J. Math. Soc. Japan
Vol. 36, No. 2, 1984

Spherical t-designs which are orbits of finite groups

Dedicated to Professor Hirosi Nagao on the occasion of his 60-th birthday

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(Received May 27, 1983)

Introduction.

A spherical t-design in $\boldsymbol{R}^d$ is a finite nonempty subset X in the unit sphere $\Omega_d=\{(x_1, \cdots, x_d)\in\boldsymbol{R}^d \mid x_1^2+\cdots+x_d^2=1\}$ such that

$$\frac{1}{|\Omega_d|}\int_{\Omega_d} f(\boldsymbol{x})d\omega(\boldsymbol{x})=\frac{1}{|X|}\sum_{\boldsymbol{x}\in X} f(\boldsymbol{x}) \tag{0.1}$$

for all polynomials $f(\boldsymbol{x})=f(x_1, x_2, \cdots, x_d)$ of degree $\leqq t$. The condition (0.1) is equivalent to the following condition:

$$\sum_{\boldsymbol{x}\in X} f(\boldsymbol{x})=0 \tag{0.2}$$

for all homogeneous harmonic polynomials $f(\boldsymbol{x})=f(x_1, \cdots, x_d)$ of degrees $1, 2, \cdots, t$.

The reader is referred to Delsarte-Goethals-Seidel [6] for the basic properties of spherical t-designs. In this paper, we study spherical t-designs X which are obtained from finite subgroups G of the real orthogonal group $O(d)$ in such a way that

$$X:=\boldsymbol{x}^G:=\{\boldsymbol{x}^g \mid g\in G\}\subset\Omega_d$$

for some $\boldsymbol{x}\in\Omega_d$. (Namely, X is a spherical t-design which is obtained as an orbit of a finite group G in $O(d)$.)

Let G be a finite subgroup of the real orthogonal group $O(d)$ acting on $\boldsymbol{R}^d$ and on Ω_d. Let ρ_i ($i=0, 1, 2, \cdots$) be the i-th spherical representation of $O(d)$, i.e., the representation of $O(d)$ on the space of homogeneous harmonic polynomials of degree i. So

$$\dim\rho_i=\binom{d+i-1}{i}-\binom{d+i-3}{i-2}.$$

In [1] the following theorem was proved:

THEOREM A (Bannai [1, Theorem 1]). (i) *Let G be a finite subgroup of*

*) Supported in part by NSF grant: MCS 79-03128A01.

$O(d)$ and let the restriction of ρ_i (=the i-th spherical representation of $O(d)$) to G remain irreducible for $i=0, 1, \cdots, s$. Then for any $\boldsymbol{x}\in\Omega_d$, the subset

$$X:=\boldsymbol{x}^G:=\{\boldsymbol{x}^g \mid g\in G\}\subset\Omega_d$$

is a spherical $2s$-design.

(ii) *If $(\rho_j, \rho_{s+1})_G=0$ for $j=0, 1, \cdots, s$ in addition to the hypothesis of* (i), *then X is a spherical $(2s+1)$-design.*

In the above theorem, the term "irreducible" meant "absolutely irreducible". Goethals and Seidel [**9**, Theorems 6.7 and 6.8] generalized Theorem A for the "real irreducible" case. In [**9**] they also proved that the converse of (i) in Theorem A is true ([**9**, Theorem 6.7]) and asked whether the converse of (ii) in Theorem A is true ([**9**, Remark 6.9]). However, their proof as well as the result itself (i.e., the converse of (i) in Theorem A) turned out to be incorrect. Thus the converse of (ii) in Theorem A also does not hold. Explicit counter examples to the converse of (i) in Theorem A will be given in § 4 of the present paper. These counter examples seem to be of independent interest.

Theorem A mentioned above was used to construct many spherical t-designs X which are obtained as orbits of finite groups G of $O(d)$. In [**1**], I claimed that it is difficult (and unlikely) to find such spherical t-designs in Ω_d if $d\geqq 3$ and t is large ([**1**, Concluding Remark (i)]). One of the purposes of the present paper is to clarify this claim further.

First, I prove the following:

THEOREM 1. *Let G be a finite subgroup of $O(d)$. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be elements in the unit sphere Ω_d in $\boldsymbol{R}^d$, and let t_i $(i=1, 2)$ be the numbers such that the set $\boldsymbol{x}_i^G$ is a t_i-design, but not (t_i+1)-design. Then we have:*

