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A classification of type I AW^*-algebras and Boolean valued analysis

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1. Introduction.

The aim of this paper is to give a complete classification of type I AW^*-algebras using Boolean valued analysis. The structure theory of type I AW^*-algebras was instituted by Kaplansky [6] as a purely algebraic generalization of type I von Neumann algebras. The structure theory of type I von Neumann algebras leads an essentially unique direct sum decomposition into homogeneous von Neumann algebras. Thus a complete system of $*$-isomorphism invariants for such an algebra is obtained as a set of cardinals together with partition of unity consisting of central projections up to automorphism of the center. Kaplansky's theory of type I AW^*-algebras succeeded in decomposing every type I AW^*-algebra into homogeneous AW^*-algebras, but his theory was not completed as he stated [**6**; p. 460], "One detail has resisted complete solution thus far: the uniqueness of the cardinal number attached to a homogeneous AW^*-algebra of type I."

In this paper, we shall show that the solution of the above cardinal uniqueness problem is negative, as conjectured by Kaplansky [**7**; p. 843, footnote]. This means that we cannot insure the uniqueness of the direct sum decomposition of type I AW^*-algebras into homogeneous algebras. Thus the structure of $*$-isomorphism invariants for type I AW^*-algebras is supposed to be more complicated. However, as we shall show in this paper, it is a surprising fact that we can find $*$-isomorphism invariants for such algebras in the objects already studied in the field of mathematical logic. Precisely, we shall show that cardinal numbers in Scott-Solovay's Boolean valued universe of sets constitute $*$-isomorphism invariants of type I AW^*-algebras.

Boolean valued analysis is our method which bridges the gap between the results of mathematical logic and the problems of analysis. This new method of analysis was introduced by D. Scott and R. Solovay when they reformulated the theory of P. J. Cohen's forcing in terms of Boolean valued models of set theory in 1966. Recently, Boolean valued analysis was developed by G. Takeuti in operator theory, harmonic analysis and operator algebras ([**12**], [**13**], [**14**],

[15]) and by the author in multiplicity theory [8]. In particular, Professor Takeuti introduced in [14] a useful machinery into the theory of operator algebras which reduces the problems of abelian von Neumann algebras to Hilbert spaces and von Neumann algebras to factors as a transfer principle. This machinery is refined in [8], the relation with multiplicity theory is obtained and in [15] it is applied to C^*-algebras. Our main tools in this paper are also along with these lines. However, the Boolean valued analysis developed so far has made a limitation on Boolean algebras to be measure algebras. For our present purpose, we have to eliminate such a limitation. The required counterexample for the cardinal uniqueness problem will be constructed from a Boolean algebra which is not a measure algebra but constructed from a notion of forcing. Thus we shall develop Boolean valued analysis in its full generality in this paper.

In Section 2, necessary preliminaries on Scott-Solovay's Boolean valued universe $V^{(B)}$ of set theory are given, where B is a complete Boolean algebra. In Section 3, representations of real numbers and complex numbers in $V^{(B)}$ are obtained. In particular, the bounded part of complex numbers in $V^{(B)}$ is a commutative AW^*-algebra such that B is isomorphic to the complete Boolean algebra of its projections, and conversely every commutative AW^*-algebra Z is $*$-isomorphic to the bounded part of complex numbers in $V^{(B)}$, where B is the complete Boolean algebra of projections in Z. In the following, let B be the complete Boolean algebra of projections in a commutative AW^*-algebra Z. In Sections 4 and 5, it is shown that the bounded part of every Hilbert space in $V^{(B)}$ is an AW^*-module over Z and conversely every AW^*-module X over Z corresponds to a Hilbert space in $V^{(B)}$ whose bounded part is isomorphic to X. The above correspondence is a functor which is an equivalence between the category of AW^*-modules over Z and bounded Z-linear maps and the category of Hilbert spaces in $V^{(B)}$ and linear operators in $V^{(B)}$ with operator bounds in Z. Combining the result obtained in [8], we can show that if Z is a W^*-algebra then the above category of AW^*-modules over Z is also equivalent to the category of non-degenerate normal $*$-representations of Z on Hilbert spaces and bounded intertwining operators. In Section 6, we shall obtain a complete system of isomorphism invariants for AW^*-modules. It is shown that there is a one-to-one correspondence between isomorphism classes of AW^*-modules over Z and cardinals in $V^{(B)}$. In this section, we shall also settle the cardinal uniqueness problem of homogeneous AW^*-algebras negatively. Precisely, we shall prove that for any infinite cardinals α and β with $\alpha<\beta$, there is an AW^*-algebra which is γ-homogeneous simultaneously for all cardinal γ such that $\alpha\leqq\gamma\leqq\beta$. In Section 7, we shall obtain a complete system of $*$-isomorphism invariants for type I AW^*-algebras. Every automorphism of B can be extended canonically to an automorphism of $V^{(B)}$. We say that two cardinals in $V^{(B)}$ are congruent if there is an automorphism of $\boldsymbol{B}$ whose canonical extension shifts one to another. Then we shall show that there is a one-to-one correspondence between $*$-isomorphism classes of type I AW^*-algebras with center isomorphic to Z and congruence classes of cardinals in $\boldsymbol{V}^{(\boldsymbol{B})}$. The exact relation between such invariants and direct sum decompositions of type I AW^*-algebras into homogeneous algebras will be also established.

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2. Preliminaries.

Let $\boldsymbol{B}$ be a complete Boolean algebra. Scott-Solovay's Boolean valued model $\boldsymbol{V}^{(\boldsymbol{B})}$ of set theory is defined in the following way [**16**; p. 59, p. 121]. For an ordinal α, we define $\boldsymbol{V}_\alpha^{(\boldsymbol{B})}$ by transfinite induction as follows:

(1) $\boldsymbol{V}_0^{(\boldsymbol{B})}=\emptyset$,

(2) $\boldsymbol{V}_\alpha^{(\boldsymbol{B})}=\{u \mid u:\mathrm{dom}(u)\to\boldsymbol{B} \text{ and } \mathrm{dom}(u)\subseteqq\bigcup_{\beta<\alpha}\boldsymbol{V}_\beta^{(\boldsymbol{B})}\}$.

Then we define $\boldsymbol{V}^{(\boldsymbol{B})}=\bigcup_{\alpha\in\mathrm{On}}\boldsymbol{V}_\alpha^{(\boldsymbol{B})}$, where On is the class of all ordinal numbers. We call elements of $\boldsymbol{V}^{(\boldsymbol{B})}$ ***B**-valued sets*. For $u, v\in\boldsymbol{V}^{(\boldsymbol{B})}$, the truth values $[\![u\in v]\!]$ and $[\![u=v]\!]$ are defined as functions from $\boldsymbol{V}^{(\boldsymbol{B})}\times\boldsymbol{V}^{(\boldsymbol{B})}$ to $\boldsymbol{B}$ satisfying the following properties:

(1) $[\![u\in v]\!]=\sup_{y\in\mathrm{dom}(v)}(v(y)\wedge[\![u=y]\!])$,

(2) $[\![u=v]\!]=\inf_{x\in\mathrm{dom}(u)}(u(x)\Rightarrow[\![x\in v]\!])\wedge\inf_{y\in\mathrm{dom}(v)}(v(y)\Rightarrow[\![y\in u]\!])$,

where $(b_1\Rightarrow b_2)=(\neg b_1)\vee b_2$ for any $b_1, b_2\in\boldsymbol{B}$. Let φ be a formula in set theory with predicate symbols $\in$ and $=$. If φ contains no free variables and all the constants in φ are members in $\boldsymbol{V}^{(\boldsymbol{B})}$, we define the truth value $[\![\varphi]\!]$ of φ by the following recursive rules.

(1) $[\![\neg\varphi]\!]=\neg[\![\varphi]\!]$,

(2) $[\![\varphi_1\wedge\varphi_2]\!]=[\![\varphi_1]\!]\wedge[\![\varphi_2]\!]$,

(3) $[\![\varphi_1\vee\varphi_2]\!]=[\![\varphi_1]\!]\vee[\![\varphi_2]\!]$,

(4) $[\![(\forall x)\varphi(x)]\!]=\inf_{u\in\boldsymbol{V}^{(\boldsymbol{B})}}[\![\varphi(u)]\!]$,

(5) $[\![(\exists x)\varphi(x)]\!]=\sup_{u\in\boldsymbol{V}^{(\boldsymbol{B})}}[\![\varphi(u)]\!]$.

The basic theorem of Scott-Solovay's Boolean valued model theory is the following [16].

THEOREM 2.1 (Scott-Solovay). *If* φ *is a theorem of ZFC, then* $[\![\varphi]\!]=1$ *is also a theorem of ZFC.*

The original universe $\boldsymbol{V}$ of ZFC can be embedded in $\boldsymbol{V}^{(\boldsymbol{B})}$ by the following operation $\check{}$ defined by the $\in$-recursion: For $y \in \boldsymbol{V}$, $\check{y}=\{\check{x} \mid x \in y\} \times\{1\}$. We say conventionally that an element $u \in \boldsymbol{V}^{(\boldsymbol{B})}$ satisfying some property exists *uniquely* if there is another u' satisfying the same property then $[\![u=u']\!]=1$. A family $\{b_\alpha\}$ of elements of $\boldsymbol{B}$ is called a *partition of unity* if $\sup_\alpha b_\alpha=1$ and $b_\alpha \wedge b_\beta=0$ for any $\alpha \neq \beta$. Let $\{b_\alpha\}$ be a partition of unity and let $\{u_\alpha\}$ be a family of $\boldsymbol{B}$-valued sets in $\boldsymbol{V}^{(\boldsymbol{B})}$. Then there is a unique element $u \in \boldsymbol{V}^{(\boldsymbol{B})}$ such that $[\![u=u_\alpha]\!] \geqq b_\alpha$ for any α. We denote this u by $\sum u_\alpha b_\alpha$ or $u_1 b_1 \oplus \cdots \oplus u_n b_n$ if α varies over $\{1, 2, \cdots, n\}$.

