J. Math. Soc. Japan
Vol. 36, No. 2, 1984

Discrete reflection groups in a parabolic subgroup of Sp(2, $\boldsymbol{R}$) and symmetrizable hyperbolic generalized Cartan matrices of rank 3

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(Received June 21, 1982)
(Revised March 28, 1983)

§ 0. Introduction.

Let Λ be a congruence subgroup of Siegel modular group Sp(2, $\boldsymbol{Z}$) acting on the Siegel upper half space $\mathfrak{S}_2$ of degree 2. Singularity and uniformizability of the Satake compactification of the factor space $\mathfrak{S}_2/\Lambda$ are studied by several authors (cf. for example, Igusa [2] and Christian [1]). In this paper, we shall study the uniformizability at the zero dimensional cusps.

Let $\mathcal{P}$ be the maximal parabolic subgroup of Sp(2, $\boldsymbol{R}$) corresponding to a zero dimensional boundary component. For a symmetrizable hyperbolic generalized Cartan matrix C of rank 3, we shall construct a discrete subgroup $\mathcal{P}(C)$ of $\mathcal{P}$, and show that the Satake compactification of the quotient space $\mathfrak{S}_2/\mathcal{P}(C)$ is non-singular by using Looijenga's theory ([4]).

The author wishes to express his deepest appreciation to Professor T. Ibukiyama and Professor T. Yamazaki who gave him unceasing encouragement and valuable comments.

§ 1. Generalized Cartan matrix and its Weyl group W.

We shall review some fundamental definition and facts about generalized Cartan matrices and their Weyl groups (see [4]).

DEFINITION. An $n\times n$ matrix $C=(C_{ij})$ is called a generalized Cartan matrix (G.C.M.) of rank n if (i) $C_{ij}\in\boldsymbol{Z}$, $C_{jj}=2$, (ii) $C_{ij}\leqq 0$ $(i\neq j)$ and (iii) $C_{ij}=0$ if and only if $C_{ji}=0$.

DEFINITION. The matrix C is said to be classical if C is the Cartan matrix of some semi-simple Lie algebra over $\boldsymbol{C}$.

DEFINITION. The matrix C is said to be euclidean if (i) it is indecomposable, (ii) $\det C=0$ and (iii) $C_k=(C_{ij})_{i,j\neq k}$ is classical for all k.

DEFINITION. The matrix C is said to be hyperbolic if (i) it is indecomposable, (ii) C is neither classical nor euclidean and (iii) C_k is classical or euclidean for all k.

DEFINITION. The matrix C is said to be symmetrizable if there exists a diagonal matrix $D=\mathrm{diag}(d_1, \cdots, d_n)$, $d_j \in \boldsymbol{Q}$, $d_j>0$ $(j=1, \cdots, n)$ such that CD is symmetric.

Let V be an n dimensional real vector space with bases α_j $(j=1, \cdots, n)$ and V^* the dual space of V. Let C be a G.C.M. of rank n. We define elements $\alpha_j^\vee$ $(j=1, \cdots, n)$ of V^* by

$$\langle \alpha_i, \alpha_j^\vee \rangle = C_{ij} \qquad i, j=1, \cdots, n$$

where $\langle\, ,\rangle$ denotes the dual pairing of V and V^*. The elements α_i $(i=1, \cdots, n)$ and $\alpha_j^\vee$ $(j=1, \cdots, n)$ are called roots and coroots, respectively. Fundamental reflection $s_j \in \mathrm{GL}(V)$ with a root α_j is defined by

$$s_j(\xi)=\xi-\langle \xi, \alpha_j^\vee \rangle \alpha_j, \qquad \xi \in V.$$

DEFINITION. The subgroup of $\mathrm{GL}(V)$ generated by $s_1, \cdots, s_n$ is called the Weyl group of C and denoted by W.

If C is symmetrizable, and if we identify $\mathrm{GL}(V)$ with the matrix representation with respect to the bases $\alpha_1, \cdots, \alpha_n$, then we have

$${}^t w(CD)w=CD, \qquad w \in W,$$

where D is a diagonal matrix mentioned above.