$$t_1\leqq 2t_2+1 \quad \text{and} \quad t_2\leqq 2t_1+1\,. \tag{0.3}$$

Theorem 1 implies that if $\boldsymbol{x}_0^G$ is a spherical t-design for some $\boldsymbol{x}_0\in\Omega_d$, then $\boldsymbol{x}^G$ are spherical $[t/2]$-designs for any $\boldsymbol{x}\in\Omega_d$. As is shown in Goethals-Seidel [**9**], this last condition has many strong character theoretical implications (see also § 3 of the present paper), and it seems that it is quite possible (by having recourse to the classification of finite simple groups) to show that if $d\geqq 3$ then the number (strength) t of such spherical t-designs is bounded by an absolute constant t_0 (which does not depend on d). [Actually I announced that this result was established by using the classification of finite simple groups at the meetings of Oberwolfach (March, 1982) and Montreal (June, 1982). However, the converse of (i) of Theorem A (which turned out to be incorrect) was used there. So my original proof of the claim needs some major repair, which is now under way.] Details on this topic will be discussed in a subsequent paper. I was able to prove a weaker version of this. Namely, if $d\geqq 3$ then $t\leqq t(d)$ where $t(d)$ is a certain function depending only on d. The details will be discussed elsewhere.

Theorem 1 will be proved in two different ways. The first proof (given in § 1) uses some properties of harmonic polynomials. The second proof (given in § 2) is more group theoretical, and the argument there, in particular the following Theorem 2, seems to have some independent interest.

THEOREM 2. *Let G be a finite subgroup of $O(d)$ and let G act (not necessarily transitively) on a finite set X of Ω_d. Let ρ_i $(i=0, 1, \cdots)$ be the i-th spherical representation of $O(d)$, and let π be the permutation representation of G on X. If X is a spherical t-design in $\boldsymbol{R}^d$, then we have*

$$(1+\rho_1+\cdots+\rho_{[t/2]})|_G \subseteqq \pi . \tag{0.4}$$

That is, the representation of G obtained by restricting the representation $1+\rho_1+\cdots+\rho_{[t/2]}$ of $O(d)$ to G is contained in the permutation representation π of G on X.

COROLLARY TO THEOREM 2. *Let $X=\boldsymbol{x}_0^G$ for a finite group G of $O(d)$ and for some $\boldsymbol{x}_0 \in \Omega_d$. If X is a spherical t-design in Ω_d, then*

$$(\rho_0, \rho_i)_G=\delta_{i0} \tag{0.5}$$

for $i=0, 1, \cdots, [t/2]$.

We remark that Corollary to Theorem 2 implies Theorem 1 and conversely Theorem 1 implies Corollary to Theorem 2. The reader will notice that Corollary to Theorem 2 is modeled on a result of Noda on ordinary t-designs (see Noda [**18**, Corollary 2]), and that the argument in Theorem 2 is similar to the one used by Stanley [**21**, Lemma 9.1] (although the argument itself was familiar to group theorists).

Our assumption that X is an orbit of a group is a strong assumption. The existence of spherical t-designs in $\boldsymbol{R}^d$ for any t and any d was just proved by Seymour and Zaslavsky [**20**]. Actually, such spherical t-designs are quite abundant and have the property that they are *deformable continuously* (see Seymour-Zaslavsky [**20**], see also Hong [**12**]). It seems interesting to explicitly construct spherical t-designs for large t, in particular to construct those which are rigid, i.e., spherical t-designs which do not allow any nontrivial deformation. In spite of the negative implication of Theorem 1 that there cannot be spherical t-designs X of the form $X=\boldsymbol{x}^G$ for large t and $d \geqq 3$, I suspect that those sets which are orbits of finite groups $G \subset O(d)$ can still serve as important components when we explicitly try to construct spherical t-designs.

Another purpose of the present paper is to study some interesting relations between spherical t-designs and various group representation theories. In § 3, the concepts of t-transitivity and t-homogeneous transitivity are introduced for (finite) subgroups G of $O(d)$, and they are compared with the corresponding concepts for permutation groups. In particular, a finite subgroup G of $O(d)$ is called a t-homogeneous linear group if the $\boldsymbol{x}^G$ are spherical t-designs for any $\boldsymbol{x}\in\Omega_d$. The content of § 3 is mainly a discussion of various interesting observations, and of whether the proposed definitions are reasonable. We remark that these concepts are also well defined for other compact symmetric spaces of rank 1 and for any group case Q-polynomial association schemes.

Counter examples for the converse of (i) of Theorem A are given in § 4. The discussion there is closely related to the concepts (just introduced) of t-transitivity and t-homogeneous transitivity for subgroups G of $O(d)$, and for subgroups G of the unitary group $U(d)$.

§ 1. The first proof of Theorem 1.

The following step (1) is due to Goethals-Seidel [**10**, Section 3].

Step (1) Let $\boldsymbol{x}_0\in\Omega_d$. Then $X=\boldsymbol{x}_0^G$ is a spherical t-design if and only if

$$f(\boldsymbol{x}_0)=0$$

for all G-invariant homogeneous harmonic polynomials $f(\boldsymbol{x})$ of degree $1, 2, \cdots, t$. (Note that for $M\in G$, $(Mf)(\boldsymbol{x})=f(M^{-1}\boldsymbol{x})$ and that f is G-invariant if and only if $Mf=f$ for all $M\in G$.)