Let $\varphi(x)$ be a formula with only x as a free variable and such that there is $v_0 \in \boldsymbol{V}^{(\boldsymbol{B})}$ with $[\![\varphi(v_0)]\!]=1$. Let $X=\{x \mid \varphi(x)\}$. We define the interpretation $X^{(\boldsymbol{B})}$ of X with respect to $\boldsymbol{V}^{(\boldsymbol{B})}$ as

$$X^{(\boldsymbol{B})}=\{\dot{u} \in \boldsymbol{V}^{(\boldsymbol{B})} \mid [\![\varphi(x)]\!]=1\},$$

where $\dot{u}$ is some representative from the equivalence class $\{v \in \boldsymbol{V}^{(\boldsymbol{B})} \mid [\![u=v]\!]=1\}$. In the sequel, we shall omit the symbol $\dot{}$ in $\dot{u}$, conventionally. Then it is known [12; p. 14] that

$$[\![(\forall x \in X) \phi(x)]\!]=\inf_{u \in X^{(\boldsymbol{B})}}[\![\phi(u)]\!],$$

$$[\![(\exists x \in X) \phi(x)]\!]=\sup_{u \in X^{(\boldsymbol{B})}}[\![\phi(u)]\!].$$

If X is a set in $\boldsymbol{V}^{(\boldsymbol{B})}$ then $X^{(\boldsymbol{B})} \times\{1\} \in \boldsymbol{V}^{(\boldsymbol{B})}$ and $[\![X=X^{(\boldsymbol{B})} \times\{1\}]\!]=1$. Let $X \in \boldsymbol{V}^{(\boldsymbol{B})}$ be definite. Then it is easy to see that $X^{(\boldsymbol{B})}=\{\sum u_\alpha b_\alpha \mid \{b_\alpha\}$ is a partition of unity and $u_\alpha \in \operatorname{dom}(X)\}$.

Let $\varphi(x, y)$ be a formula with only x and y as free variables such that (i) $(\exists x, y) \varphi(x, y)$, and (ii) $(\forall x, y, z) \varphi(x, y) \wedge \varphi(x, z) \Rightarrow y=z$ hold. Let $F=\{\langle x, y\rangle \mid \varphi(x, y)\}$ and $\mathbf{dom}(F)=\{x \mid(\exists y) \varphi(x, y)\}$. We define the interpretation $F(\cdot)_{\boldsymbol{B}}$ of the function F with respect to $\boldsymbol{V}^{(\boldsymbol{B})}$ as follows: For any $u \in \mathbf{dom}(F)^{(\boldsymbol{B})}$, $F(u)_{\boldsymbol{B}}$ is a unique $v \in \boldsymbol{V}^{(\boldsymbol{B})}$ such that $[\![\varphi(u, v)]\!]=1$ or equivalently $F(\cdot)_{\boldsymbol{B}}: \mathbf{dom}(F)^{(\boldsymbol{B})} \rightarrow \boldsymbol{V}^{(\boldsymbol{B})}$ such that $[\![F(u)_{\boldsymbol{B}}=F(u)]\!]=1$ for any $u \in \mathbf{dom}(F)^{(\boldsymbol{B})}$. Let $d \subseteqq \boldsymbol{V}^{(\boldsymbol{B})}$. A function $g: d \rightarrow \boldsymbol{V}^{(\boldsymbol{B})}$ is called *extensional* if for any $x, x' \in d$, $[\![x=x']\!] \leqq [\![g(x)=g(x')]\!]$. A $\boldsymbol{B}$-valued set $u \in \boldsymbol{V}^{(\boldsymbol{B})}$ is called *definite* if for any $x \in \operatorname{dom}(u)$, $u(x)=1$. Then it is known [16] that for definite $u, v \in \boldsymbol{V}^{(\boldsymbol{B})}$, there is a bijective correspondence between functions f in $\boldsymbol{V}^{(\boldsymbol{B})}$ such that $[\![f: u \rightarrow v]\!]=1$ and extensional maps $g: \operatorname{dom}(u) \rightarrow v^{(\boldsymbol{B})}$. The correspondence is given by the relation $[\![f(x)=g(x)]\!]=1$ for any $x \in \operatorname{dom}(u)$. In this case, $\operatorname{dom}(u) \subseteqq \mathbf{dom}(f)^{(\boldsymbol{B})}$ and $g(x)=f(x)_{\boldsymbol{B}}$ for any $x \in \operatorname{dom}(u)$.

LEMMA 2.2. *Let R be a ring with unit in $V^{(B)}$, i.e., $[\![R$ is a ring with unit$]\!]=1$. Then $R^{(B)}$ is a ring with unit and there is an embedding j from $\boldsymbol{B}$ into the center of $R^{(B)}$ such that*

(1) $j(b)x=xb\oplus 0(\neg b)$,

(2) $j(b)j(c)=j(b\wedge c)$,

(3) $j(b)+j(c)=j(b\vee c)+j(b\wedge c)$,

(4) $j(\neg b)=1-j(b)$, *for any* $x\in R^{(B)}$, $b, c\in \boldsymbol{B}$.

PROOF. It is easy to see that $R^{(B)}$ is a ring. For any $b\in\boldsymbol{B}$, consider the partition $\langle b, \neg b\rangle$ of unity and the family $\langle I, 0\rangle$ in $R^{(B)}$, where I is the unit of $R^{(B)}$ and 0 is the zero of $R^{(B)}$, and let $j(b)=Ib\oplus 0(\neg b)$. Then it is easy to see that $j: \boldsymbol{B}\to R^{(B)}$ has the required properties. QED

A subset S of an $R^{(B)}$-module is called ***B*-convex** if $bx+(1-b)y\in S$ for any $b\in\boldsymbol{B}$ and $x, y\in S$.

LEMMA 2.3. *Let M and N be unital R-modules in $V^{(B)}$. Then $M^{(B)}$ and $N^{(B)}$ are unital $R^{(B)}$-modules. If M is definite and if* $\mathrm{dom}(M)$ *is $\boldsymbol{B}$-convex then every mapping f from* $\mathrm{dom}(M)$ *into $N^{(B)}$ such that $f(bx+(1-b)y)=bf(x)+(1-b)f(y)$ for any $x, y\in\mathrm{dom}(M)$, $b\in\boldsymbol{B}$ is extensional.*

PROOF. It is obvious that $M^{(B)}$ and $N^{(B)}$ are unital $R^{(B)}$-modules. Let $f:\mathrm{dom}(M)\to N^{(B)}$ satisfy the required properties. Let $x, y\in\mathrm{dom}(M)$ and $b\in\boldsymbol{B}$. Then it is easy to see that $bx+(1-b)y=xb\oplus y(\neg b)$ and that $bf(x)+(1-b)f(y)=f(x)b\oplus f(y)(\neg b)$. Thus by [8; Theorem 2.3] f is extensional. QED

3. Real and complex numbers in $V^{(B)}$.

Let $\boldsymbol{B}$ be a complete Boolean algebra and $V^{(B)}$ be Scott-Solovay's Boolean valued universe of ZFC. It is known that natural numbers $\boldsymbol{N}$ and rational numbers $\boldsymbol{Q}$ are absolute in $V^{(B)}$, i.e., $[\![\boldsymbol{N}=\check{\boldsymbol{N}}]\!]=1$ and $[\![\boldsymbol{Q}=\check{\boldsymbol{Q}}]\!]=1$. In this section, we shall consider real numbers and complex numbers in $V^{(B)}$.

We define a real number to be the lower half line of a Dedekind cut without the end point. That is, the formula 'a is a real number' is expressed as follows:

$$a\subseteq\boldsymbol{Q}\wedge(\exists s\in\boldsymbol{Q})[s\in a]\wedge(\exists s\in\boldsymbol{Q})[s\notin a]$$
$$\wedge(\forall s\in\boldsymbol{Q})[s\in a\Leftrightarrow(\exists t\in\boldsymbol{Q})[s<t\wedge t\in a]].$$

Denote by $\boldsymbol{R}$ and $\boldsymbol{C}$ the sets of all real numbers and complex numbers, respectively. Let $\boldsymbol{R}^{(B)}$ be the interpretation of $\boldsymbol{R}$ in $V^{(B)}$, i.e.,

$$\boldsymbol{R}^{(B)}=\{u\in V^{(B)}\mid [\![u \text{ is a real number}]\!]=1\},$$

and let $\boldsymbol{C}^{(B)}$ be the interpretation of $\boldsymbol{C}$ in $V^{(B)}$, i.e.,

$$C^{(B)}=\{u\in V^{(B)} \mid [\![u \text{ is a complex number}]\!]=1\}\ .$$

Then $C^{(B)}=R^{(B)}+\check{i}R^{(B)}$.

Let L be a normed linear space in $V^{(B)}$. Denote by $L^{(B)}$ the interpretation of L. The *bounded part* $L_\infty^{(B)}$ of $L^{(B)}$ is defined as follows:

$$L_\infty^{(B)}=\{u\in L^{(B)}\mid \exists M\in R,\ [\![\|u\|<\check{M}]\!]=1\}\ .$$

LEMMA 3.1. $[\![L_\infty^{(B)}\times\{1\}=L]\!]=1$.

PROOF. Obviously, $[\![L^{(B)}\times\{1\}=L]\!]=1$ and we have only to show that $[\![L_\infty^{(B)}\times\{1\}\supseteq L^{(B)}\times\{1\}]\!]=1$. Let $x\in L^{(B)}$. Then $[\![(\exists n\in N)n\leqq\|x\|<n+1]\!]=1$, and hence $\{b_n\}$ is a partition of unity of B where

$$b_n=[\![\check{n}\leqq\|x\|<(n+1)^\vee]\!]\ ,$$

for any $n\in N$. Let x_n be such that $x_n=xb_n\oplus 0(\neg b_n)$. Then it is easy to see that $x=\sum_{n\in N}x_nb_n$ and $x_n\in L_\infty^{(B)}$. Thus the conclusion follows immediately. QED

Now we shall obtain representations of $R^{(B)}$, $R_\infty^{(B)}$, $C^{(B)}$ and $C_\infty^{(B)}$ coherent with the representations of B. Let Ω be a topological space and B the complete Boolean algebra of all regular open subsets of Ω. In the sequel, we shall use the following notations for any subset A of Ω : $A^c=\Omega-A$, A^-=the closure of A, A°=the interior of A. We shall say that a subset A of Ω is *congruent* with a subset B of Ω and write $A\sim B$ if $(A-B)\cup(B-A)$ is a meager set.

