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Fixed point theorems for families of nonexpansive mappings on unbounded sets

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§ 1. Introduction.

Let C be a nonempty closed convex subset of a real Banach space B. Then, a mapping $T: C\rightarrow C$ is called nonexpansive on C, if

$$\|Tx-Ty\|\leqq\|x-y\| \quad \text{for all} \quad x, y\in C.$$

Let $F(T)$ be the set of fixed points of T, that is,

$$F(T)=\{z\in C : Tz=z\}.$$

The theorem of Browder-Göhde-Kirk [2], [5], [8] assures that if B is uniformly convex and if C is bounded, closed, and convex, then such a mapping must have a fixed point. Recently, Kirk-Ray [9], Pazy [11] and Takahashi [14] studied the problem of the existence of fixed points for nonexpansive mappings defined on unbounded sets. On the other hand, Baillon [1] has shown the first nonlinear ergodic theorem: If B is a Hilbert space and C is bounded, closed and convex, then, the Cesàro means

$$S_n(x)=\frac{1}{n}\sum_{k=0}^{n-1}T^k x$$

converge weakly to an element of $F(T)$ for each $x\in C$. Later, Takahashi [14] considered the nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in Hilbert spaces.

Our purpose in this paper is to obtain a necessary and sufficient condition for a noncommutative semigroup of nonexpansive mappings defined on unbounded sets in Banach spaces to have a common fixed point. This is a generalization of Kirk-Ray [9] and Takahashi [14]. Furthermore, we deal with the problem relative to nonlinear ergodic theory for a noncommutative semigroup of nonexpansive mappings in Banach spaces.

§ 2. Preliminaries.

Let S be an abstract semigroup with identity and $m(S)$ the Banach space of all bounded real valued functions on S with the supremum norm. For each $s\in S$ and $f\in m(S)$, we define elements f_s and f^s in $m(S)$ given by $f_s(t)=f(st)$ and $f^s(t)=f(ts)$ for all $t\in S$. An element $\mu\in m(S)^*$ (the dual space of $m(S)$) is called a mean on S if $\|\mu\|=\mu(1)=1$. Let μ be a mean on S and $f\in m(S)$. Then we denote by $\mu(f)$ the value of μ at the function f. According to the time and circumstances, we write by $\mu_t(f(t))$ the value $\mu(f)$. A mean μ is called left [right] invariant if $\mu(f_s)=\mu(f)$ $[\mu(f^s)=\mu(f)]$ for all $f\in m(S)$ and $s\in S$. An invariant mean is a left and right invariant mean. A semigroup which has a left [right] invariant mean is called left [right] amenable. A semigroup which has an invariant mean is called amenable. A semigroup S is called left [right] reversible if for every pair of elements $a, b\in S$, there exists a pair $c, d\in S$ such that $ac=bd$ $[ca=db]$. Day [**4**] proved that a commutative semigroup is amenable. Granirer [**6**], [**7**] showed that every left [right] amenable semigroup is left [right] reversible. We also know that $\mu\in m(S)^*$ is a mean on S if and only if

$$\inf\{f(s) : s\in S\}\leqq\mu(f)\leqq\sup\{f(s) : s\in S\}$$

for every $f\in m(S)$. Furthermore we have the following: Let S be a left amenable semigroup and μ be a left invariant mean on S. Then, we have

$$\sup_s\inf_t f(st)\leqq\mu(f)\leqq\inf_s\sup_t f(st)$$

for every $f\in m(S)$. In fact, let f be an element of $m(S)$ and μ be a left invariant mean on S. Then we have

$$\mu(f)=\mu(f_s)\leqq\sup_t f_s(t)=\sup_t f(st)$$

and hence $\mu(f)\leqq\inf_s\sup_t f(st)$. Similarly, we can prove $\sup_s\inf_t f(st)\leqq\mu(f)$.

We also have that if S is a right amenable semigroup and μ is a right invariant mean on S, then we have

$$\sup_s\inf_t f(ts)\leqq\mu(f)\leqq\inf_s\sup_t f(ts)$$

for every $f\in m(S)$.