PROOF. Since G is transitive on X, we have

$$\frac{1}{|X|}\sum_{\boldsymbol{x}\in X}f(\boldsymbol{x})=\frac{1}{|G|}(\sum_{M\in G}Mf)(\boldsymbol{x}_0)\,.$$

Here $\frac{1}{|G|}\sum_{M\in G}(Mf)$ is clearly a G-invariant homogeneous harmonic polynomial if f is a homogeneous harmonic polynomial. So we have ($\Leftarrow$). If f is G-invariant, then clearly $\frac{1}{|G|}\sum_{M\in G}(Mf)=f$. So we have ($\Rightarrow$).

Step (2) Completion of the proof of Theorem 1.

PROOF. Let $\boldsymbol{x}_i$ $(i=1, 2)$ and t_i $(i=1, 2)$ be as in the hypothesis of Theorem 1. Without loss of generality we may assume that $t_1\leqq t_2$, and we will prove that $t_2\leqq 2t_1+1$. Since $\boldsymbol{x}^G$ is not always a (t_1+1)-design for every $\boldsymbol{x}\in\Omega_d$, there exists a G-invariant harmonic polynomial f_1 of degree t_1+1. We want to show that there exists some G-invariant (non constant) homogeneous harmonic polynomial g of degree $\leqq 2(t_1+1)$ such that $g(\boldsymbol{x}_2)\neq 0$. It is known that any homogeneous polynomial $f(\boldsymbol{x})$ of degree i is uniquely decomposed as

$$f=p_i+\|\boldsymbol{x}\|^2p_{i-2}+\|\boldsymbol{x}\|^4p_{i-4}+\cdots+\|\boldsymbol{x}\|^{2[i/2]}p_{i-2[i/2]}\,,$$

where $\|\boldsymbol{x}\|^2=x_1^2+\cdots+x_d^2$ and the p_j are homogeneous harmonic polynomials of degree j (see, e.g., [8, p. 366]). It is also known that

$$\int_{\Omega_d} f(\boldsymbol{x})d\omega(\boldsymbol{x})=0$$

for any homogeneous harmonic polynomial $f(\boldsymbol{x})$ of degree $\geqq 1$, (i.e., the mean value property of harmonic polynomials). Now, let us consider the above mentioned decomposition of f_1^2:

$$f_1^2=p_{2(t_1+1)}+\|\boldsymbol{x}\|^2 p_{2t_1}+\|\boldsymbol{x}\|^4 p_{2t_1-2}+\cdots+\|\boldsymbol{x}\|^{2t_1}p_2+\|\boldsymbol{x}\|^{2(t_1+1)}p_0,$$

where the p_j are homogeneous harmonic polynomials of degree j. Then, since f_1 and $\|\boldsymbol{x}\|^2$ are G-invariant, we have

$$f_1^2=q_{2(t_1+1)}+\|\boldsymbol{x}\|^2 q_{2t_1}+\|\boldsymbol{x}\|^4 q_{2t_1-2}+\cdots+\|\boldsymbol{x}\|^{2t_1}q_2+\|x\|^{2(t_1+1)}q_0,$$

where

$$q_j=\frac{1}{|G|}\sum_{M\in G}(Mp_j) \qquad \text{for } j=2,4,\cdots,2(t_1+1).$$

So the q_j are homogeneous harmonic G-invariant polynomials for $j=0,2,4,\cdots,2(t_1+1)$. Since $f_1\not\equiv 0$ (because $f_1(\boldsymbol{x}_1)\neq 0$), we have

$$\int_{\Omega_d} f_1^2 d\omega=q_0\int_{\Omega_d} d\omega>0, \qquad \text{and so we have } q_0=p_0>0.$$

Now we assume that $f(\boldsymbol{x}_2)=0$ for all (non constant) homogeneous harmonic G-invariant polynomials f of degree $\leqq 2(t_1+1)$, and we will get a contradiction. Since the degree of $f_1\leqq t_1+1$, we have $f_1(\boldsymbol{x}_2)=0$ by our assumption. This implies that, since $\|\boldsymbol{x}_2\|^2=1$ and $q_0>0$, we have $q_j(\boldsymbol{x}_2)\neq 0$ for some $j=2,4,\cdots,2(t_1+1)$. But this is a contradiction. So we proved $t_2\leqq 2t_1+1$. Hence the proof of Theorem 1 is completed.

§ 2. Proof of Theorem 2 and the second proof of Theorem 1.

We first prove Theorem 2.

Let $\{S_{ij} \mid 0\leqq j\leqq Q_i(1)\}$ be an orthonormal basis of the space Harm(i) of the homogeneous harmonic polynomials of degree i in $\boldsymbol{R}^d$, (see [8, § 11.4], or [6, Theorem 3.3]). So

$$Q_i(1)=\text{degree of } \rho_i=\binom{d+i-1}{i}-\binom{d+i-3}{i-2}.$$

Then any $g\in O(d)$ acts on the space Harm(i) by

$$\rho_i(g)f(\boldsymbol{x})=f(\rho_1(g^{-1})\boldsymbol{x}), \tag{2.1}$$

where ρ_i denotes the i-th spherical representation of $O(d)$. Then $\rho_i(g)\in O(Q_i(1))$ for any $g\in O(d)$ (see [8, § 11.4, Lemma 5]).