LEMMA 3.2. *Let $\{A_i\}$ be a family of open sets. Then* $(\bigcup_i A_i^{-\circ})^-=(\bigcup_i A_i)^-$.

PROOF. Since A_i is open, $A_i\subseteqq A_i^{-\circ}$ and hence $(\bigcup_i A_i)^-\subseteqq(\bigcup_i A_i^{-\circ})^-$. Since $(\bigcup_i A_i^{-\circ})^-$ is the smallest closed set containing all $A_i^{-\circ}$'s, the conclusion follows from the obvious relation $A_i^{-\circ}\subseteqq(\bigcup_i A_i)^-$ for all i. QED

LEMMA 3.3. *Let f be an extended real valued lower semicontinuous function on Ω and let $a\in V^{(B)}$. Suppose that*

$$[\![\check{s}\in a]\!]=\{\omega\in\Omega\mid s<f(\omega)\}^{-\circ}\ ,$$

for all $s\in Q$. Then we have the following:

(1) $[\![(\exists s\in Q)s\in a]\!]=\Omega$ *if and only if* $\{\omega\in\Omega\mid f(\omega)=-\infty\}$ *is nowhere dense.*

(2) $[\![(\exists s\in Q)s\notin a]\!]=\Omega$ *if and only if* $\bigcup_{n\in N}\{\omega\in\Omega\mid f(\omega)\leqq n\}^\circ$ *is dense.*

PROOF. (1): By the easy computations, we have

$$\begin{aligned}\{\omega\in\Omega\mid f(\omega)=-\infty\}^{-\circ}&=(\bigcup_{s\in Q}\{\omega\in\Omega\mid s<f(\omega)\})^{-c}\\&=(\bigcup_{s\in Q}\{\omega\in\Omega\mid s<f(\omega)\}^{-\circ})^{-c}\ , \quad \text{by Lemma 3.2}\end{aligned}$$

$$=[\![(\exists s\in \boldsymbol{Q})s\in a]\!]^{-c}$$

$$=[\![\neg(\exists s\in \boldsymbol{Q})s\in a]\!].$$

Thus $\{\omega\in\Omega \mid f(\omega)=-\infty\}$ is nowhere dense if and only if $[\![(\exists s\in \boldsymbol{Q})s\in a]\!]=\Omega$.
(2): By the easy computations, we have

$$\begin{aligned}[\![(\exists s\in \boldsymbol{Q})s\notin a]\!] &=(\bigcup_{s\in \boldsymbol{Q}}\{\omega\in\Omega \mid s<f(\omega)\})^{c\circ-\circ})^{-\circ}\\ &=(\bigcup_{s\in \boldsymbol{Q}}\{\omega\in\Omega \mid s<f(\omega)\}^{c\circ})^{-\circ}, \quad \text{by Lemma 3.2}\\ &=(\bigcup_{s\in \boldsymbol{Q}}\{\omega\in\Omega \mid f(\omega)\leqq s\}^{\circ})^{-\circ}\\ &=(\bigcup_{n\in \boldsymbol{N}}\{\omega\in\Omega \mid f(\omega)\leqq n\}^{\circ})^{-\circ}.\end{aligned}$$

It follows that $[\![(\exists s\in \boldsymbol{Q})s\notin a]\!]=\Omega$ if and only if $\bigcup_{n\in \boldsymbol{N}}\{\omega\in\Omega \mid f(\omega)\leqq s\}^{\circ}$ is dense. QED

Let $LC(\Omega)$ be the space of all extended real valued lower semicontinuous functions on Ω satisfying that

(1) $\{\omega\in\Omega \mid f(\omega)=-\infty\}$ is nowhere dense,

(2) $\bigcup_{n\in \boldsymbol{N}}\{\omega\in\Omega \mid f(\omega)\leqq n\}^{\circ}$ is dense.

THEOREM 3.4. *The relation*

$$[\![\check{s}\in a]\!]=\{\omega\in\Omega \mid s<f(\omega)\}^{-\circ}, \tag{3.1}$$

for all $s\in \boldsymbol{Q}$, *sets up a surjective mapping* $\Phi: f\mapsto a$ *from* $LC(\Omega)$ *onto* $\boldsymbol{R}^{(B)}$ *such that* $\Phi(f)=\Phi(g)$ *if and only if* $\{\omega\in\Omega \mid f(\omega)\neq g(\omega)\}$ *is meager. Moreover, f is bounded if and only if* $\Phi(f)\in \boldsymbol{R}_{\infty}^{(B)}$.

PROOF. Let $f\in LC(\Omega)$. Let $a\in \boldsymbol{V}^{(B)}$ be such that $\mathrm{dom}(a)=\mathrm{dom}(\check{\boldsymbol{Q}})$ and that $a(\check{s})=\{\omega\in\Omega \mid s<f(\omega)\}^{-\circ}$ for any $s\in \boldsymbol{Q}$. Then obviously, $[\![\check{s}\in a]\!]=\{\omega\in\Omega \mid s<f(\omega)\}^{-\circ}$ for any $s\in \boldsymbol{Q}$. Then we have

$$\begin{aligned}[\![\check{s}\in a]\!] &=\{\omega\in\Omega \mid s<f(\omega)\}^{-\circ}\\ &=(\bigcup_{s<t\in \boldsymbol{Q}}\{\omega\in\Omega \mid t<f(\omega)\})^{-\circ}\\ &=(\bigcup_{s<t\in \boldsymbol{Q}}[\![\check{t}\in a]\!])^{-\circ}, \quad \text{by Lemma 3.2}\\ &=[\![(\exists t\in \boldsymbol{Q})[\check{s}<t\wedge t\in a]]\!],\end{aligned}$$

whence $[\![(\forall s\in \boldsymbol{Q})s\in a\Leftrightarrow(\exists t\in \boldsymbol{Q})[s<t\wedge t\in a]]\!]=\Omega$. Thus by Lemma 3.3, we have $[\![a$ is a real number$]\!]=\Omega$, and hence there is a unique $\Phi(f)\in \boldsymbol{R}^{(B)}$ satisfying the relation (3.1). Conversely, let $a\in \boldsymbol{R}^{(B)}$. Let $f:\Omega\to\overline{\boldsymbol{R}}$ be an extended real valued function such that

$$f(\omega)=\sup\{s\in \boldsymbol{Q} \mid \omega\in\bigcup_{s<t\in \boldsymbol{Q}}[\![\check{t}\in a]\!]\},$$

for all $\omega \in \boldsymbol{Q}$. Then for any $r \in \boldsymbol{Q}$ and $\omega \in \Omega$,

$$\begin{aligned} r<f(\omega) &\Longleftrightarrow r<\sup\{s\in\boldsymbol{Q} \mid \omega\in \bigcup_{s<t\in\boldsymbol{Q}} [\![\check{t}\in a]\!]\} \\ &\Longleftrightarrow (\exists s,\, t\in\boldsymbol{Q})[r<s<t \wedge \omega\in[\![\check{t}\in a]\!]] \\ &\Longleftrightarrow (\exists t\in\boldsymbol{Q})[r<t\wedge\omega\in[\![\check{t}\in a]\!]] \\ &\Longleftrightarrow \omega\in \bigcup_{r<t\in\boldsymbol{Q}} [\![\check{t}\in a]\!]\,, \end{aligned}$$

whence $\{\omega\in\Omega \mid s<f(\omega)\} = \bigcup_{s<t\in\boldsymbol{Q}} [\![t\in a]\!]$. Thus f is lower semicontinuous. Since $a\in\boldsymbol{R}^{(\boldsymbol{B})}$, $\{\omega\in\Omega \mid s<f(\omega)\}^{-\circ}=[\![\check{s}\in a]\!]$ for any $s\in\boldsymbol{Q}$. By Lemma 3.3, $f\in LC(\Omega)$. It follows that Φ is surjective. By the relation (3.1), it is obvious that $\Phi(f)=\Phi(g)$ if and only if $\{\omega\in\Omega \mid f(\omega)\neq g(\omega)\}$ is meager. QED

A topological space Ω is called a *Baire space* if every meager open subset is empty. All locally compact Hausdorff spaces and complete metric spaces are Baire spaces. Let $\mathcal{B}(\Omega)$ be the space of all complex valued Borel functions on Ω and let $\mathcal{N}(\Omega)$ be the space of all functions in $\mathcal{B}(\Omega)$ vanishing outside a meager Borel set. Then $\mathcal{B}(\Omega)$ is a $*$-algebra and $\mathcal{N}(\Omega)$ is a $*$-ideal of $\mathcal{B}(\Omega)$ by the pointwise operations. Let $B(\Omega)$ be the quotient space $\mathcal{B}(\Omega)/\mathcal{N}(\Omega)$. On the other hand, $C^{(\boldsymbol{B})}$ is also a $*$-algebra by the operations defined in $\boldsymbol{V}^{(\boldsymbol{B})}$.

