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Capacities and Bergman kernels for Riemann surfaces and Fuchsian groups

Dedicated to Professor Yûsaku Komatu on his 70th birthday

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1. Riemann surfaces.

Let Ω be a Riemann surface with $\Omega\notin O_{G}$. Let $k_{0}(w, \omega)dwd\bar{\omega}$ denote the Bergman kernel of the Hilbert space of square integrable abelian differentials $a(w)dw$ on Ω. It has the reproducing property

$$a(\omega)=\frac{1}{\pi}\iint_{\Omega}a(w)\overline{k_{0}(w, \omega)}dudv\,. \tag{1.1}$$

We use the notation of Ahlfors and Sario [1, p. 302] which differs from that of Sario and Oikawa [7, p. 104] by a factor π.

Let $c_{\beta}(\omega)|d\omega|$ denote the capacity metric of the ideal boundary of Ω [7, p. 55]. If $g(w, \omega)$ denotes the Green's function of Ω with pole at ω then

$$g(w, \omega)=-\log|w-\omega|-\log c_{\beta}(\omega)+o(1) \quad \text{as} \quad w\rightarrow\omega\,. \tag{1.2}$$

The second author [8] conjectured that

$$k_{0}(\omega, \omega)\geqq c_{\beta}(\omega)^{2} \quad \text{for} \quad \omega\in\Omega \tag{1.3}$$

and proved this for the special case that Ω is a doubly connected plane domain.

We shall prove a weaker inequality. Let $\lambda(\omega)|d\omega|$ denote the Poincaré metric of Ω which has constant curvature -4.

THEOREM 1. *If $\Omega\notin O_{G}$ then, for $\omega\in\Omega$,*

$$k_{0}(\omega, \omega)\geqq c_{\beta}(\omega)^{2}\Big/\left(8\log\frac{\lambda(\omega)}{c_{\beta}(\omega)}+6\log 2\right). \tag{1.4}$$

We shall reformulate this theorem for Fuchsian groups and then prove it in that form.

If Ω is a Riemann surface such that $c_{\beta}(\omega)/\lambda(\omega)$ is bounded below then (1.4) implies $k_{0}(\omega, \omega)\geqq\text{const.}\,c_{\beta}(\omega)^{2}$. This assumption holds, in particular, if Ω is a plane domain with uniformly perfect boundary [5]. Examples are given by the complement of the Cantor set or the limit set of finitely generated Fuchsian groups.

It was proved in [8] (where $\Omega\notin O_G$ should have been assumed) that

$$k_0(\omega, \omega)=\frac{\partial^2}{\partial\omega\partial\bar{\omega}}\log c_\beta(\omega) \qquad (\omega\in\Omega). \tag{1.5}$$

Hence the conjecture (1.3) can be rewritten as

$$-\frac{4}{c_\beta(\omega)^2}\frac{\partial^2}{\partial\omega\partial\bar{\omega}}\log c_\beta(\omega)\leqq -4;$$

this would mean that the Riemannian metric $c_\beta(\omega)|d\omega|$ has curvature $\leqq -4$

We derive now (1.5) directly from Schiffer's identity [7, p. 105]

$$k_0(w, \omega)=-2\frac{\partial^2}{\partial\omega\partial\bar{\omega}}g(w, \omega) \tag{1.6}$$

without using Fuchsian groups as in [8].

It is sufficient to prove (1.5) at the origin of a parametric disk. By (1.2), the symmetric function

$$h(w, \omega)=\begin{cases} g(w, \omega)+\log|w-\omega| & \text{for } w\neq\omega, \\ -\log c_\beta(\omega) & \text{for } w=\omega \end{cases} \tag{1.7}$$

is harmonic in each variable. The quadratic terms in the development around (0, 0) are of the form

$$a(w^2+\omega^2)+\bar{a}(\bar{w}^2+\bar{\omega}^2)+bw\omega+\bar{b}\bar{w}\bar{\omega}+c(w\bar{\omega}+\bar{w}\omega). \tag{1.8}$$

Since $\partial^2 h/\partial w\partial\bar{\omega}=\partial^2 g/\partial w\partial\bar{\omega}$ by (1.7), we see from (1.6) and (1.8) that $k_0(0, 0)=-2c$. On the other hand, if we put $w=\omega$ in (1.8), we obtain that $-\partial^2\log c_\beta(\omega)/\partial\omega\partial\bar{\omega}$ has also the value $-2c$ for $\omega=0$.