Let H_i be the characteristic matrix of a subset X of Ω_d, i.e., H_i is the $|X|\times Q_i(1)$ matrix whose rows are parametrized by the elements of X and columns are parametrized by $j\in\{1, 2, \cdots, Q_i(1)\}$ and whose $(\boldsymbol{x}, j)$-entry is given by $S_{ij}(\boldsymbol{x})$. Now, let

$$R_s(1)=Q_0(1)+Q_1(1)+\cdots+Q_s(1)=\binom{d+s-1}{s}+\binom{d+s-2}{s-1},$$

and let H be the $|X|\times R_s(1)$ matrix obtained by arranging as follows:

$$H=[H_0, H_1, \cdots, H_s].$$

LEMMA 2.1. *If X is a spherical 2s-design in $\Omega_d\subset\boldsymbol{R}^d$, then*

$$\det({}^tH\cdot H)\neq 0.$$

So, H has the maximum possible rank $R_s(1)$.

PROOF. By [6, Theorem 5.3]. (Note that the orthonormal basis $\{S_{ij}\}$ is a fixed scalar multiple of the orthogonal (but not normal) basis $\{W_{ij}\}$ used in [6].)

LEMMA 2.2. *Let G be a finite subgroup of $O(d)$, and let G fix a subset X of Ω_d as a set. Let ρ_i $(i=0, 1, \cdots)$ be the i-th spherical representation of G, and let π be the permutation representation of G on X. Then we have*

$$\pi(g)\cdot H_i\cdot{}^t\rho_i(g)=H_i \qquad \textit{for all } g\in G$$

for $i=0, 1, 2, \cdots$. Furthermore, if we set

$$\tilde{\rho}=\rho_0\dot{+}\rho_1\dot{+}\cdots\dot{+}\rho_s \quad (\textit{direct sum}),$$

then we have

$$\pi(g)\cdot H=H\cdot\tilde{\rho}(g), \qquad \textit{for all } g\in G.$$

PROOF. Since $\rho_i(g)S_{ij}(\boldsymbol{x})=S_{ij}(\rho_1(g^{-1})\boldsymbol{x})$, we have the first assertion. The second assertion is obtained from the fact that the $\rho_i(g)$ are elements in $O(Q_i(1))$ and from the definitions of H and $\tilde{\rho}$.

PROOF OF THEOREM 2. Since the matrix H intertwines the two representations π and $\tilde{\rho}=\rho_0+\rho_1+\cdots+\rho_s$, and since H has the maximum possible rank $R_s(1)$ by Lemma 2.1, we get the desired result by Schur's lemma.

PROOF OF COROLLARY TO THEOREM 2. $(1_G, \pi)_G$ is the number of orbits of G on X. Since $\pi\supseteqq\rho_0+\rho_1+\cdots+\rho_s$ and since $(1_G, \pi)_G=1$ in our case, we have $(\rho_0, \rho_i)_G=\delta_{0i}$ for $i=0, 1, \cdots, s$, where $\rho_0=1_G$.

Next we prove:

Theorem 1 $\Leftrightarrow$ Corollary to Theorem 2.

PROOF. ($\Rightarrow$) By Theorem 1, $\boldsymbol{x}^G$ must be a spherical $[t/2]$-design for any $\boldsymbol{x}\in\Omega_d$. So we have the desired result by [9, Theorem 6.10, (i)$\Rightarrow$(iii)]. ($\Leftarrow$) Let

x_1^G be a spherical t_1 design but not (t_1+1)-design. If x_2^G is a spherical $2(t_1+1)$ design, then by Corollary to Theorem 2, $(\rho_0, \rho_i)_G=\delta_{i0}$ for $i=0, 1, \cdots, (t_1+1)$. So x_2^G is a spherical (t_1+1)-design by [**9**, Theorem 6.10, (iii)⇒(i)]. But this is a contradiction.

REMARKS. (i) The argument given here works for the compact symmetric spaces of rank 1. (Proofs work exactly the same way word for word.)

(ii) Let $\mathfrak{X}=(X, \{R_i\}_{0\leq i\leq d})$ be a symmetric association scheme of class d which comes from a generously transitive permutation group G of rank d. If $\mathfrak{X}$ is a Q-polynomial scheme, then the concept of t-design is well defined for any positive integer t, and the representation ρ_i corresponds to the irreducible representation of G (appearing in the permutation representation) which corresponds to the primitive idempotent E_i. Using this definition of t-design and the representations ρ_i (and by replacing $O(d)$ by the group G and Ω_d by X), a result similar to Theorem 2 is obtained. (Again the proof is exactly the same.)