Let B be a real Banach space and let B^* be its dual, that is, the space of all continuous linear functionals f on B. The value of $f\in B^*$ at $x\in B$ will be denoted by $\langle x, f\rangle$. With each $x\in B$, we associate the set

$$J(x)=\{f\in B^* : \langle x, f\rangle=\|x\|^2=\|f\|^2\}.$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x)\neq\emptyset$ for any $x\in B$. The multi-valued operator $J : B\to B^*$ is called the duality mapping of B. A Banach space B is said to be smooth provided

$$\lim_{t\to 0}\frac{\|x+th\|-\|x\|}{t} \qquad (*)$$

exists for each $x, h\in B$. When this is the case, the norm of B is said to be Gâteaux differentiable. The space B is said to have uniformly Gâteaux differentiable norm if for each $h\in B$, the limit $(*)$ is attained uniformly for x with $\|x\|=1$. It is well known that if B is smooth, then the duality mapping J is single valued. It is also known that if B has uniformly Gâteaux differentiable norm, J is uniformly continuous on bounded sets when B has its strong topology while B^* has its weak star topology; see [3]. Let K be a subset of B. Then, we denote by $\delta(K)$ the diameter of K. A point $x\in K$ is a diametral point of K provided

$$\sup\{\|x-y\| : y\in K\}=\delta(K).$$

A closed convex subset C of a Banach space B is said to have normal structure, if for each closed bounded convex subset K of C, which contains at least two points, there exists an element of K which is not a diametral point of K. A Banach space B is called uniformly convex if the modulus of convexity

$$\delta(\varepsilon)=\inf\left\{1-\frac{1}{2}\|x+y\| : \|x\|, \|y\|\leqq 1, \|x-y\|\geqq\varepsilon\right\}$$

is positive in its domain of definition $\{\varepsilon : 0<\varepsilon\leqq 2\}$. A closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure.

§ 3. Fixed point theorems.

Before proving fixed point theorems for a noncommutative semigroup of nonexpansive mappings defined on unbounded sets in a Banach space, we prove the following Lemmas.

LEMMA 1. *Let C be a nonempty closed convex subset of a Banach space B, let S be a left reversible semigroup of nonexpansive mappings t of C into itself, and suppose that $\{tx : t\in S\}$ for some $x\in C$ is bounded. Then, the real valued functions f and g on B defined by $f(y)=\inf_s \sup_t \|stx-y\|$ and $g(y)=\inf_s \sup_t \|stx-y\|^2$ for each $y\in B$ are continuous and convex.*

PROOF. Let $y\in B$ and $r>0$. Then, there exists a positive number M such that

$$\begin{aligned}
&\|stx-y\|^2-\|stx-z\|^2\\
&\quad=(\|stx-y\|+\|stx-z\|)(\|stx-y\|-\|stx-z\|)\\
&\quad\leqq(\|stx-y\|+\|stx-z\|)\|y-z\|\\
&\quad\leqq M\|y-z\|
\end{aligned}$$

for all $s, t\in S$ and $z\in S_r(y)=\{v\in B : \|y-v\|<r\}$. So, we have

$$\sup_t \|stx-y\|^2 \leqq \sup_t \|stx-z\|^2 + M\|y-z\|$$

and hence $g(y) \leqq g(z)+M\|y-z\|$. Similarly, we have $g(z) \leqq g(y)+M\|z-y\|$. Therefore, $|g(y)-g(z)| \leqq M\|y-z\|$ for all $z \in S_r(y)$. This implies that g is continuous on B. Let α and β be nonnegative numbers with $\alpha+\beta=1$. Then, since

$$\|stx-(\alpha y+\beta z)\|^2 \leqq \alpha\|stx-y\|^2+\beta\|stx-z\|^2 ,$$

we have

$$\inf_s \sup_t \|stx-(\alpha y+\beta z)\|^2 \leqq \inf_s (\alpha \sup_t \|stx-y\|^2+\beta \sup_t \|stx-z\|^2) .$$