2. Fuchsian groups.

There is a Fuchsian group Γ without elliptic elements such that $\boldsymbol{D}/\Gamma$ is conformally equivalent to Ω. We can choose Γ such that $0\in\boldsymbol{D}$ corresponds to $\omega\in\Omega$. Since $\Omega\notin O_G$ the group Γ is of convergence type.

The space of square integrable abelian differentials corresponds to the Bers space $A_1^2(\Gamma)$ of Γ-automorphic forms of weight 1 with

$$\|f\|^2=\frac{1}{\pi}\iint_F|f(z)|^2dxdy<\infty \tag{2.1}$$

where F denotes a fundamental domain of Γ with area $\partial F=0$. The Bergman kernel function of $A_1^2(\Gamma)$ specialized to the origin is

$$q(z)\equiv q(z, 0)=\sum_{\gamma\in\Gamma}\gamma'(z) \qquad (z\in\boldsymbol{D}); \tag{2.2}$$

see [2] [3, p. 602]. It has the reproducing property

(2.3) $$q(0)=\langle f, q\rangle\equiv\frac{1}{\pi}\iint_F f(z)\overline{q(z)}dxdy \qquad \text{for} \quad f\in A_1^2(\Gamma).$$

We also consider the Blaschke product

(2.4) $$b(z)=\prod_{\gamma\in\Gamma,\gamma\neq\iota}\frac{|\gamma(0)|}{\gamma(0)}\gamma(z) \qquad (z\in \boldsymbol{D})$$

where ι denotes the identity. By Myrberg's theorem [9, p. 522], $g(w, \omega)$ corresponds to $-\log|b(z)|$. The conjecture (1.3) can now be expressed as

(2.5) $$q(0)\leqq b'(0)^2$$

which, by (2.2) and (2.4), is equivalent to

(2.6) $$\sum_{\gamma\in\Gamma}\gamma'(0)\geqq\prod_{\gamma\in\Gamma,\gamma\neq\iota}|\gamma(0)|^2.$$

This is perhaps the simplest form of the conjecture (1.3).

Theorem 1 is contained in the following result where we allow elliptic elements.

THEOREM 2. *If Γ is a Fuchsian group of convergence type, then*

(2.7) $$q(0)\geqq b'(0)^2\Big/\left(8\log\frac{1}{b'(0)}+6\log 2\right)$$

or, equivalently,

(2.8) $$\sum_{\gamma\in\Gamma}\gamma'(0)\geqq\prod_{\gamma\in\Gamma,\gamma\neq\iota}|\gamma(0)|^2\Big/\left(8\sum_{\gamma\neq\iota}\log\frac{1}{|\gamma(0)|}+6\log 2\right).$$

Applying (2.8) to a conjugate group, we obtain, for $\zeta\in\boldsymbol{D}$,

(2.9) $$\sum_{\gamma\in\Gamma}\frac{(1-|\zeta|^2)^2\gamma'(\zeta)}{(1-\bar{\zeta}\gamma(\zeta))^2}\geqq\prod_{\gamma\neq\iota}\left|\frac{\gamma(\zeta)-\zeta}{1-\bar{\zeta}\gamma(\zeta)}\right|^2\Big/\left(8\sum_{\gamma\neq\iota}\log\left|\frac{1-\bar{\zeta}\gamma(\zeta)}{\gamma(\zeta)-\zeta}\right|+6\log 2\right)$$

as the conformally invariant form of (2.8).

3. Proof of Theorem 2.

The following lemma is a more precise form of a well-known result; see e.g. [6, p. 637].

LEMMA. *If Γ is of convergence type, then*

(3.1) $$\sum_{\in\Gamma}(1-|\gamma(z)|^2)\leqq 4\log\frac{1}{b'(0)}+6\log 2.$$

PROOF. Let $F=\{z\in \boldsymbol{D}: |z|<|\gamma(z)|$ for $z\in\Gamma,\ \gamma\neq\iota\}$ be the Ford fundamental domain of Γ and let $\{|z|<\rho\}$ be the largest disk in F around 0. If $\beta\in\Gamma\setminus\{\iota\}$ is chosen such that $|\beta(0)|$ is minimal then $|\beta(0)|<2\rho$ and thus, by (2.4),

$$b'(0)\leqq|\beta(0)|<2\rho\,. \tag{3.2}$$

It was proved in [4, p. 301] that

$$|b(z)|\geqq\frac{1}{4}b'(0)\min(\rho,\ |z|)\qquad \text{for}\quad z\in F. \tag{3.3}$$