(iii) A special case of (ii), i.e., a result similar to Corollary to Theorem 2 (which is a special case of Theorem 2) was previously obtained by Noda [**18**, Theorem 1 and Corollary 2] for ordinary t-designs, i.e., for Johnson association schemes. (See also Dowling [**7**], where he formulated a result similar to Noda's in a slightly more general setting.) A generalization of it, i.e., a representation theoretical result similar to Theorem 2 was formulated by Stanley [**21**, Lemma 9.1] in terms of bipartite graphs, although the argument itself was a kind of folklore in group theory. Our proof of Theorem 2 is based on the same argument. Although there is no graph available in our case, the characteristic matrix H plays a role similar to that of the incidence matrix of bipartite graph.

§ 3. t-homogeneity and t-transitivity for (finite) subgroups G of $O(d)$.

A finite permutation group G on a finite set X (with $|X|=n$) (i.e., a finite subgroup G of S_n, the symmetric group of degree n) is said to be t-transitive (resp. t-homogeneous) if G acts transitively on the set of ordered t-tuples (resp. unordered t-tuples). So a t-transitive group is t-homogeneous by the definition. In what follows (unless otherwise stated) we always assume that $t\leqq n/2$ when we discuss t-homogeneity. The reader is referred to Neumann [**17**] for further references on these concepts.

Let $\chi^{(n_1, n_2, \cdots, n_l)}$, with $n_1+\cdots+n_l=n$ and $n_1\geqq n_2\geqq\cdots\geqq n_l>0$, be the irreducible representation of the symmetric group S_n corresponding to the Young diagram of type $(n_1, n_2, \cdots, n_l)$. The *level* of $\chi=\chi^{(n_1, n_2, \cdots, n_l)}$ is defined to be the number $n_2+n_3+\cdots+n_l$ $(=n-n_1)$. The following characterization of t-transitivity and t-homogeneity for a subgroup G of S_n is well known (and easily verified) (cf. [**17**]).

(i) A subgroup G of S_n is t-transitive if and only if

$$(3.1)\qquad (\chi^{(n_1,\cdots,n_l)}, \chi^{(n_1',n_2',\cdots,n_{l'}')})_G=\delta_{(n_1,n_2,\cdots,n_l),(n_1',n_2',\cdots,n_{l'}')}$$

$$(=(\chi^{(n_1,n_2,\cdots,n_l)}, \chi^{(n_1',n_2',\cdots,n_{l'}')})_{S_n})$$

for all irreducible characters $\chi=\chi^{(n_1,n_2,\cdots,n_l)}$ and $\chi'=\chi^{(n_1',n_2',\cdots,n_{l'}')}$ such that level$(\chi)+$level$(\chi')\leqq t$.

(ii) A subgroup G of S_n is t-homogeneous if and only if

$$(3.2)\qquad (\chi^{(n)}, \chi^{(n-l,l)})=\delta_{0,l}$$

$$(=(\chi^{(n)}, \chi^{(n-l,l)})_{S_n})\qquad \text{for } l=0, 1, \cdots, t.$$

In this section, I would like to propose the definitions of t-transitivity and t-homogeneity for subgroups G of $O(d)$. To begin with, let us recall the representation theory of $O(d)$.

It is well known (cf. [**23, 16**]) that there is a one to one correspondence between the set of irreducible representations of $O(d)$ and the set of Young diagrams $(n_1, n_2, \cdots, n_l)$ of any size such that $l_1+l_2\leqq d$, where the number l_i $(i=0, 1, \cdots, n_1)$ denotes the size of the i-th column of the Young diagram $(n_1, n_2, \cdots, n_l)$. In what follows we will call a Young diagram $(n_1, n_2, \cdots, n_l)$ *admissible* if $l_1+l_2\leqq d$. It is also well known that the i-th spherical representation ρ_i $(i=0, 1, \cdots)$ corresponds to the Young diagram of type (i).

We propose the following definitions.

DEFINITION 3.1. (i) A (finite) subgroup G of $O(d)$ is called t-transitive (or linearly t-transitive) if and only if

$$(3.3)\qquad (\chi^{(n_1,n_2,\cdots,n_l)}, \chi^{(n_1',n_2',\cdots,n_{l'}')})_G=\delta_{(n_1,\cdots,n_l),(n_1',\cdots,n_{l'}')}$$

$$(=(\chi^{(n_1,n_2,\cdots,n_l)}, \chi^{(n_1',n_2',\cdots,n_{l'}')})_{O(d)})$$

for all admissible diagrams $(n_1, \cdots, n_l)$ and $(n_1', \cdots, n_{l'}')$ such that

$$\text{size}(n_1, \cdots, n_l)+\text{size}(n_1', \cdots, n_{l'}')\leqq t.$$

(ii) A (finite) subgroup G of $O(d)$ is called t-homogeneous (or t-linearly homogeneous) if and only if

$$(3.4)\qquad (\chi^{\emptyset}, \chi^{(i)})_G=\delta_{0,i}$$

$$(=(\chi^{\emptyset}, \chi^{(i)})_{O(d)})\qquad \text{for } i=0, 1, \cdots, t.$$