Put $a=\inf_s \sup_t \|stx-y\|^2$ and $b=\inf_s \sup_t \|stx-z\|^2$, and let $\varepsilon>0$. Then, there exist $s_1, s_2 \in S$ such that $\sup_t \|s_1tx-y\|^2 < a+\varepsilon$ and $\sup_t \|s_2tx-z\|^2 < b+\varepsilon$. Since S is left reversible, we obtain $u_1, u_2 \in S$ with $s_1u_1=s_2u_2$. So, if $s_0=s_1u_1=s_2u_2$, we have

$$\sup_t \|s_0tx-y\|^2 < a+\varepsilon \quad \text{and} \quad \sup_t \|s_0tx-z\|^2 < b+\varepsilon$$

and hence

$$\begin{aligned}
&\inf_s (\alpha \sup_t \|stx-y\|^2+\beta \sup_t \|stx-z\|^2) \\
&\leqq \alpha \sup_t \|s_0tx-y\|^2+\beta \sup_t \|s_0tx-z\|^2 \\
&< \alpha(a+\varepsilon)+\beta(b+\varepsilon) \\
&= \alpha a+\beta b+(\alpha+\beta)\varepsilon .
\end{aligned}$$

Since ε is arbitrary, we have

$$g(\alpha y+\beta z) \leqq \alpha g(y)+\beta g(z) .$$

This implies that g is convex on B.

By the same method, we can prove that the function f is continuous and convex on B.

LEMMA 2. *Let C be a nonempty closed convex subset of a Banach space B and S be a semigroup of nonexpansive mappings t of C into itself. Let $\{tx : t \in S\}$ be a bounded subset of C and μ be a mean on S. Then, the real valued functions f and g on B given by $f(y)=\mu_t\|tx-y\|$ and $g(y)=\mu_t\|tx-y\|^2$ for each $y \in B$ are continuous and convex.*

PROOF. Since

$$-\|y-z\| \leqq \|tx-y\|-\|tx-z\| \leqq \|y-z\|$$

for $y, z \in B$ we have

$$-\|y-z\| \leqq \mu_t\|tx-y\|-\mu_t\|tx-z\| \leqq \|y-z\| .$$

Therefore, f is continuous in y. By linearity of μ and convexity of norm $\|\cdot\|$,

f is convex in y.

By the same method, we can prove that the function g is continuous and convex on B.

THEOREM 1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space B and S be a left reversible semigroup of nonexpansive mappings t of C into itself. Then, $F(S)\neq\emptyset$ if and only if there exists an element $x\in C$ such that $\{tx : t\in S\}$ is bounded.*

PROOF. Suppose that $\{tx : t\in S\}$ is bounded for some $x\in C$ and define the real valued function f on B by

$$f(y)=\inf_s \sup_t \|stx-y\| .$$

Then, by Lemma 1, f is continuous and convex. Furthermore, define

$$r=\inf\{f(y) : y\in C\} \quad \text{and} \quad K=\{y\in C : f(y)\leqq r+1\} .$$

Then, it is obvious that K is nonempty, closed and convex. Since $\{tx : t\in S\}$ is bounded, K is also bounded. Since f is weakly lower semicontinuous and K is weakly compact, we obtain that $C_0=\{u\in C : r=f(u)\}$ is nonempty. If $r=0$, then since $\|u-v\|\leqq f(u)+f(v)=0$ for $u, v\in C_0$, we have that C_0 consists of a single point. Let $r>0$. Suppose that $\|u-v\|=\varepsilon>0$ for some u and v in C_0 and choose a positive number a such that

$$[1-\delta(\varepsilon/r+a)](r+a)<r .$$

Since $u, v\in C_0$, there exist $s_1, s_2\in S$ such that

$$\sup_t \|s_1tx-u\|<r+a \quad \text{and} \quad \sup_t \|s_2tx-v\|<r+a .$$

Since S is left reversible, there exist $u_1, u_2\in S$ such that $s_1u_1=s_2u_2$. So, if $s_0=s_1u_1=s_2u_2$, we have

