw}{e^w-1}.$$

The integral can be written as

$$-\int_L\exp\left(\frac{i}{4\pi}(w+2m\pi i-i\eta)^2+\left(\frac{\eta}{2\pi}-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}\right)(w+2m\pi i-i\eta)-mw\right)\times(w+2m\pi i-i\eta)^h\frac{dw}{e^w-1},$$

where L is a line in the direction $\arg w=\frac{\pi}{4}$, passing between 0 and $2\pi i$. This is $h!$ times the coefficient of ξ^h in Taylor's expansion of the function

$$-\int_L \exp\Big(\frac{i}{4\pi}(w+2m\pi i-i\eta)^2+\Big(\frac{\eta}{2\pi}-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}+\xi\Big)(w+2m\pi i-i\eta)-mw\Big)\frac{dw}{e^w-1}$$

$$=-\exp\Big(i(2m\pi-\eta)\Big(\frac{3\eta}{4\pi}-\frac{m}{2}-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}+\xi\Big)\Big)$$

$$\times\int_L \exp\Big(\frac{i}{4\pi}w^2+\Big(\frac{\eta}{\pi}-2m-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}+\xi\Big)w\Big)\frac{dw}{e^w-1}$$

at the point $\xi=0$. We now recall ([3], p. 26) that

$$\int_L \frac{e^{(i/4\pi)w^2+aw}}{e^w-1}dw=2\pi e^{i\pi(a^2/2-5/8)}\Psi(a).$$

Therefore the function is

$$2\pi(-1)^{m-1}\exp\Big(-\frac{it}{2}-\frac{5\pi i}{8}-\frac{i}{4}\rho_k^2 t^{-2r}\Big)\Psi\Big(\frac{\eta}{\pi}-2m-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}+\xi\Big)$$

$$\times\exp\Big(\frac{i\pi}{2}\xi^2+\sqrt{\frac{\pi}{2}}\rho_k t^{-r}\xi\Big)$$

$$=2\pi(-1)^{m-1}\exp\Big(-\frac{it}{2}-\frac{5\pi i}{8}-\frac{i}{4}\rho_k^2 t^{-2r}\Big)\sum_{j=0}^{\infty}\Psi^{(j)}\Big(\frac{\eta}{\pi}-2m-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}\Big)\frac{\xi^j}{j!}$$

$$\times\sum_{q=0}^{\infty}\Big(\frac{i\pi}{2}\xi^2\Big)^q\frac{1}{q!}\sum_{u=0}^{\infty}\Big(\sqrt{\frac{\pi}{2}}\rho_k t^{-r}\xi\Big)^u\frac{1}{u!}.$$

Hence we get

$$(16)\quad I_3=(i\eta)^{s-1}\sum_{k=1}^{g}\sum_{h=0}^{n_k} b_{kh}(i\sqrt{2\pi})^{-h}h!\,2\pi(-1)^{m-1}\exp\Big(-\frac{it}{2}-\frac{5\pi i}{8}-\frac{i}{4}\rho_k^2 t^{-2r}\Big)$$

$$\times\sum_{q=0}^{[h/2]}\sum_{j=0}^{h-2q}\Psi^{(j)}\Big(\frac{\eta}{\pi}-2m-\frac{i}{\sqrt{2\pi}}\rho_k t^{-r}\Big)$$

$$\times\frac{1}{j!}\Big(\frac{i\pi}{2}\Big)^q\frac{1}{q!}\Big(\sqrt{\frac{\pi}{2}}\rho_k t^{-r}\Big)^{h-j-2q}\frac{1}{(h-j-2q)!}.$$

For simplicity we have considered only the case where $|e^w-1|>A$ on C_2. In this case, if we take account of the fact that $I=I_1+I_2+I_3$, then we obtain the theorem from (11), (12), (13), (15), and (16). In the remaining case where the path C_2 goes near a pole of $1/(e^w-1)$, we have also the same result, by using the technique in Titchmarsh ([3], p. 73). This completes the proof.

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