THEOREM 3.5. *Let Ω be a Baire space. Then there is a $*$-isomorphism between $B(\Omega)$ and $\boldsymbol{C}^{(\boldsymbol{B})}$.*

PROOF. Since $\boldsymbol{C}^{(\boldsymbol{B})}=\boldsymbol{R}^{(\boldsymbol{B})}+\check{i}\boldsymbol{R}^{(\boldsymbol{B})}$, we have only to show the existence of an isomorphism between the real part of $B(\Omega)$ and $\boldsymbol{R}^{(\boldsymbol{B})}$. Let $f:\Omega\to\boldsymbol{R}$ be a Borel function. For any $s\in\boldsymbol{Q}$, let B_s be a unique regular open set such that $B_s\sim\{\omega\in\Omega \mid s<f(\omega)\}$. Let $g:\Omega\to\boldsymbol{R}$ be such that

$$g(\omega)=\sup\{s\in\boldsymbol{Q} \mid \omega\in \bigcup_{s<t\in\boldsymbol{Q}} B_t\}\,.$$

Then $\{\omega\in\Omega \mid s<g(\omega)\} = \bigcup_{s<t\in\boldsymbol{Q}} B_t$ and g is an extended real valued lower semicontinuous function. We have

$$\begin{aligned} \{\omega\in\Omega \mid g(\omega)=-\infty\}^{-\circ} &= (\bigcap_{s\in\boldsymbol{Q}} B_s^c)^\circ \\ &\sim \bigcap_{s\in\boldsymbol{Q}} \{\omega\in\Omega \mid f(\omega)\leqq s\}=\emptyset\,, \end{aligned}$$

whence $\{\omega\in\Omega \mid g(\omega)=-\infty\}^{-\circ}=\emptyset$ by the Baire property of Ω. We have

$$\begin{aligned} (\bigcup_n \{\omega\in\Omega \mid g(\omega)\leqq n\}^\circ)^- &= (\bigcup_n B_n^{c\circ})^- \\ &\sim \bigcup_n \{\omega\in\Omega \mid f(\omega)\leqq n\}=\Omega\,, \end{aligned}$$

whence, $\bigcup_n \{\omega\in\Omega \mid g(\omega)\leqq n\}^\circ$ is dense by the Baire property of Ω. It follows that $g\in LC(\Omega)$. Conversely, let $g\in LC(\Omega)$. Then

$$\{\omega\in\Omega\,|\,g(\omega)=\infty\}=(\bigcup_n\{\omega\in\Omega\,|\,g(\omega)\leqq n\})^c$$
$$\sim(\bigcup_n\{\omega\in\Omega\,|\,g(\omega)\leqq n\}^{\circ})^{c\circ}=\emptyset\,,$$

whence $\{\omega\in\Omega\,|\,g(\omega)=\infty \text{ or } g(\omega)=-\infty\}$ is a meager Borel set. Thus there is a real valued Borel function f which differs from g only on a meager set. It follows that the correspondence Φ obtained in Theorem 3.4 can be extended to all real valued Borel functions in such a way that $\Phi(f)=\Phi(g)$ if and only if $\{\omega\in\Omega\,|\,f(\omega)\neq g(\omega)\}$ is meager. Now it is easy to see that this extension induces an isomorphism between the real part of $B(\Omega)$ and $\boldsymbol{R}^{(\boldsymbol{B})}$. QED

A topological space is called a *Stonean space* if it is a compact Hausdorff space in which the closure of every open set is open. In Stonean spaces every regular open set is clopen and the Stone representation space of the complete Boolean algebra $\boldsymbol{B}$ of regular open subsets is homeomorphic to the original space. Denote by $C(\Omega)$ the space of all complex valued continuous functions on Ω. If Ω is a Stonean space then $C(\Omega)$ is a commutative AW^*-algebra. Conversely, the maximal ideal space Ω of any commutative AW^*-algebra Z is a Stonean space and by the Gelfand isomorphism Z is $*$-isomorphic to $C(\Omega)$. In this case, the Stone representation space of the complete Boolean algebra of projections in Z is also homeomorphic to Ω.

THEOREM 3.6. *Let Ω be a Stonean space. Then there is a $*$-isomorphism Φ between $C(\Omega)$ and $\boldsymbol{C}_\infty^{(\boldsymbol{B})}$ satisfying*

$$\{\omega\in\Omega\,|\,s<f(\omega)\}^{-\circ}=[\![\check{s}\in\Phi(f)]\!]\,,$$

for all real valued $f\in C(\Omega)$ and $s\in\boldsymbol{Q}$.

PROOF. It is known [**11**; p. 104] that every bounded real valued lower semicontinuous function on a Stonean space coincides with a unique continuous function except on a meager set. Thus the restriction of Φ obtained in Theorem 3.4 on bounded real valued continuous functions is one-to-one and onto $\boldsymbol{R}_\infty^{(\boldsymbol{B})}$. In this case, obviously Φ is an isomorphism and its complexification is the required $*$-isomorphism. QED

COROLLARY 3.7. *Every commutative AW^*-algebra Z is $*$-isomorphic to $\boldsymbol{C}_\infty^{(\boldsymbol{B})}$, where $\boldsymbol{B}$ is the complete Boolean algebra of projections in Z.*

PROOF. Immediate from Theorem 3.6. QED

In the sequel, we shall denote by $\hat{Z}$ the $*$-algebra $\boldsymbol{C}^{(\boldsymbol{B})}$, where $\boldsymbol{B}$ is the complete Boolean algebra of projections in a commutative AW^*-algebra Z. Then Z is a $*$-subalgebra of $\hat{Z}$ and coincides with the bounded part of $\hat{Z}$.

4. AW^*-modules and Hilbert spaces in $V^{(B)}$.

Let Z be a commutative AW^*-algebra and $\boldsymbol{B}$ the complete Boolean algebra of projections in Z. We identify Z with $C_\infty^{(B)}$ and $\hat{Z}$ with $C^{(B)}$; only notations Z and $\hat{Z}$ will be used hereafter.

A unital Z-module X is called a *pre-C^*-module* if there is defined on X a Z-valued inner product such that

(1) $(x, y)=(y, x)^*$,

(2) $(x, x)\geqq 0$ and is 0 only for $x=0$,

(3) $(ax+y, z)=a(x, z)+(y, z)$,

for all $x, y, z\in X$ and $a\in Z$. A pre-C^*-module X over Z is called a *C^*-module* if it is complete with respect to the norm $\|\cdot\|$ on X defined by $\|x\|=\|(x, x)\|^{1/2}$ for all $x\in X$. A pre-C^*-module X is called *self-dual* if every bounded Z-linear map $f: X\to Z$ is of the form $f(\cdot)=(\cdot, y)$ for some y in X. A C^*-module X over Z is called an *AW^*-module* if for any partition $\{b_\alpha\}$ of unity in $\boldsymbol{B}$ and norm bounded family $\{x_\alpha\}$ in X there is in X an element x with $b_\alpha x=b_\alpha x_\alpha$ for any α (cf. [**7**]). It should be remarked that the condition (a) in the definition of AW^*-modules in [**7**; p. 482] is satisfied automatically for any pre-C^*-module over Z. It is easily shown that every self-dual pre-C^*-module is a C^*-module. It is shown in [**7**; Theorem 5] that every AW^*-module is self-dual.

Let X and Y be two C^*-modules over Z. Denote by Hom(X, Y) the space of all bounded Z-linear maps T which possess the bounded adjoints with respect to Z-valued inner product, and denote Hom(X, X) by End(X). A Z-linear map T is called *unitary* if it preserves the Z-valued inner product. It is known [**9**; Proposition 3.4] that if X is self-dual then Hom(X, Y) is the space of all bounded Z-linear maps. If X is an AW^*-module then End(X) is a type I AW^*-algebra with center isomorphic to Z and every type I AW^*-algebra with center isomorphic to Z arises in this way [**7**].

Let H be a Hilbert space in $V^{(B)}$, i.e., $[\![H$ is a Hilbert space$]\!]=1$. Denote by $(\cdot, \cdot)_B$ the inner product on H in $V^{(B)}$ and by $\|\cdot\|_B$ the norm on H in $V^{(B)}$. Let $H^{(B)}$ be the interpretation of H and $H_\infty^{(B)}$ the bounded part of $H^{(B)}$. Then

$$H^{(B)}=\{x\in V^{(B)} \mid [\![x\in H]\!]=1\},$$

$$H_\infty^{(B)}=\{x\in H^{(B)} \mid \|x\|_B\in Z\},$$

and for any $x, y\in H^{(B)}$, $(x, y)_B\in\hat{Z}$ and $\|x\|_B\in\hat{Z}$. By Lemma 3.1, we have $[\![H_\infty^{(B)}\times\{1\}=H]\!]=1$.

Let W be another Hilbert space in $V^{(B)}$. We can identify $\boldsymbol{B}$-valued sets f such that $[\![f: H\to W]\!]=1$ with the corresponding extensional maps $\tilde{f}: H^{(B)}\to W^{(B)}$

such that $[\![\tilde{f}(x)=f(x)]\!]=1$ for all $x\in H^{(B)}$. Let $\mathcal{L}(H, W)^{(B)}$ be the set of all bounded linear maps $f: H\to W$ in $V^{(B)}$. Denote by $\|f\|_B$ the operator bound of $f\in\mathcal{L}(H, W)^{(B)}$ in $V^{(B)}$. Then it is easy to see that

$$\|f\|_B=\inf\{a\in \boldsymbol{R}^{(B)} \mid \|f(x)\|_B\leqq a\|x\|_B \text{ for all } x \text{ in } H^{(B)}\}.$$

Let $\mathcal{L}(H, W)_{\infty}^{(B)}$ be the set of all $f\in\mathcal{L}(H, W)^{(B)}$ such that $\|f\|_B\in Z$.

THEOREM 4.1. *Let H and W be Hilbert spaces in $V^{(B)}$. Then we have the following.*

(1) *$H_{\infty}^{(B)}$ (and $W_{\infty}^{(B)}$) is an AW^*-module.*

(2) *Every bounded Z-linear map $f: H_{\infty}^{(B)}\to W_{\infty}^{(B)}$ can be uniquely extended to $f\in\mathcal{L}(H, W)_{\infty}^{(B)}$ and every $f\in\mathcal{L}(H, W)_{\infty}^{(B)}$ arises in this way.*

(3) *Under the extension of* (2), $\mathrm{Hom}(H_{\infty}^{(B)}, W_{\infty}^{(B)})\cong\mathcal{L}(H, W)_{\infty}^{(B)}$.