By the definition, if a subgroup G of $O(d)$ is t-transitive, then it is t-homogeneous. A combinatorial characterization of t-homogeneous finite subgroups G of $O(d)$ is given by Theorem 6.10 of Goethals-Seidel [**9**]. That is, a finite subgroup G of $O(d)$ is t-homogeneous if and only if the set $X=\boldsymbol{x}^G$ is a spherical t-design in $\boldsymbol{R}^d$ for any $\boldsymbol{x}\in\Omega_d$. (Right now we do not know any similar combinatorial characterization of t-transitive subgroups G of $O(d)$.)

Now, let us recall some important results on t-transitive and t-homogeneous *permutation* groups.

(a) If G is a t-homogeneous permutation group, then it is $(t-1)$-transitive.

(b) Let $t\geqq 5$. If G is a t-homogeneous permutation group, then it is t-transitive, (Livingston-Wagner [**15**]).

(c) For $t\leqq 4$, t-homogeneous but not t-transitive permutation groups are classified, (Kantor [**13**]).

(d) By assuming the classification of finite simple groups, t-transitive groups are classified for $t\geqq 2$. (Curtis-Kantor-Seitz [**4**], Hering [**11**], O'Nan (unpublished), etc.)

PROBLEMS AND CONJECTURES. As in the permutation group case, I expect that for large t there is no *finite* subgroup G of $O(d)$ (with $d\geqq 3$) which is either t-transitive or t-homogeneous. (I believe that I have succeeded in proving the non-existence of any t-transitive finite subgroup G of $O(d)$ (with $d\geqq 3$) for large t by having recourse to the classification of finite simple groups. I am now working on the t-homogeneous case.) However, contrary to the permutation group case, the corresponding assertion to (a) is not true for finite subgroups of $O(d)$ (at least for some t). So, this makes it difficult to obtain results corresponding to (b), (c), (d) above for finite subgroups G of $O(d)$, although I do believe that it is not absolutely impossible to obtain such results (by using the classification of finite simple groups).

In the above definition 3.1 (i) and (ii), we did not use the finiteness of the group G. If G is a compact subgroup of $O(d)$, then the definitions (i) and (ii) still make sense. If we allow compact (non finite) subgroups G of $O(d)$, then we do have examples which are (linearly) t-homogeneous for any t, but not 4-transitive. Namely, take a unitary group $U(d')$ of degree d'. Then $U(d')$ is naturally imbedded in $O(2d')$ as a subgroup. This subgroup is shown to be t-homogeneous for any t, but not 4-transitive (see §4 for the details).

The above mentioned definitions of t-transitivity and t-homogeneity are generalized for other situations, in particular for compact symmetric spaces of rank 1 and for Q-polynomial association schemes which come from generously transitive permutation groups.

DEFINITION 3.2. Let $M=\tilde{G}/\tilde{H}$ be a compact symmetric space of rank 1 (i.e., M is a compact 2-point homogeneous space: Such spaces together with the compact Lie group $\tilde{G}$ and subgroup $\tilde{H}$ are classified, see [**24**], [**26**]). Then the i-th spherical representations ρ_i of $\tilde{G}$ are well defined for $i=0, 1, \cdots$.

(i) A (finite) subgroup G of $\tilde{G}$ is t-transitive if and only if

$$(\rho^m, \rho^n)_G=(\rho^m, \rho^n)_{\tilde{G}} \tag{3.3'}$$

for any m and n with $m+n\leqq t$, where $\rho^i=\rho\otimes\cdots\otimes\rho$ (i ρ's) denotes the i-th tensor product of ρ. (If $\tilde{G}=O(d)$ and $M=\Omega_d$, then the definition (3.3′) is shown to be equivalent to (3.3).)

(ii) A (finite) subgroup G of $\tilde{G}$ is t-homogeneous if and only if

$$(\rho_0, \rho_i)_G=(\rho_0, \rho_i)_{\tilde{G}} \tag{3.4'}$$

for all $i=0, 1, \cdots, t$. (If $\tilde{G}=O(d)$ and $M=\Omega_d$, then the definition (3.4′) is exactly the same as (3.4).) (There may be room of debate whether the definition (i) of t-transitivity is the best one or not.)

DEFINITION 3.3. Let $\mathfrak{X}=(X, \{R_i\}_{0\leq i\leq d})$ be a Q-polynomial association scheme of class d which comes from a generously transitive permutation group $\tilde{G}$ of rank $d+1$ on a finite set X.