$$\sup_t \|s_0tx-u\|<r+a \quad \text{and} \quad \sup_t \|s_0tx-v\|<r+a .$$

Since B is uniformly convex, we have that for any $t\in S$,

$$\left\|\frac{u+v}{2}-s_0tx\right\|\leqq[1-\delta(\varepsilon/r+a)](r+a)<r$$

and hence $f(u+v/2)=\inf_s \sup_t \|stx-(u+v)/2\|<r$. This is a contradiction. Therefore C_0 is a single point; say z. Since for each $s_0\in S$

$$\begin{aligned}\inf_s \sup_t \|stx-s_0z\| &\leqq \inf_s \sup_t \|s_0stx-s_0z\| \\ &\leqq \inf_s \sup_t \|stx-z\|=r ,\end{aligned}$$

we have $s_0z=z$ and hence the point z is a common fixed point of S. The converse is obvious.

As a direct consequence of Theorem 1, we have the following [**14**]:

COROLLARY 1. *Let C be a nonempty closed convex subset of a Hilbert space H and S be a left amenable semigroup of nonexpansive mappings t of C into itself. Then, $F(S)\neq\emptyset$ if and only if there exists $x\in C$ such that $\{tx : t\in S\}$ is bounded.*

By using Lim's fixed point theorem [**10**], we can prove a generalization of Theorem 1.

THEOREM 2. *Let C be a nonempty closed convex subset of a reflexive Banach space which has normal structure and S be a left reversible semigroup of nonexpansive mappings t of C into itself. Then, $F(S)\neq\emptyset$ if and only if there exists an element $x\in C$ such that $\{tx : t\in S\}$ is bounded.*

PROOF. Suppose that $\{tx : t\in S\}$ is bounded for some $x\in C$. Then, as in the proof of Theorem 1, define

$$f(y)=\inf_s \sup_t \| stx-y\| ,$$

$$r=\inf\{f(y) : y\in C\} ,$$

and

$$K=\{y\in C : f(y)\leqq r+1\} .$$

Then, it is obvious that K is nonempty, closed, bounded, and convex. Let $z\in K$ and $t_0\in S$. Then, since $t_0z\in C$ and

$$\inf_s \sup_t \| stx-t_0z\| \leqq \inf_s \sup_t \| t_0stx-t_0z\|$$

$$\leqq \inf_s \sup_t \| stx-z\| \leqq r+1 ,$$

we have that K is S-invariant. So, from Lim's fixed point theorem, there exists a common fixed point for the semigroup S. The converse is obvious.

Similarly, we can prove the following:

THEOREM 3. *Let B be a Banach space whose bounded closed convex subsets have the common fixed point property relative to left reversible semigroups of nonexpansive mappings, let C be a nonempty closed convex subset of B, and suppose that S is a left reversible semigroup of nonexpansive mappings t of C into itself. Then, $F(S)\neq\emptyset$ if and only if there exists an element $x\in C$ such that $\{tx : t\in S\}$ is bounded.*

COROLLARY 2 (Kirk-Ray [**9**]). *Let B be a Banach space whose bounded closed convex subsets have the fixed point property relative to nonexpansive selfmappings, let C be a closed convex subset of B, and suppose $T : C\to C$ is a nonexpansive mapping. If there exists $u\in C$ such that the set $G(u, Tu)=\{z\in C : \|z-u\|\geqq \|z-Tu\|\}$ is bounded, then T has a fixed point in C.*

PROOF. Let $a=\sup\{\|y-z\| : y, z\in G(u, Tu)\}$ and $x=Tu$. Then, by mathematical induction, we prove that $\{T^n x : n=0, 1, 2, \cdots\}$ is bounded. In fact, it is obvious that $\|x-Tu\|\leqq 3a$. Let $\|T^{k-1}x-Tu\|\leqq 3a$. If $T^{k-1}x\in G(u, Tu)$, then, since $Tu\in G(u, Tu)$ and $(1/2)(u+Tu)\in G(u, Tu)$, we have

$$\|T^k x-Tu\|\leqq\|T^{k-1}x-u\|\leqq\|T^{k-1}x-Tu\|+\|Tu-u\|\leqq a+2a=3a .$$

If $T^{k-1}x\notin G(u, Tu)$, we have

$$\|T^k x-Tu\|\leqq\|T^{k-1}x-u\|<\|T^{k-1}x-Tu\|\leqq 3a .$$

Using Theorem 3 here, we complete the proof.