PROOF. Let $x, y\in H_{\infty}^{(B)}$ and $a\in Z$. Then $|(x, y)_B|\leqq\|x\|_B\|y\|_B$, $\|ax\|_B\leqq |a|_B\|x\|_B$ and $\|x+y\|_B\leqq\|x\|_B+\|y\|_B$ by the direct interpretation. Thus it is easy to see that $H_{\infty}^{(B)}$ is a pre-C^*-module. Let $f: H_{\infty}^{(B)}\to W_{\infty}^{(B)}$ be a bounded Z-linear map. Then by Lemma 2.3, f is extensional, and hence we have a unique extension $\tilde{f}: H^{(B)}\to W^{(B)}$ of f. Since $H_{\infty}^{(B)}$ and $W_{\infty}^{(B)}$ are pre-C^*-modules, we have $(f(x), f(x))_B\leqq\|f\|^2(x, x)_B$ for any x in $H_{\infty}^{(B)}$ from [**9**; Theorem 2.8]. Thus it is easy to see that $f\in\mathcal{L}(H, W)_{\infty}^{(B)}$. Conversely, let $f\in\mathcal{L}(H, W)_{\infty}^{(B)}$. Then for any $x\in H_{\infty}^{(B)}$, we have $(f(x), f(x))_B\leqq\|f\|_B^2(x, x)_B$, so that $f(x)\in W_{\infty}^{(B)}$ and f is a bounded Z-linear map. This concludes (2). Let $f: H_{\infty}^{(B)}\to Z$ be a bounded Z-linear map. Then $f\in\mathcal{L}(H, C)_{\infty}^{(B)}$ by (2). Thus by the Riesz Theorem in $V^{(B)}$, there is some $y\in H^{(B)}$ such that $f(x)_B=(x, y)_B$ for any x in $H^{(B)}$. Since f is bounded, it is easy to see that $y\in H_{\infty}^{(B)}$. It follows that $H_{\infty}^{(B)}$ is self-dual and hence it is a C^*-module. Since $H_{\infty}^{(B)}$ is self-dual, every bounded Z-linear map has its bounded adjoint so that we have $\mathrm{Hom}(H_{\infty}^{(B)}, W_{\infty}^{(B)})=\mathcal{L}(H, W)_{\infty}^{(B)}$ by (2). Thus (3) holds. Let $\{b_\alpha\}$ be a partition of unity in $\boldsymbol{B}$ and $\{x_\alpha\}$ be a norm bounded family in $H^{(B)}$. Let $x=\sum_{\alpha} x_\alpha b_\alpha$. Then it is easily seen that $b_\alpha x=b_\alpha x_\alpha$ for any α. Let $r\in\boldsymbol{R}$ be such that $\sup_\alpha\|x_\alpha\|_B\leqq r$. Then $[\![\|x_\alpha\|\leqq\check{r}]\!]=1$ and hence $[\![\|x\|\leqq\check{r}]\!]\geqq b_\alpha$ for any α. Thus $x\in H_{\infty}^{(B)}$. Therefore, $H_{\infty}^{(B)}$ is an AW^*-module and hence (1) holds. QED

5. Construction of Hilbert spaces in $V^{(B)}$.

Let S be a set. A kernel $K: S\times S\to\boldsymbol{C}$ is called *positive definite* if the following inequality

$$\sum_{i,j=1}^{n} c_i\bar{c}_j K(x_i, x_j)\geqq 0 \tag{5.1}$$

holds for every choice of $x_1, \cdots, x_n$ in S and for every choice of complex numbers $c_1, \cdots, c_n$. In the sequel, we call a function $K: S\times S\to\hat{Z}$ as a *$\hat{Z}$-kernel* and identify it with $\tilde{K}\in \boldsymbol{V}^{(\boldsymbol{B})}$ such that $[\![\tilde{K}:\check{S}\times\check{S}\to\boldsymbol{C}]\!]=1$ and that $[\![\tilde{K}(\check{x}, \check{y})=K(x, y)]\!]=1$ for any $x, y\in S$. A $\hat{Z}$-kernel K is called *positive definite* if the following inequality

$$\sum_{i,j=1}^{n} c_i\overline{c_j}K(x_i, x_j)\geqq 0 \tag{5.2}$$

holds for every choice of $x_1, \cdots, x_n\in S$, and for every choice of complex numbers $c_1, \cdots, c_n$.

PROPOSITION 5.1. *A $\hat{Z}$-kernel K on $S\times S$ is a positive definite kernel in $\boldsymbol{V}^{(\boldsymbol{B})}$, i.e., $[\![\tilde{K}:\check{S}\times\check{S}\to\boldsymbol{C}$ is positive definite$]\!]=1$, if and only if K is a positive definite $\hat{Z}$-kernel.*

PROOF. The necessity is obvious. To prove the sufficiency, note that a $\boldsymbol{C}$-valued kernel is positive definite if the inequality (5.1) holds for rational complex numbers $c_1, \cdots, c_n$. Since $[\![\check{\boldsymbol{Q}}+\check{i}\check{\boldsymbol{Q}}=\boldsymbol{Q}+i\boldsymbol{Q}]\!]=1$ by the absoluteness of rational numbers, the sufficiency follows from the partition argument (i.e., for rational complex numbers $c_1, \cdots, c_n$ in $\boldsymbol{V}^{(\boldsymbol{B})}$ there is a partition $\{b_\alpha\}$ of unity and rational complex numbers $\{d_{1\alpha}, \cdots, d_{n\alpha}\}$ for any α such that

$$\left[\!\!\left[\sum_{i,j=1}^{n} c_i\overline{c_j}K(x_i, x_j)=\sum_{i,j=1}^{n} d_{i\alpha}\overline{d_{j\alpha}}K(x_i, x_j)\right]\!\!\right]\geqq b_\alpha .\Big)$$

QED

It should be noticed that the above proposition relaxes the condition for positive definiteness appeared in [**13**; Ch. 4, Theorem 1.4].

THEOREM 5.2. *Let K be a positive definite $\hat{Z}$-kernel on $S\times S$. For any $x\in S$, define $\tilde{x}\in\boldsymbol{V}^{(\boldsymbol{B})}$ by the relations*

$$\mathrm{dom}(\tilde{x})=\mathrm{dom}(\check{S}) \qquad \textit{and} \quad \tilde{x}(\check{y})=[\![K(x, x)+K(y, y)=2\,\mathcal{R}_e K(x, y)]\!]$$

for any $y\in S$. Let $\tilde{S}$ be such that $\tilde{S}=\{\tilde{x}\mid x\in S\}\times\{1\}$. Then there is a Hilbert space H in $\boldsymbol{V}^{(\boldsymbol{B})}$ such that $[\![\tilde{S}\subseteqq H]\!]=1$ and that $(\tilde{x}, \tilde{y})_{\boldsymbol{B}}=K(x, y)$ for any $x, y\in S$.

PROOF. Since $\tilde{K}$ is a positive definite kernel in $\boldsymbol{V}^{(\boldsymbol{B})}$, by the interpretation of the usual theory of positive definite kernels [1], we have

$$[\![(\exists H)(\exists J)\ H \text{ is a Hilbert space}$$
$$\wedge J:\check{S}\to H\wedge(\forall x, y\in\check{S})\tilde{K}(x, y)=(J(x), J(y))_{\boldsymbol{B}}]\!]=1 .$$

Thus it is sufficient to show that there is a one-to-one correspondence between $\tilde{S}$ and $\{J(x)\mid x\in\check{S}\}$ in $\boldsymbol{V}^{(\boldsymbol{B})}$. To show this we have only to prove that $[\![J(\check{x})=J(\check{y})]\!]=[\![\tilde{x}=\tilde{y}]\!]$ for any $x, y\in X$. It is easy to see that

$$\tilde{x}(\check{y})=[\![(J(\check{x})-J(\check{y}), J(\check{x})-J(\check{y}))_{\boldsymbol{B}}=0]\!]$$
$$=[\![J(\check{x})=J(\check{y})]\!]=0$$

for any $x, y \in S$. Thus we have

$$
\begin{aligned}
&[\![J(\check{x})=J(\check{y})]\!] \\
&\quad = [\![(\forall z \in \check{S})[J(\check{x})=J(z) \Leftrightarrow J(\check{y})=J(z)]]\!] \\
&\quad = \inf_{z \in S}([\![J(\check{x})=J(\check{z})]\!] \Leftrightarrow [\![J(\check{y})=J(\check{z})]\!]) \\
&\quad = \inf_{z \in S}(\tilde{x}(\check{z}) \Leftrightarrow \tilde{y}(\check{z})) \\
&\quad = [\![\tilde{x}=\tilde{y}]\!],
\end{aligned}
$$

for any $x, y \in S$. QED

We call $\tilde{S} \in V^{(B)}$ obtained in the above theorem the *Boolean embedding* of S with respect to a positive definite $\hat{Z}$-kernel K, and $J: S \to H$ the *embedding map*.

Obviously, a Z-valued inner product of a pre-C^*-module is a positive definite $\hat{Z}$-kernel. Applying Theorem 5.2 to it, we have that for any pre-C^*-module X there is a Hilbert space H in $V^{(B)}$ such that $[\![\tilde{X} \subseteq H]\!]=1$.

THEOREM 5.3. *Let X be a self-dual C^*-module over Z and $\tilde{X}$ be its Boolean embedding with respect to the Z-valued inner product of X. Then $\tilde{X}$ is a Hilbert space in $V^{(B)}$ and the relation $U_X x=\tilde{x}$ for any $x \in X$ defines a unitary Z-linear map U_X from X onto $\tilde{X}_\infty^{(B)}$.*

PROOF. Denote by K the Z-valued inner product of X. Let H be a Hilbert space in $V^{(B)}$ obtained in Theorem 5.2. Since $(\tilde{x}, \tilde{y})_B=K(x, y)$ for any $x, y \in X$, it is easy to check that the addition and action of Z on X coincide with those of $\tilde{X}$. Thus, we have $[\![\tilde{X}$ is a linear subspace of $H]\!]=1$, under the operation defined on X. Let $f \in V^{(B)}$ be a bounded linear functional on $\tilde{X}$ in $V^{(B)}$. Then there is a partition $\{b_\alpha\}$ of unity such that $\|b_\alpha f\|_B \in Z$ for any α. It is easy to see that the function $g: X \to Z$ defined by $g(x)=b_\alpha f(x)$ for $x \in X$ is a bounded Z-linear map on X. By the self-duality of X there is some $y_\alpha \in X$ such that $g(x)=K(x, y_\alpha)$ for any $x \in X$. It follows that $[\![\tilde{y}_\alpha \in \tilde{X}]\!]=1$ and $[\![(\forall x \in \tilde{X}) f(x)=(x, \tilde{y}_\alpha)_B]\!] \geqq b_\alpha$ for any α. This shows that for $y=\sum_\alpha \tilde{y}_\alpha b_\alpha$, we have $[\![y \in \tilde{X}$ and $(\forall x \in \tilde{X}) f(x)=(x, y)_B]\!]=1$. Therefore $\tilde{X}$ is a self-dual inner product space in $V^{(B)}$ and hence $[\![\tilde{X}$ is a Hilbert space$]\!]=1$. Since it is obvious that U_X is a unitary Z-linear map from X into $X_\infty^{(B)}$, we have only to show that U_X is surjective. Let $y \in X_\infty^{(B)}$. Then obviously, the function f such that $f(x)=(\tilde{x}, y)_B$ for $x \in X$ is a bounded Z-linear map on X. Thus there is some $z \in X$ such that $(\tilde{x}, y)_B=K(x, z)=(\tilde{x}, \tilde{z})_B$ for any $x \in X$. This shows that $[\![z=y]\!]=1$ and hence U_X is surjective. QED

By the above theorem we obtain the following characterization of AW^*-modules.