We put

$$C=\{\xi \in V \mid \langle \xi, \alpha_j^\vee \rangle >0\}: \text{ Weyl chamber},$$

$$I=\bigcup_{w \in W} w\bar{C}: \text{ Tits cone},$$

$$\mathring{I}: \text{ the interior of } I,$$

$$Q:=\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_n: \text{ root lattice},$$

$$\widetilde{W}=W \ltimes Q,$$

$$\Omega=V+\sqrt{-1}\,\mathring{I} \subset V \otimes_{\boldsymbol{R}} \boldsymbol{C}.$$

The groups W and $\widetilde{W}$ acts properly discontinuously on $\mathring{I}$ and Ω, respectively. Here the lattice Q acts on Ω as real displacements.

§ 2. Symmetrizable hyperbolic G.C.M. of rank 3.

LEMMA 2.1. *Let* $C=(C_{ij})_{i,j=1,2,3}$ *be a G.C.M. of rank* 3. *The matrix* C *is symmetrizable if and only if* $C_{12}C_{23}C_{31}=C_{21}C_{32}C_{13}$. *If* C *is indecomposable and symmetrizable,* C *is hyperbolic if and only if*

$$\det C_k \geqq 0 \qquad k=1, 2, 3$$

and

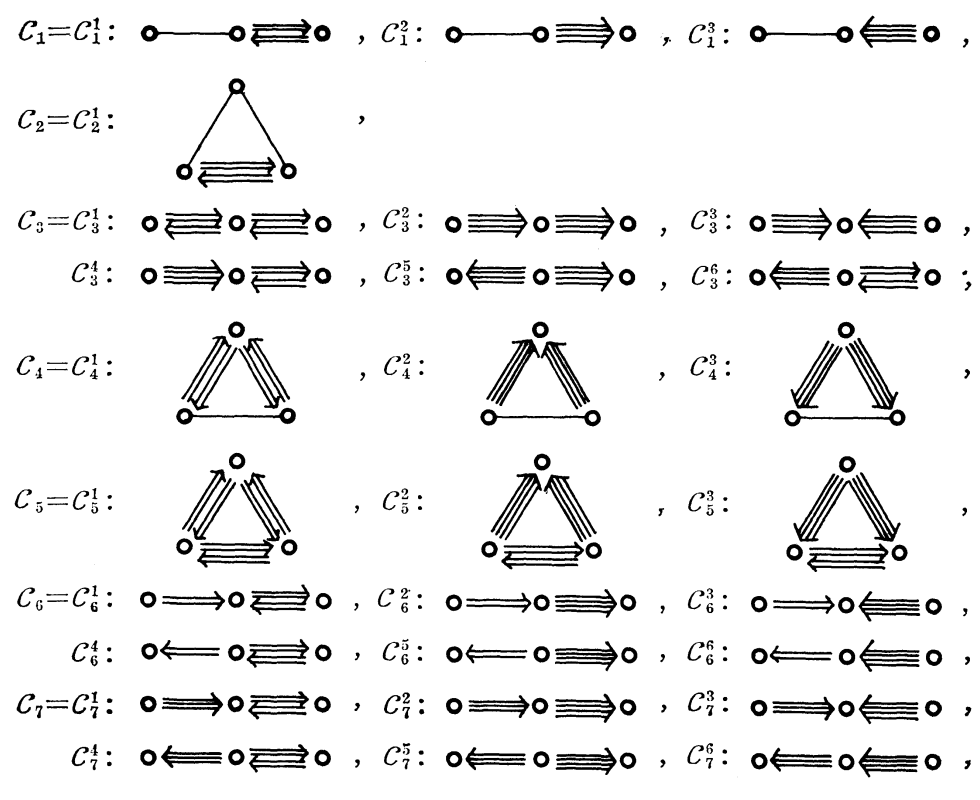
$$C_{12}C_{21}+C_{23}C_{32}+C_{31}C_{13}-C_{12}C_{23}C_{31}-4>0 .$$

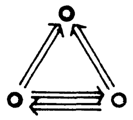
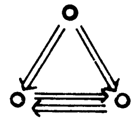
PROOF. Let $D=\mathrm{diag}(d_1, d_2, d_3)$. The matrix CD is symmetric if and only if $C_{ij}d_j-C_{ji}d_i=0$ $(i, j=1, 2, 3)$. This system of linear equations with respect to d_1, d_2 and d_3 has a non-zero solution if and only if $C_{12}C_{23}C_{31}=C_{21}C_{32}C_{13}$. To prove the latter part, we have only to notice the following equality

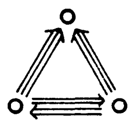
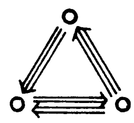
$$\det C=8+C_{12}C_{23}C_{31}+C_{21}C_{32}C_{13}-2C_{12}C_{21}-C_{23}C_{32}-C_{31}C_{13}$$

and recall the fact (cf. [5]) that a G.C.M. is classical or euclidean if and only if it is positive semi-definite. Q.E.D.