§ 4. Ergodic theorems.

Let C be a closed convex subset of a Banach space B and S be a semigroup of nonexpansive mappings t of C into itself. Then, if $\{tx : t\in S\}$ for some $x\in C$ is bounded and μ is a mean on S, we can define the real valued continuous convex function g on B by $g(y)=\mu_t\|tx-y\|^2$ for each $y\in B$; see Lemma 2. So, let us define

$$M(x, \mu)=\{z\in C : g(z)=\inf_{y\in C} g(y)\} .$$

THEOREM 4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space B, let S be an amenable semigroup of nonexpansive mappings t of C into itself, and let μ be an invariant mean on S. Suppose that*

$$F(S)=\bigcap\{F(t) : t\in S\}\neq\emptyset .$$

Then, the set $M(x, \mu)\cap F(S)$ consists of a single point for each x and the point is independent of μ.

PROOF. Let μ be an invariant mean on S and $x\in C$. Then, since $F(S)\neq\emptyset$, $\{tx : t\in S\}$ is bounded and hence we can define the set $M(x, \mu)$. Consider

$$r=\inf\{\mu_t\|tx-y\|^2 : y\in C\}$$

and

$$C_1=\{y\in C : \mu_t\|tx-y\|^2\leqq r+1\} .$$

Then, it is obvious that C_1 is nonempty, closed and convex. The set C_1 is also bounded. In fact, if $y_0\in C_1$, we have

$$\inf_t\|tx-y_0\|^2\leqq\mu_t\|tx-y_0\|^2<r+2 .$$

So, there exists $t_0\in S$ such that $\|t_0x-y_0\|^2<r+2$. Hence C_1 is bounded. Since $y\mapsto\mu_t\|tx-y\|^2$ is weakly lower semicontinuous and C_1 is weakly compact, we have that

$$M(x, \mu)=\{z\in C_1 : r=\mu_t\|tx-z\|^2\}$$

is nonempty, closed and convex. If $z\in M(x, \mu)$ and $s\in S$, then since $sz\in C$ and

$$\mu_t\|tx-sz\|^2=\mu_t\|stx-sz\|^2\leqq\mu_t\|tx-z\|^2=r,$$

we have that $M(x, \mu)$ is S-invariant. So, by Theorem 1, there exists an element u in $M(x, \mu)$ such that $su=u$ for all $s\in S$.

Now, we show that the set $M(x, \mu)\cap F(S)$ is a single point. Let $u, v\in M(x, \mu)\cap F(S)$. If $r=0$, then since

$$\|u-v\|^2\leqq 2\|u-tx\|^2+2\|tx-v\|^2,$$

we have

$$\|u-v\|^2\leqq 2\mu_t\|u-tx\|^2+2\mu_t\|tx-v\|^2=0$$

and hence $u=v$. So, let $r>0$. Let $\|u-v\|=\varepsilon>0$ and choose a positive number a such that

$$[1-\delta(\varepsilon/\sqrt{r}+a)](\sqrt{r}+a)<\sqrt{r},$$

where δ is the modulus of convexity of the norm. Since $u, v\in M(x, \mu)$, there exist $t_0, t_1\in S$ such that

$$\|t_0-u\|<\sqrt{r}+a \quad \text{and} \quad \|t_1x-v\|<\sqrt{r}+a.$$

Since S is right amenable, there exist $u_0, u_1\in S$ such that $u_0t_0=u_1t_1=s_0$. For each $t\in S$, we have

$$\|ts_0x-u\|=\|tu_0t_0x-u\|\leqq\|t_0x-u\|<\sqrt{r}+a$$

and

$$\|ts_0x-v\|=\|tu_1t_1x-v\|\leqq\|t_1x-v\|<\sqrt{r}+a.$$

Since X is uniformly convex, we have

$$\|(u+v)/2-ts_0x\|\leqq[1-\delta(\varepsilon/\sqrt{r}+a)](\sqrt{r}+a)<\sqrt{r}$$

and hence

$$\mu_t\left\|tx-\frac{u+v}{2}\right\|^2=\mu_t\left\|ts_0x-\frac{u+v}{2}\right\|^2<r.$$