THEOREM 5.4. *A pre-C*-module over Z is an AW^*-module if and only if it is self-dual.*

PROOF. The necessity is proved in [**7**; Theorem 5]. The sufficiency follows immediately from Theorem 4.2, (1) and Theorem 5.3. QED

Let X and Y be two AW^*-modules, $\tilde{X}$ and $\tilde{Y}$ be their Boolean embeddings, respectively. In general, the set $\tilde{X}_{\infty}^{(B)}$ depends on the selection of the representatives from $\{v\in V^{(B)} \mid [\![u=v]\!]=1\}$ for $u\in V^{(B)}$ such that $[\![u\in\tilde{X}]\!]=1$. However, by the proof of Theorem 5.3, for any $u\in\tilde{X}_{\infty}^{(B)}$ there is exactly one $x\in X$ such that $[\![u=\tilde{x}]\!]=1$. Thus we can choose the representatives in such a way that $\tilde{X}_{\infty}^{(B)}=\mathrm{dom}(\tilde{X})$. By the same reason, we can suppose that $\tilde{Y}_{\infty}^{(B)}=\mathrm{dom}(\tilde{Y})$. Let $T\in\mathrm{Hom}(X, Y)$ and define a map $T_0:\mathrm{dom}(\tilde{X})\to\mathrm{dom}(\tilde{Y})$ by the relation $T_0\tilde{x}=(Tx)^{\sim}$ for any $x\in X$. Then $T_0=U_Y T U_X^{-1}$ and that $T_0\in\mathrm{Hom}(X_{\infty}^{(B)}, Y_{\infty}^{(B)})$ by Theorem 5.3. Therefore, from Theorem 4.1, (3) there is a unique $\tilde{T}\in\mathcal{L}(\tilde{X}, \tilde{Y})^{(B)}$ such that $[\![\tilde{T}\tilde{x}=(Tx)^{\sim}]\!]=1$ for any $x\in X$. We call this $\tilde{T}$ the *Boolean embedding* of T. By [**9**; Remark 2.9], we have $\|T\|=\|\,\|\tilde{T}\|_B\|=\inf\{r\in\boldsymbol{R} \mid \|\tilde{T}\|_B\leqq r\}$ and it is easy to see that $(T^*)^{\sim}=(\tilde{T})^*$.

Let $\mathbf{Hilbert}_{\infty}^{(B)}$ be the category of Hilbert spaces in $V^{(B)}$ and bounded linear maps f in $V^{(B)}$ such that $\|f\|_B\in Z$ and let $\boldsymbol{AW^*}\text{-}\mathbf{mod}^Z$ be the category of AW^*-modules over Z and bounded Z-linear maps. Then the following theorem summarizes the functorial properties of Boolean embeddings.

THEOREM 5.5. *The Boolean embedding $E:X\to\tilde{X}$, $E:T\to\tilde{T}$ is a functor from $\boldsymbol{AW^*}\text{-}\mathbf{mod}^Z$ to $\mathbf{Hilbert}_{\infty}^{(B)}$ which is an equivalence of these two categories, and which satisfies the following properties:*

(1) $E(aT+S)=aE(T)+E(S)$,

(2) $E(T^*)=E(T)^*$,

(3) $\|\,\|E(T)\|_B\|=\|T\|$,

(4) *T is a unitary Z-linear map if and only if*

$$[\![E(T) \text{ is a unitary transformation}]\!]=1,$$

for any $a\in Z$, and $T, S\in\mathrm{Hom}(X, Y)$ and $X, Y\in\boldsymbol{AW^*}\text{-}\mathbf{mod}^Z$. Its adjoint functor is $R:H\to H_{\infty}^{(B)}$, $R:f\to f|_{H_{\infty}^{(B)}}$ obtained in Theorem 4.1, (3). The natural isomorphism $RE=1$ on $\boldsymbol{AW^*}\text{-}\mathbf{mod}^Z$ is $\{U_X \mid X\in\boldsymbol{AW^*}\text{-}\mathbf{mod}^Z\}$ obtained in Theorem 5.3.

Consider the case that a commutative AW^*-algebra Z is a von Neumann algebra. As in [**10**] and [8], we denote by **Normod-**Z the category of non-degenerate normal $*$-representations of Z on Hilbert spaces and bounded intertwining operators. Then we have the following.

THEOREM 5.6. *Let Z be a commutative von Neumann algebra. Then the two categories $\boldsymbol{AW^*}\text{-}\mathbf{mod}^Z$ and **Normod-**Z are equivalent.*

PROOF. By [8; Theorem 5.5], we have obtained that **Hilbert**$_\infty^{(B)}$ is equivalent to **Normod-**Z. Thus the assertion follows immediately from Theorem 5.5. QED

6. A classification of AW^*-modules.

Let Z be a commutative AW^*-algebra and $\boldsymbol{B}$ be the complete Boolean algebra of projections in Z.

For any set S in $\boldsymbol{V}^{(\boldsymbol{B})}$, denote by $\mathrm{card}(S)_{\boldsymbol{B}}$ the cardinality of S in $\boldsymbol{V}^{(\boldsymbol{B})}$. For any Hilbert space H in $\boldsymbol{V}^{(\boldsymbol{B})}$, denote by $\dim(H)_{\boldsymbol{B}}$ the dimension of H in $\boldsymbol{V}^{(\boldsymbol{B})}$. By the direct interpretation, we have that $\mathrm{card}(S)_{\boldsymbol{B}}=\mathrm{card}(S')_{\boldsymbol{B}}$ if and only if $[\![$There is a bijection from S to $S']\!]=1$ and that $\dim(H)_{\boldsymbol{B}}=\dim(W)_{\boldsymbol{B}}$ if and only if $[\![$There is a unitary transformation from H onto $W]\!]=1$.

We say that two AW^*-modules X and Y over Z are *unitarily equivalent* if there is a unitary Z-linear map from X onto Y. Denote by $l^2(S)$ the l^2-space of S, i.e., the set of square summable complex valued functions on S. Denote by $l^2(S)^{(\boldsymbol{B})}$ the interpretation of $l^2(S)$ and $l^2(S)_\infty^{(\boldsymbol{B})}$ the bounded part of $l^2(S)^{(\boldsymbol{B})}$, i.e.,

$$l^2(S)^{(\boldsymbol{B})}=\{x\in \boldsymbol{V}^{(\boldsymbol{B})} \mid [\![x: S\to \boldsymbol{C} \text{ and } \sum_{s\in S}|x(s)|^2<\infty]\!]=1\},$$

$$l^2(S)_\infty^{(\boldsymbol{B})}=\{x\in l^2(S)^{(\boldsymbol{B})} \mid \exists M\in \boldsymbol{R},\ [\![\sum_{s\in S}|x(s)|^2<\check{M}]\!]=1\}.$$

By the direct interpretation, $l^2(S)^{(\boldsymbol{B})}$ is a Hilbert space in $\boldsymbol{V}^{(\boldsymbol{B})}$ and by Theorem 4.2, $l^2(S)_\infty^{(\boldsymbol{B})}$ is an AW^*-module over Z.

Now we shall obtain a complete system of unitary invariants of AW^*-modules

For any AW^*-module X over Z, denote by $\mathrm{Dim}(X)$ the *Z-dimension* of X defined by

$$\mathrm{Dim}(X)=\dim(\tilde{X})_{\boldsymbol{B}},$$

where $\tilde{X}$ is the Boolean embedding of X (cf. Theorem 5.2, Theorem 5.3).

THEOREM 6.1. *Two AW^*-modules are unitarily equivalent if and only if they have the same Z-dimension. For any cardinal α in $\boldsymbol{V}^{(\boldsymbol{B})}$, there is an AW^*-module X whose Z-dimension is α.*

PROOF. Let X and Y be AW^*-modules over Z. Suppose that there is a unitary Z-linear map U from X onto Y. Then it follows from Theorem 5.5,

$$[\![U \text{ is a unitary transformation from } \tilde{X} \text{ onto } \tilde{Y}]\!]=1,$$

and hence $\dim(\tilde{X})_{\boldsymbol{B}}=\dim(\tilde{Y})_{\boldsymbol{B}}$. Thus $\mathrm{Dim}(X)=\mathrm{Dim}(Y)$. Conversely, suppose that $\mathrm{Dim}(X)=\mathrm{Dim}(Y)$. Then we have

$$[\![(\exists U)\ U \text{ is a unitary transformation from } \tilde{X} \text{ onto } \tilde{Y}]\!]=1.$$

Let T be such that $T=U|_{X_\infty^{(B)}}$. Then by Theorem 5.5, we have that $U_Y^{-1}TU_X$ is a unitary Z-linear map from X onto Y. Thus X and Y are unitarily equivalent. The rest of the assertions follows immediately from the fact that $\mathrm{Dim}(l^2(\alpha)_\infty^{(B)})=\alpha$ for any cardinal α in $V^{(B)}$. QED

A *base* of an AW^*-module X over Z is a family $\{e_i\}$ such that (1) $\langle e_i, e_i\rangle=1$ for any i, (2) $\langle e_i, e_j\rangle=0$ if $i\neq j$, (3) for any $x\in X$, if $\langle x, e_i\rangle=0$ for any i then $x=0$. For a cardinal α, an AW^*-module is called *α-homogeneous* if it has a base with cardinality α.