It follows that, if $z\in F,\ |z|\geqq\rho$,

$$\begin{aligned}\sum_{\gamma\in\Gamma}(1-|\gamma(z)|^2)&\leqq 2\sum_{\gamma\in\Gamma}\log\frac{1}{|\gamma(z)|}=2\log\frac{1}{|b(z)|}\\ &\leqq 2\log\frac{4}{\rho b'(0)}\leqq 4\log\frac{\sqrt{8}}{b'(0)}\end{aligned} \tag{3.4}$$

because of (3.2). If $|z|\leqq\rho$ then we see from (3.3) that

$$\begin{aligned}\sum_{\gamma\in\Gamma}(1-|\gamma(z)|^2)&\leqq 1+2\sum_{\gamma\neq\iota}\log\frac{1}{|\gamma(z)|}=1+2\log\left|\frac{z}{b(z)}\right|\\ &\leqq 1+2\log\frac{4}{b'(0)}\,,\end{aligned}$$

and this bound is smaller than that in (3.4). Hence (3.1) follows because the left-hand side is Γ-invariant.

PROOF OF THEOREM 2. Using an idea of Rao [6], we consider the Poincaré theta series

$$f(z)=\theta\left[\frac{b(z)}{z}\right]\equiv\sum_{\gamma\in\Gamma}\frac{b(\gamma(z))}{\gamma(z)}\gamma'(z)\qquad \text{for}\quad z\in\boldsymbol{D}. \tag{3.5}$$

Since b is bounded, we have $f\in A_1^2(\Gamma)$ [2] [3, p. 596]. The reason for this choice is that $f(0)=b'(0)$, by (2.4). Hence we obtain from (2.3) and (2.1) by Schwarz's inequality that

$$b'(0)^2=|f(0)|^2\leqq\|f\|^2\|q\|^2=\|f\|^2q(0)\,; \tag{3.6}$$

the identity $q(0)=\|q\|^2$ follows from (2.3) and (2.1).

We write

$$f(z)=\sum_{k=0}^{\infty}a_kz^k\,,\quad \frac{b(z)}{z}=\sum_{k=0}^{\infty}b_kz^k\qquad (z\in\boldsymbol{D})\,.$$

It follows from (3.5) by the scalar product formula [3, p. 596] that

$$\|f\|^2=\langle f(z),\ \theta[z^{-1}b(z)]\rangle=\sum_{k=0}^{\infty}\frac{a_k\bar{b}_k}{k+1}\,.$$

Hence Schwarz's inequality shows that

$$\|f\|^4 \leqq \sum_{k=0}^{\infty} \frac{|a_k|^2}{(k+1)^2} \sum_{k=0}^{\infty} |b_k|^2 \leqq \sum_{k=0}^{\infty} \frac{|a_k|^2}{(k+1)^2}$$

because $|z^{-1}b(z)| \leqq 1$. It follows that

$$\|f\|^4 \leqq 2\sum_{k=0}^{\infty} \frac{|a_k|^2}{(k+1)(k+2)}$$
$$= \frac{2}{\pi} \iint_D |f(z)|^2 (1-|z|^2) dxdy$$

as we see from Parseval's formula. Since D is the disjoint union of the sets $\gamma(F)$ $(\gamma \in \Gamma)$ except for a set of zero area, we obtain that

$$\|f\|^4 = \frac{2}{\pi} \sum_{\gamma \in \Gamma} \iint_F |f(z)|^2 (1-|z|^2) dxdy$$
$$= \frac{2}{\pi} \iint_F |f(z)|^2 \sum_{\gamma \in \Gamma} (1-|\gamma(z)|^2) dxdy,$$

where we have used that $f(\gamma(z))\gamma'(z) = f(z)$.

We apply now the lemma. It follows from (3.1) and (2.1) that

$$\|f\|^4 \leqq \left(8 \log \frac{1}{b'(0)} + 6 \log 2\right) \|f\|^2.$$

Dividing through by $\|f\|^2$ we obtain (2.7) because $q(0) \geqq b'(0)/\|f\|^2$ by (3.6).

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