(i) A subgroup G of $\tilde{G}$ is t-transitive if and only if

$$(\rho_1^m, \rho_1^n)_G=(\rho_1^m, \rho_1^n)_{\tilde{G}} \tag{3.1)'}$$

for any m and n with $m+n\leqq t$, where ρ_1 is the representation of $\tilde{G}$ which correspond to the primitive idempotent E_1 of the Q-polynomial structure, and $\rho_1^i=\rho_1\otimes\cdots\otimes\rho_1$ (i ρ_1's).

(ii) A subgroup G of $\tilde{G}$ is t-homogeneous if and only if

$$(\rho_0, \rho_i)_G=\delta_{0i}\ (=(\rho_0, \rho_i)_{\tilde{G}}) \tag{3.2)'}$$

for $i=0, 1, \cdots, t$. (ρ_i is the representation of $\tilde{G}$ corresponding to the primitive idempotents E_i of the Q-polynomial structure.) (If $\mathfrak{X}$ is the Johnson association scheme $J(v, k)$ and $\tilde{G}=S_v$, then the definitions (3.1)′ and (3.2)′ are equivalent to (3.1) and (3.2) respectively. There may be room of debate whether the definition (i) is the best one or not.)

REMARKS. (i) We point out that Theorem 2 (in §2) is also true for finite subgroup G of $\tilde{G}$ of a compact symmetric space M of rank 1 and t-designs there. (Note that the concept of t-design is well defined there.) Theorem 2 is also true for subgroup G of $\tilde{G}$ of a Q-polynomial association scheme which comes from a generously transitive permutation group $\tilde{G}$. (Note that in the above two cases, the non-singularity of the matrix corresponding to H in Theorem 2 is proved, exactly the same way as for spherical t-designs and for ordinary t-designs, for compact symmetric spaces of rank 1 and for Q-polynomial schemes.) Also, note that the combinatorial characterization of t-homogeneous groups G of $\tilde{G}$ works for these two cases.

(ii) In the Johnson scheme case, i.e., in the permutation group case, there is no nontrivial subgroup G which is t-homogeneous for large t. Also t-homogeneity implies $(t-1)$-transitivity. However, in some other Q-polynomial association schemes (which come from generously transitive permutation groups), the situation is quite different. For example, in the Hamming scheme $H(n, q)$ (the group $\tilde{G}=S_q \wr S_n$), a regular subgroup G of order q^n is t-homogeneous for $t=0, 1, \cdots, n$, but not 2-transitive. It would be interesting to know which Q-polynomial schemes (coming from generously transitive permutation groups) are like Johnson schemes and which are not.

§ 4. Counter examples.

In this section we will construct some counter examples to the converse of (i) of Theorem A (i.e., to Theorem 6.7 of [**9**]). Namely, we will show the following:

PROPOSITION 4.1. *The fact that a finite subgroup G of $O(d)$ is t-homogeneous does not always imply*

$$(\rho_i, \rho_i)_G=1 \qquad \textit{for all } i=0, 1, \cdots, \left[\frac{t}{2}\right], \quad \textit{and for all } t.$$

[In the proof of Theorem 6.7 in Goethals and Seidel [**9**], page 267 line 4 up, it is claimed that "This polynomial cannot be a constant on Ω_d". But this seems to be not always true.]

Our counter examples given below are based on the fact that $U(d')$, naturally imbedded in $O(2d')$, is t-homogeneous for all t, but is not 4-transitive. (That is, ρ_2 of $O(2d')$ is not (real) irreducible when restricted to $U(d')$.)

Suppose that $d=2d'$ (d even). Then the unitary group $U(d')$ (of matrix size d') is embedded in $O(d)$ naturally by the map

$$U=A+\sqrt{-1}\,B \longmapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

(with A, B real matrices). Let $z_1, \cdots, z_{d'}$ be the complex variables $z_i=x_i+\sqrt{-1}\,y_i$. $\mathrm{Hom}(i, j)$ denotes the space of all $\boldsymbol{C}$-coefficient homogeneous polynomials in $z_1, \cdots, z_{d'}, \bar{z}_1, \cdots, \bar{z}_{d'}$ of degree i in $z_1, \cdots, z_{d'}$ and of degree j in $\bar{z}_1, \cdots, \bar{z}_{d'}$. Then the Laplacian

$$\Delta=\frac{\partial^2}{\partial z_1 \partial \bar{z}_1}+\cdots+\frac{\partial^2}{\partial z_{d'} \partial \bar{z}_{d'}}$$

where

$$\frac{\partial}{\partial \bar{z}_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}+\sqrt{-1}\,\frac{\partial}{\partial y_i}\right)$$

and

$$\frac{\partial}{\partial z_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}-\sqrt{-1}\frac{\partial}{\partial y_i}\right)$$

induces an *onto* homomorphism from $\mathrm{Hom}(i, j)$ to $\mathrm{Hom}(i-1, j-1)$. The kernel $\mathrm{Ker}\Delta\cap\mathrm{Hom}(i, j)$ is denoted by $\mathrm{Harm}(i, j)$. We have

$$\begin{aligned}\dim \mathrm{Harm}(i, j)&=\dim \mathrm{Hom}(i, j)-\dim \mathrm{Hom}(i-1, j-1)\\ &=\binom{d'+i-1}{i}\binom{d'+j-1}{j}-\binom{d'+i-2}{i-1}\binom{d'+j-2}{j-1}.\end{aligned}$$