By the lemma we can find every symmetrizable hyperbolic G.C.M. of rank 3. The corresponding Dynkin diagrams are the following:

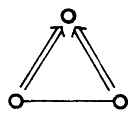
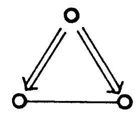


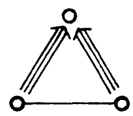
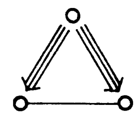
$C_8 = C_8^1$:  , C_8^2:  ,

$C_9 = C_9^1$:  , C_9^2:  ,

$C_{10} = C_{10}^1 : \circ\Rightarrow\circ\Rightarrow\circ$, $C_{10}^2 : \circ\Rightarrow\circ\Leftarrow\circ$, $C_{10}^3 : \circ\Leftarrow\circ\Rightarrow\circ$, $C_{10}^4 : \circ\Leftarrow\circ\Leftarrow\circ$,

$C_{11} = C_{11}^1 : \circ\Rrightarrow\circ\Rrightarrow\circ$, $C_{11}^2 : \circ\Rrightarrow\circ\Lleftarrow\circ$, $C_{11}^3 : \circ\Lleftarrow\circ\Rrightarrow\circ$,

$C_{12} = C_{12}^1$:  , C_{12}^3:  ,

$C_{13} = C_{13}^1$:  , C_{13}^2:  .

Notice that the Weyl group of C_ν^j coincides with that of C_ν. We give the corresponding matrices $\{C_\nu\}$, for later use.

$$C_1=\begin{pmatrix} 2 & -1 & \\ -1 & 2 & -2 \\ & -2 & 2 \end{pmatrix},\quad C_2=\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix},\quad C_3=\begin{pmatrix} 2 & -2 & \\ -2 & 2 & -2 \\ & -2 & 2 \end{pmatrix},$$

$$C_4=\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix},\quad C_5=\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},\quad C_6=\begin{pmatrix} 2 & -1 & \\ -2 & 2 & -2 \\ & -2 & 2 \end{pmatrix},$$

$$C_7=\begin{pmatrix} 2 & -1 & \\ -3 & 2 & -2 \\ & -2 & 2 \end{pmatrix},\quad C_8=\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -2 \\ -2 & -1 & 2 \end{pmatrix},\quad C_9=\begin{pmatrix} 2 & -1 & -2 \\ -3 & 2 & -3 \\ -2 & -1 & 2 \end{pmatrix},$$

$$C_{10}=\begin{pmatrix} 2 & -1 & \\ -2 & 2 & -1 \\ & -3 & 2 \end{pmatrix},\quad C_{11}=\begin{pmatrix} 2 & -1 & \\ -3 & 2 & -1 \\ & -3 & 2 \end{pmatrix},\quad C_{12}=\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix},$$

$$C_{13}=\begin{pmatrix} 2 & -1 & -1 \\ -3 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}.$$

Throughout this paper, C stands for a symmetrizable hyperbolic G.C.M. of rank 3, and D a diagonal matrix with the properties mentioned in § 1. If C is symmetric, we put $D=E$ (the unit matrix).