THEOREM 6.2 *An AW^*-module X over Z is α-homogeneous if and only if* $\mathrm{Dim}(X)=\mathrm{card}(\check{\alpha})_B$.

PROOF. Let X be an α-homogeneous AW^*-module over Z, and $\{e_i\}$ be a base of X with cardinality α. Consider the Boolean embedding $\tilde{X}$ and the embedding map $J\in \boldsymbol{V}^{(B)}$ (cf. Theorem 5.2). Denote by K the $\hat{Z}$-valued inner product of $\tilde{X}$. Let e be the corresponding family in $\boldsymbol{V}^{(B)}$ to $\{e_i\}$, i.e., $[\![e:\check{\alpha}\to\check{X}]\!]=1$ and $[\![e(\check{i})=e_i]\!]=1$ for any $i\in\alpha$. By the proof of Theorem 5.2, $[\![J:\check{X}\to\tilde{X}]\!]=1$ and $[\![K(e_i, e_j)=(J(e(\check{i})), J(e(\check{j})))]\!]=1$ for any $i, j\in\alpha$. It follows that $\{J(e(\check{i}))\}$ is a base of $\tilde{X}$ indexed by $\check{\alpha}$ in $\boldsymbol{V}^{(B)}$. Thus $[\![\dim(\tilde{X})=\mathrm{card}(\check{\alpha})]\!]=1$. Conversely, let X be an AW^*-module over Z such that $\mathrm{Dim}(X)=\mathrm{card}(\check{\alpha})_B$. Let $e\in\boldsymbol{V}^{(B)}$ be a base of $\tilde{X}$ in $\boldsymbol{V}^{(B)}$. Then it is easy to see that $\{e(\check{i})\,|\,i\in\alpha\}$ is a base of the AW^*-module $\tilde{X}_\infty^{(B)}$ with cardinality α. Since X and $\tilde{X}_\infty^{(B)}$ are unitarily equivalent, X is α-homogeneous. QED

In [6], Kaplansky showed that the cardinality of the base of an AW^*-module over Z is unique if Z satisfies the countable chain condition locally but he conjectured that the uniqueness may fail otherwise [7; p. 844, footnote]. Now we shall construct examples in which the uniqueness fails, using some known results on forcing.

THEOREM 6.3. *For any pair of infinite cardinals α and β with $\alpha<\beta$, there is an AW^*-module which is γ-homogeneous for any cardinal γ such that $\alpha\leqq\gamma\leqq\beta$.*

PROOF. Let $\boldsymbol{P}$ be the set of all functions p such that

(1) $\mathrm{dom}(p)\subseteqq\alpha$ and $\mathrm{card}(\mathrm{dom}(p))<\alpha$,

(2) $\mathrm{ran}(p)\subseteqq\beta$,

and let $p\leqq q$ if and only if p is an extension of q for $p, q\in\boldsymbol{P}$. For any $p\in\boldsymbol{P}$, let $[p]$ be such that $[p]=\{q\in\boldsymbol{P}\,|\,q\leqq p\}$, and define the topology on $\boldsymbol{P}$ whose open base is $\{[p]\,|\,p\in\boldsymbol{P}\}$. Let $\boldsymbol{B}$ be the Boolean algebra of all regular open subsets of $\boldsymbol{P}$, and consider the Boolean valued universe $\boldsymbol{V}^{(B)}$. It is easily seen that $[p]\in\boldsymbol{B}$ for any $p\in\boldsymbol{P}$, and so we suppose that $\boldsymbol{P}\subseteqq\boldsymbol{B}$ by the identification $[p]$ with p. Let $F\in\boldsymbol{V}^{(B)}$ be such that $\mathrm{dom}(F)=\mathrm{dom}(\check{\boldsymbol{P}})$, and that $F(\check{p})=[p]$ for every $p\in\boldsymbol{P}$. Then by [4; Theorem 43 (b)],

$$[\![F \text{ is a generic filter of } \boldsymbol{P} \text{ over } \boldsymbol{M}]\!]=1,$$

where $\boldsymbol{M}$ is a predicate defined by $\boldsymbol{M}(a)=\sum_{x\in V}[\![a=\check{x}]\!]$. Since $[\![\boldsymbol{M}$ is a standard transitive model of ZFC$]\!]=1$, interpreting the forcing argument [**4**; p. 183], we have

$$[\![\bigcup F \text{ is a function from } \check{\alpha} \text{ onto } \check{\beta}]\!]=1\,.$$

Let γ be a cardinal such that $\alpha\leqq\gamma\leqq\beta$. Then $[\![\check{\alpha}\subseteqq\check{\gamma}\subseteqq\check{\beta}]\!]=1$. It follows that $[\![\mathrm{card}(\check{\alpha})=\mathrm{card}(\check{\gamma})=\mathrm{card}(\check{\beta})]\!]=1$. Let X be an AW^*-module such that $X=l^2(\check{\alpha})_\infty^{(\boldsymbol{B})}$. Then $\mathrm{Dim}(X)=\dim(l^2(\check{\alpha}))_{\boldsymbol{B}}=\mathrm{card}(\check{\alpha})_{\boldsymbol{B}}=\mathrm{card}(\check{\gamma})_{\boldsymbol{B}}$. Therefore the AW^*-module X is γ-homogeneous by Theorem 6.2. QED

COROLLARY 6.4. *For any pair of infinite cardinals α and β with $\alpha<\beta$, there is an AW^*-algebra which is γ-homogeneous for any cardinal γ such that $\alpha\leqq\gamma\leqq\beta$.*

PROOF. Since End(X) is an $\aleph$-homogeneous AW^*-algebra if X is an $\aleph$-homogeneous AW^*-module [**7**; Theorem 7], the assertion follows immediately from Theorem 6.3. QED

7. A classification of type I AW^*-algebras.

Let A be a type I AW^*-algebra, Z be its center, $\boldsymbol{B}$ be the complete Boolean algebra of all central projections in A.

Let π be an automorphism of $\boldsymbol{B}$. Then π can be extended to an automorphism $\pi: \boldsymbol{V}^{(\boldsymbol{B})}\to\boldsymbol{V}^{(\boldsymbol{B})}$ such that for every formula φ and $u_1, \cdots, u_n\in\boldsymbol{V}^{(\boldsymbol{B})}$,

$$\pi([\![\varphi(u_1, \cdots, u_n)]\!])=[\![\varphi(\pi(u_1), \cdots, \pi(u_n))]\!]\,,$$

(cf. [**16**; Theorem 19.3]). In particular, if α is a cardinal in $\boldsymbol{V}^{(\boldsymbol{B})}$, then $\pi(\alpha)$ is also a cardinal in $\boldsymbol{V}^{(\boldsymbol{B})}$. Two cardinals α and β in $\boldsymbol{V}^{(\boldsymbol{B})}$ are called *congruent*, if there is an automorphism π of $\boldsymbol{B}$ such that $[\![\alpha=\pi(\beta)]\!]=1$.

It was shown by Kaplansky [7] that a type I AW^*-algebra A with center isomorphic to Z is isomorphic to End(X) for some AW^*-module X over Z. The following theorem provides easily a complete system of $*$-isomorphism invariants of type I AW^*-algebras.

THEOREM 7.1. *Let X and Y be two AW^*-modules over Z. Then* End(X) *and* End(Y) *are $*$-isomorphic if and only if* Dim(X) *and* Dim(Y) *are congruent.*

PROOF. Suppose that Dim(X) and Dim(Y) are congruent. Then there is a cardinal α in $\boldsymbol{V}^{(\boldsymbol{B})}$ and an automorphism π of $\boldsymbol{B}$ such that Dim$(X)=\alpha$ and Dim$(Y)=\pi(\alpha)$. Thus we have only to show that $\mathcal{L}(l^2(\alpha))_\infty^{(\boldsymbol{B})}$ and $\mathcal{L}(l^2(\pi(\alpha)))_\infty^{(\boldsymbol{B})}$ are $*$-isomorphic. Obviously, $[\![\mathcal{L}(l^2(\pi(\alpha)))=\pi(\mathcal{L}(l^2(\alpha)))]\!]=1$ and hence π is a one-to-one correspondence between $\mathcal{L}(l^2(\alpha))^{(\boldsymbol{B})}$ and $\mathcal{L}(l^2(\pi(\alpha)))^{(\boldsymbol{B})}$. It is easy to see that $\pi(T+S)=\pi(T)+\pi(S)$, $\pi(TS)=\pi(T)\pi(S)$, $\pi(\check{c}T)=\check{c}\pi(T)$, $\pi(T^*)=\pi(T)^*$ and that $\|\pi(T)\|_{\boldsymbol{B}}=\pi(\|T\|_{\boldsymbol{B}})$, for any $T, S\in\mathcal{L}(l^2(\alpha))^{(\boldsymbol{B})}$, $c\in\boldsymbol{C}$. It follows that π is a

$*$-isomorphism from $\mathcal{L}(l^2(\alpha))_\infty^{(B)}$ onto $\mathcal{L}(l^2(\pi(\alpha)))_\infty^{(B)}$. Conversely, suppose that End(X) and End(Y) are $*$-isomorphic. Let $\alpha=\mathrm{Dim}(X)$ and $\beta=\mathrm{Dim}(Y)$. Then there is a $*$-isomorphism Φ from $\mathcal{L}(l^2(\alpha))_\infty^{(B)}$ onto $\mathcal{L}(l^2(\beta))_\infty^{(B)}$. Then their centers coincide with the bounded parts of scalar multiplications on l^2-spaces in $\boldsymbol{V}^{(\boldsymbol{B})}$. Let π be an automorphism on $\boldsymbol{B}$ such that $\pi(b)=\Phi(b)$ for any $b\in B$. Then we can extend π to $\boldsymbol{V}^{(\boldsymbol{B})}$, so that $\pi(a)=\Phi(a)$ for any $a\in Z$. Let Ψ be a map from $\mathcal{L}(l^2(\pi(\alpha)))_\infty^{(B)}$ into $\mathcal{L}(l^2(\beta))_\infty^{(B)}$ such that $\Psi(T)=\Phi(\pi^{-1}(T))$ for any $T\in\mathcal{L}(l^2(\pi(\alpha)))_\infty^{(B)}$. Then for any $a\in Z$ and $T\in\mathcal{L}(l^2(\pi(\alpha)))_\infty^{(B)}$, we have

$$\Psi(aT)=\Phi(\pi^{-1}(aT))=\Phi(\pi^{-1}(a)\pi^{-1}(T))=\Phi(\pi^{-1}(a))\Psi(T)=a\Psi(T).$$