$U(d')$ acts on $\mathrm{Harm}(i, j)$ naturally, and this representation, here we denote it by $\binom{(i)}{(j)}$, is irreducible for any i and j. It is well known that ρ_i (=the i-th spherical representation of $O(d)$) is decomposed as follows when restricted to the subgroup $U(d')$:

$$\rho_i=\bigoplus_{k=0}^{i}\binom{(k)}{(i-k)}\qquad \text{(as } U(d')\text{-spaces)}.$$

In particular,

$$\rho_1=\binom{(1)}{\emptyset}+\binom{\emptyset}{(1)}$$

and

$$\rho_2=\binom{(2)}{\emptyset}+\binom{(1)}{(1)}+\binom{\emptyset}{(2)}.$$

Now, $\binom{(2)}{\emptyset}+\binom{\emptyset}{(2)}$ is a real irreducible representation of $U(d')$. So $U(d')\subset O(2d')$ cannot be 4-transitive, because

$$(\rho_2, \rho_2)_{U(d')}\neq 1\qquad \text{(when } U(d')\subset O(2d')\text{)}$$

Also, we can see that

$$(\rho_0, \rho_i)_{U(d')}=\delta_{0i}\qquad \text{for } i=0, 1, \cdots.$$

This fact suggests that if we can find an (absolutely) irreducible finite subgroup G of $U(d')$ such that (ρ_1), ρ_2, ρ_3, ρ_4 of $O(2d')$ restricted to G do not contain the identity character $(=\rho_0)$ of G, then such G is 4-homogeneous, but ρ_2 is not irreducible, and so G is a desired example.

An explicit such example G is given by the 6-fold covering of the $PSU_4(3)$ in $U(6)$ (in $O(12)$). The above mentioned properties are easily checked by using the character table of the 6-fold covering group of $PSU_4(3)$ obtained by Lindsey [14]. First, this was checked by computations. The following argument will replace the computations. Actually what we need is only the character table of

$PSU_4(3)$. The fact that $\binom{(2)}{\varnothing}$, $\binom{\varnothing}{(2)}$, $\binom{(3)}{\varnothing}$, $\binom{(2)}{(1)}$, $\binom{(1)}{(2)}$, $\binom{\varnothing}{(3)}$, $\binom{(4)}{\varnothing}$, $\binom{(3)}{(1)}$, $\binom{(1)}{(3)}$ and $\binom{\varnothing}{(4)}$ do not contain ρ_0 (=the identity representation) when restricted to G is clear by considering the restrictions of the representations to the center Z (of order 6) of G. So we need only to prove that $\binom{(1)}{(1)}$ and $\binom{(2)}{(2)}$ do not contain ρ_0 when restricted to G. $\binom{(1)}{(1)}$ and $\binom{(2)}{(2)}$ must be representations of $PSU_4(3)$ $(=G/Z)$, and it is easily seen from the character table (Lindsey [**14**]) that $\binom{(1)}{(1)}$ (of degree 35) remain irreducible. We know that as the representations of $U(d')$ the tensor product of two $\binom{(1)}{(1)}$'s does contain $\binom{(2)}{(2)}$ and that (since $\binom{(1)}{(1)}$ has a real character) the tensor product of two $\binom{(1)}{(1)}$'s contains ρ_0 multiplicity one. Therefore $\binom{(2)}{(2)}$ does not contain ρ_0 when restricted to G. The same argument shows that the 6-fold covering of the sporadic Suzuki simple group S in $U(12)$ is another example (see Wright [**25**] for the character table of S). In this case, $\binom{(1)}{(1)}$ is of degree 343. (I believe that there are some other examples. But I believe that for large t there is no t-homogeneous finite subgroup G of $O(d)$, and so there are no counter examples for large t.)

It would be very interesting to know to what extent the converse of (i) of Theorem A is true.

REMARKS. (i) The result corresponding to the converse of (i) of Theorem A is also not true for the Johnson scheme, i.e., for subgroups of S_n. An explicit counter example is $P\Gamma L(2, 32)$ acting on 33 letters. This permutation group is 4-homogeneous, but the representation $\rho_2=\chi^{(31,2)}$ of S_{33} restricted to $P\Gamma L(2, 32)$ is not irreducible. (Since t-homogeneous but not t-transitive groups are classified, it would be possible to list all such counter examples.)

(ii) As the converse of (i) of Theorem A is not correct, the answer to the question, Remark 6.9 in [**9**], is also negative.

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