This is a contradiction. Therefore, the set $M(x, \mu)\cap F(S)$ is a single point. Let $u\in F(S)$. Then, we know that

$$\sup_s\inf_t\|tsx-u\|^2\leqq\mu_t\|tx-u\|^2\leqq\inf_s\sup_t\|tsx-u\|^2.$$

Put $a=\inf_s\sup_t\|tsx-u\|^2$ and let ε be an arbitrary positive number. Then, we have

$$\sup_t\|tsx-u\|^2\geqq a>a-\varepsilon$$

for all $s\in S$. Fix $s\in S$. Then, for each $t\in S$, there exists a $t_0\in S$ such that $\|t_0tsx-u\|^2>a-\varepsilon$. Since t_0 is nonexpansive, we obtain $\|tsx-u\|^2\geqq\|t_0tsx-u\|^2>$

$a-\varepsilon$. So, we have $\inf_t \|tsx-u\|^2 \geqq a-\varepsilon$ and hence

$$\sup_s \inf_t \|tsx-u\|^2 \geqq a = \inf_s \sup_t \|tsx-u\|^2 .$$

Therefore, we have

$$\sup_s \inf_t \|tsx-u\|^2 = \mu_t \|tx-u\|^2 = \inf_s \sup_t \|tsx-u\|^2 .$$

This implies that $M(x, \mu) \cap F(S) = \{z\}$ is independent of μ.

THEOREM 5. *Let C be a nonempty closed convex subset of a Banach space B with a uniformly Gâteaux differentiable norm, let S be an amenable semigroup of nonexpansive mappings t of C into itself, and let μ be an invariant mean on S. Suppose that*

$$F(S) = \bigcap \{F(t) : t \in S\} \neq \emptyset$$

and let $u, x \in C$. Then, $u \in M(x, \mu)$ if and only if $\mu_t \langle z-u, J(tx-u)\rangle \leqq 0$ for all $z \in C$, where J is the duality mapping of B.

PROOF. For z in C and $0 \leqq \lambda \leqq 1$, we have

$$\begin{aligned} \|tx-u\|^2 &= \|tx-\lambda u-(1-\lambda)z+(1-\lambda)(z-u)\|^2 \\ &\geqq \|tx-\lambda u-(1-\lambda)z\|^2 + 2(1-\lambda)\langle z-u, J(tx-\lambda u-(1-\lambda)z)\rangle . \end{aligned}$$

Let $\varepsilon > 0$ be given. Since the norm of B is uniformly Gâteaux differentiable, the duality map is uniformly continuous on bounded subsets of B from the strong topology of B to the weak star topology of B^*. Therefore,

$$|\langle z-u, J(tx-\lambda u-(1-\lambda)z)-J(tx-u)\rangle| < \varepsilon$$

if λ is close enough to 1. Consequently, we have

$$\begin{aligned} \langle z-u, J(tx-u)\rangle &< \varepsilon + \langle z-u, J(tx-\lambda u-(1-\lambda)z)\rangle \\ &\leqq \varepsilon + \frac{1}{2(1-\lambda)}\{\|tx-u\|^2 - \|tx-\lambda u-(1-\lambda)z\|^2\} \end{aligned}$$

and hence

$$\mu_t \langle z-u, J(tx-u)\rangle \leqq \varepsilon + \frac{1}{2(1-\lambda)}\{\mu_t \|tx-u\|^2 - \mu_t \|tx-\lambda u-(1-\lambda)z\|^2\} \leqq \varepsilon .$$

Therefore, we have $\mu_t \langle z-u, J(tx-u)\rangle \leqq 0$ for all $z \in C$.