For any $T, S\in\mathcal{L}(l^2(\pi(\alpha)))_\infty^{(B)}$, it is easy to see that $\Psi(TS)=\Psi(T)\Psi(S)$, $\Psi(T+S)=\Psi(T)+\Psi(S)$ and that $\Psi(T^*)=\Psi(T)^*$. Thus Ψ is an extensional map and $[\![\Psi$ is a $*$-isomorphism from $\mathcal{L}(l^2(\pi(\alpha)))$ onto $\mathcal{L}(l^2(\beta))]\!]=1$. Interpreting the theorem that every $*$-isomorphism of type I factor is spacial, we have $[\![\Psi$ is implimented by a unitary transformation from $l^2(\pi(\alpha))$ onto $l^2(\beta)]\!]=1$. It follows that $[\![\pi(\alpha)=\beta]\!]=1$. QED

For any cardinal α, donote by $[\alpha]$ the congruence class of α, i.e.,

$$[\alpha]=\{\pi(\alpha)\mid\pi\in\mathrm{Aut}(\boldsymbol{B})\}.$$

For any type I AW^*-algebra A with center isomorphic to Z, denote by Deg(A) the *degree* of A defined by

$$\mathrm{Deg}(A)=[\mathrm{Dim}(X)],$$

where X is an AW^*-module over Z such that A is $*$-isomorphic to End(X). By Theorem 7.1, Deg(A) is uniquely defined.

THEOREM 7.2. *Two type I AW^*-algebras are $*$-isomorphic if and only if their centers are $*$-isomorphic and they have the same degree. For any non-zero cardinal α in $\boldsymbol{V}^{(\boldsymbol{B})}$, there is a type I AW^*-algebra A with center isomorphic to Z such that* Deg$(A)=[\alpha]$.

PROOF. Obvious consequence of the previous results. QED

For a cardinal α, a type I AW^*-algebra is *α-homogeneous* if there is a partition of unity with equivalent abelian projections whose cardinality is α.

THEOREM 7.3. *A type I AW^*-algebra A is α-homogeneous if and only if* Deg$[A]=[\mathrm{card}(\check{\alpha})_{\boldsymbol{B}}]$.

PROOF. A type I AW^*-algebra A is α-homogeneous if and only if it is $*$-isomorphic with End(X) for an α-homogeneous AW^*-module over the center of A. In this case, Dim$(X)=\mathrm{card}(\check{\alpha})_{\boldsymbol{B}}$. Thus A is α-homogeneous if and only if Deg$(A)=[\mathrm{card}(\check{\alpha})_{\boldsymbol{B}}]$. QED

It is known [**6**] that every type I AW^*-algebra admits a direct sum decomposition into homogeneous subalgebras. Let A be a type I AW^*-algebra and $\boldsymbol{B}$

the complete Boolean algebra of central projections of A. A *cardinal series* of A is a family $\{\alpha_i, b_i \mid i\in\beta\}$ of cardinals α_i and central projections b_i of A indexed by a cardinal β such that

(1) $\alpha_i<\alpha_j$, for any $i, j\in\beta$ with $i<j$,

(2) $\{b_i \mid i\in\beta\}$ is a partition of unity of $\boldsymbol{B}$.

A cardinal series $\{\alpha_i, b_i \mid i\in\beta\}$ is called a *decomposition series* if

$$A=\sum_{i\in\beta}^{\oplus} A_i\,, \tag{3}$$

where A_i is an α_i-homogeneous subalgebra and b_i is the unit of A_i.

If A satisfies countable chain condition locally, then decomposition series of A are essentially unique in the following sense: If $\{\alpha'_i, b'_i \mid i\in\beta'\}$ is another decomposition series of A. Then $\beta=\beta'$, $\alpha_i=\alpha'_i$ for any $i\in\beta$ and there is an automorphism π of $\boldsymbol{B}$ such that $b_i=\pi(b'_i)$ for any $i\in\beta$. In general, the situation is not so simple as shown in Corollary 6.4. Our next theorem determines all possible decomposition series of A in terms of our invariants.

THEOREM 7.4. *Let A be a type I AW*-algebra. Then a cardinal series $\{\alpha_i, b_i \mid i\in\beta\}$ of A is a decomposition series of A if and only if*

$$\mathrm{Deg}[A]=\left[\mathrm{card}\Big(\sum_{i\in\beta}\check{\alpha}_i b_i\Big)_{\boldsymbol{B}}\right].$$

For the proof of the above theorem, we shall use the following lemma, although we shall omit its tedious proof (cf. [8; Theorem 6.3]).

LEMMA 7.5. *Let $S\in \boldsymbol{V}^{(\boldsymbol{B})}$, $S_i\in \boldsymbol{V}^{(\boldsymbol{B})}$ for $i\in I$, and $\{b_i \mid i\in I\}$ be a partition of unity of $\boldsymbol{B}$. If $\left[\!\left[S=\sum_{i\in I} S_i b_i\right]\!\right]=1$ then*

(1) $l^2(S)_\infty^{(\boldsymbol{B})}\cong \sum_{i\in I}^{\oplus} l^2(S_i)_\infty^{(b_i\boldsymbol{B})}$,

(2) $\mathcal{L}(l^2(S))_\infty^{(\boldsymbol{B})}\cong \sum_{i\in I}^{\oplus} \mathcal{L}(l^2(S_i))_\infty^{(b_i\boldsymbol{B})}$.

PROOF OF THEOREM 7.4. Let $\{\alpha_i, b_i \mid i\in\beta\}$ be a cardinal series of a type I AW*-algebra A and $\boldsymbol{B}$ the complete Boolean algebra of central projections in A. Then we have a direct sum decomposition

$$A=\sum_{i\in\beta}^{\oplus} A_i\,,$$

where A_i is an α_i-homogeneous subalgebra and b_i is the unit of A_i for any $i\in\beta$. By Theorem 7.3 and Lemma 7.5, (2), we have

$$A\cong\sum_{i\in\beta}^{\oplus}\mathcal{L}(l^2(\check{\alpha}_i))_\infty^{(b_i\boldsymbol{B})}\cong\mathcal{L}\Big(l^2\Big(\sum_{i\in\beta}\check{\alpha}_i b_i\Big)\Big)_\infty^{(\boldsymbol{B})}.$$

Thus $\mathrm{Deg}[A]=\left[\mathrm{card}\Big(\sum_{i\in\beta}\check{\alpha}_i b_i\Big)_{\boldsymbol{B}}\right]$. Conversely, suppose that

$$\operatorname{Deg}[A]=\left[\operatorname{card}\left(\sum_{i\in\beta}\check{\alpha}_i b_i\right)_{\boldsymbol{B}}\right].$$

Then we have by Lemma 7.5, (2),

$$A\cong\sum_{i\in\beta}^{\oplus}\mathcal{L}(l^2(\alpha_i))_\infty^{(b_i\boldsymbol{B})}.$$

Let A_i' be the image of $\mathcal{L}(l^2(\alpha_i))_\infty^{(b_i\boldsymbol{B})}$ by the above $*$-isomorphism. Then $A=\sum_{i\in\beta}^{\oplus}A_i'$. In this case A_i' is α_i-homogeneous and there is an automorphism π of $\boldsymbol{B}$ such that the unit of A_i' is $\pi(b_i)$. Let π^{-1} the inverse of π. Then by [**6**; Theorem 7.1], π^{-1} can be extended to a $*$-automorphism λ on A. Let $A_i=\lambda(A_i')$. Then we obtain the required direct sum decomposition $A=\sum_{i\in\beta}^{\oplus}A_i$, where A_i is α_i-homogeneous and b_i is the unit of A_i. QED

References

[1] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., **68** (1950), 337-404.
[2] S. K. Berberian, Baer $*$-rings, Springer, Berlin, 1972.
[3] P. R. Halmos, Lectures on Boolean algebras, van Nostrand, New York, 1963.
[4] T. Jech, Set theory, Academic Press, New York, 1978.
[5] I. Kaplansky, Projections in Banach algebras, Ann. of Math., **53** (1951), 235-249.
[6] I. Kaplansky, Algebras of type I, Ann. of Math., **56** (1952), 460-472.
[7] I. Kaplansky, Modules over operator algebras, Amer. J. Math., **75** (1953), 839-858.
[8] M. Ozawa, Boolean valued interpretation of Hilbert space theory, J. Math. Soc. Japan, **35** (1983), 609-627.
[9] W. L. Paschke, Inner product modules over B^*-algebras, Trans. Amer. Math. Soc., **182** (1973), 443-468.
[10] M. A. Rieffel, Morita equivalence for C^*-algebras and W^*-algebras, J. Pure Appl. Alg., **5** (1974), 51-96.
[11] M. Takesaki, Theory of operator algebras I, Springer, New York, 1979.
[12] G. Takeuti, Two applications of logic to mathematics, Iwanami and Princeton University Press, Tokyo and Princeton, 1978.
[13] G. Takeuti, A transfer principle in harmonic analysis, J. Symbolic Logic, **44** (1979), 417-440.
[14] G. Takeuti, Von Neumann algebras and Boolean valued analysis, J. Math. Soc. Japan, **35** (1983), 1-21.
[15] G. Takeuti, C^*-algebras and Boolean valued analysis, Japan. J. Math., **9** (1983), 207-246.
[16] G. Takeuti and W. M. Zaring, Axiomatic set theory, Springer, Heidelberg, 1973.

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Added in proof. A direct proof of Corollary 6.4 will be published by the author in "Non-uniqueness of the cardinality attached to homogeneous AW^*-algebras" (to appear in Proc. Amer. Math. Soc.).