Since for $z, u \in C$,

$$\|tx-z\|^2 - \|tx-u\|^2 \geqq 2\langle u-z, J(tx-u)\rangle ,$$

and $\mu_t \langle z-u, J(tx-u)\rangle \leqq 0$ for all $z \in C$, then $u \in M(x, \mu)$.

THEOREM 6. *Let C be a nonempty closed convex subset of a Hilbert space H, S be an amenable semigroup of nonexpansive mappings t of C into itself and*

μ be an invariant mean on S. Suppose that

$$F(S)=\bigcap\{F(t) : t\in S\}\neq\emptyset .$$

Then, the set $M(x, \mu)$ consists of a single point x_0 and the point x_0 is independent of μ.

Furthermore, putting $Px=M(x, \mu)=x_0$ for each $x\in C$, then, P is a nonexpansive retraction of C onto $F(S)$ such that $Pt=tP=P$ for every $t\in S$ and $Px\in\bigcap_{s\in S}\overline{\mathrm{co}}\{stx : t\in S\}$ for every $x\in C$, where $\overline{\mathrm{co}}A$ is the closure of the convex hull of A.

PROOF. Let μ be an invariant mean on S and $x\in C$. Then, since $F(S)\neq\emptyset$, $\{tx : t\in S\}$ is bounded and hence, for each y in H, the real-valued function $t\mapsto\langle tx, y\rangle$ is in $m(S)$. Denote by $\mu_t\langle tx, y\rangle$ the value of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t\langle tx, y\rangle|\leqq\|\mu\|\cdot\sup_t|\langle tx, y\rangle|\leqq(\sup_t\|tx\|)\cdot\|y\| ,$$

it is continuous in y, so by the Riesz theorem, there exists an $x_0\in H$ such that

$$\mu_t\langle tx, y\rangle=\langle x_0, y\rangle$$

for every $y\in H$. If $x_0\notin\bigcap_{s\in S}\overline{\mathrm{co}}\{stx : t\in S\}$, then we have $x_0\notin\overline{\mathrm{co}}\{s_0tx : t\in S\}$ for some s_0 in S. By the separation theorem there exists a y_0 in H such that

$$\langle x_0, y_0\rangle<\inf\{\langle z, y_0\rangle : z\in\overline{\mathrm{co}}\{s_0tx : t\in S\}\} .$$

So, we have

$$\begin{aligned}\inf_t\langle s_0tx, y_0\rangle&\leqq\mu_t\langle s_0tx, y_0\rangle=\mu_t\langle tx, y_0\rangle\\&=\langle x_0, y_0\rangle\\&<\inf\{\langle z, y_0\rangle : z\in\overline{\mathrm{co}}\{s_0tx : t\in S\}\}\\&\leqq\inf_t\langle s_0tx, y_0\rangle .\end{aligned}$$

This is a contradiction. Therefore, we have

$$x_0\in\bigcap_{s\in S}\overline{\mathrm{co}}\{stx : t\in S\} .$$

Let $u\in C$. Then, since

$$\|x_0-u\|^2=\|tx-u\|^2-\|tx-x_0\|^2-2\langle tx-x_0, x_0-u\rangle$$

for every $t\in S$ and hence

$$\begin{aligned}\|x_0-u\|^2&=\mu_t(\|tx-u\|^2-\|tx-x_0\|^2-2\langle tx-x_0, x_0-u\rangle)\\&=\mu_t\|tx-u\|^2-\mu_t\|tx-x_0\|^2\geqq 0 ,\end{aligned}$$

we have $x_0\in M(x, \mu)$. If $u\in M(x, \mu)$, then since

$$\mu_t\|tx-u\|^2-\mu_t\|tx-x_0\|^2\leqq 0,$$

we have $u=x_0$. Setting $Px=x_0$, it follows from [**14**] and above that P is a nonexpansive retraction of C onto $F(S)$ such that $Pt=tP=P$ for every $t\in S$ and

$$Px\in\bigcap_{s\in S}\overline{\mathrm{co}}\{stx:t\in S\}$$

for every $x\in C$.

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