For each C, put

$$V_-=\{\xi=\xi_1\alpha_1+\xi_2\alpha_2+\xi_3\alpha_3\in V \mid (\xi_1, \xi_2, \xi_3)(CD)^t(\xi_1, \xi_2, \xi_3)<0\}.$$

LEMMA 2.2. *Notations being as above, we have*

$$C\subset V_-.$$

PROOF. We first note that the sign of CD is $(+, +, -)$ and so the domain V_- is a disjoint union of two convex cones in V. Since C is a cone in V bounded by three planes $\langle\xi, \alpha_j\rangle=0$ $(j=1, 2, 3)$, we have only to show that the intersection of any two planes in question is contained in V_-. Let $\xi=\xi_1\alpha_1+\xi_2\alpha_2+\xi_3\alpha_3\in V$ satisfies

$$\langle\xi, \alpha_1^\vee\rangle=C_{11}\xi_1+C_{21}\xi_2+C_{31}\xi_3\neq 0,$$

$$\langle\xi, \alpha_2^\vee\rangle=C_{12}\xi_1+C_{22}\xi_2+C_{32}\xi_3=0,$$

$$\langle\xi, \alpha_3^\vee\rangle=C_{13}\xi_1+C_{23}\xi_2+C_{33}\xi_3=0.$$

Since these equalities imply

$$\xi_1=\langle\xi, \alpha_1^\vee\rangle\begin{vmatrix} 2 & C_{32} \\ C_{23} & 2 \end{vmatrix}\Big/\det C,$$

we have

$$\begin{aligned}(\xi_1, \xi_2, \xi_3)(CD)^t(\xi_1, \xi_2, \xi_3)&=\sum C_{ji}d_i\xi_i\xi_j\\ &=\langle\xi, \alpha_1^\vee\rangle d_1\xi_1\\ &=\langle\xi, \alpha_1\rangle^2 d_1\det C_1/\det C.\end{aligned}$$

This proves the lemma. Q.E.D.

COROLLARY 2.3. *The cone $\mathring{I}$ is a connected component of V_-.*

PROOF. We consider the situation $C\subset V_-$ modulo the multiplicative group $\boldsymbol{R}^+$. Recall that CD is the W-invariant indefinite form, then we know that the closure of $C/\boldsymbol{R}^+$ in $V_-/\boldsymbol{R}^+$ is a fundamental domain of the triangle group W acting on the Klein model of the hyperbolic space $(V_-/\boldsymbol{R}^+, CD)$. Q.E.D.

§ 3. Representation of W into GL(2, $\boldsymbol{R}$).

Set

$$H=\begin{pmatrix} & & -1 \\ & 2 & \\ -1 & & \end{pmatrix}.$$

LEMMA 3.1. *For each C, there exists a 3×3 real matrix A such that $D={}^tAHA$, and with respect to the bases e_1, e_2, e_3 of V defined by $(e_1, e_2, e_3)=(\alpha_1, \alpha_2, \alpha_3)A^{-1}$, the cone $\mathring{I}$ is represented by*

$$\mathring{I}=\{\xi=ue_1+we_2+ve_3\in V \mid u>0,\ w^2-uv<0\}.$$

PROOF. Since both the symmetric matrices CD and H have the same signature $(+, +, -)$, there exists a matrix A such that $CD={}^tAHA$. If we change the sign of A, if necessary, we conclude by Lemma 2.2 and Corollary 2.3 that the cone $\mathring{I}$ is represented as in the lemma. Q.E.D.

The matrix A is by no means uniquely determined. To fix an idea, for each matrix $C=C_\nu$, we choose and fix matrices D and A as follows:

$$A_1=\begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ -1 & & \end{pmatrix},\quad A_2=\begin{pmatrix} -1 & 1 & -1 \\ -1 & & 1 \\ & -1 & \end{pmatrix},\quad A_3=\begin{pmatrix} -1 & & \\ 1 & -1 & 1 \\ & & -2 \end{pmatrix},$$

$$A_4=\begin{pmatrix} -1 & & \\ 1 & -1 & 1 \\ & & -3 \end{pmatrix},\quad A_5=\begin{pmatrix} & -2 & \\ 1 & -1 & -1 \\ & & -2 \end{pmatrix},\quad A_6=\begin{pmatrix} 1/\sqrt{2} & -\sqrt{2} & \\ & \sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & & \end{pmatrix},$$

$$A_7=\begin{pmatrix} 1/\sqrt{3} & -\sqrt{3} & \\ & \sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & & \end{pmatrix},\quad A_8=\begin{pmatrix} 1 & -1 & 1 \\ -1 & & 1 \\ & 2 & \end{pmatrix},\quad A_9=\begin{pmatrix} 1 & -1 & 1 \\ -1 & & 1 \\ & 3 & \end{pmatrix},$$

$$A_{10}=\begin{pmatrix} 1/\sqrt{2} & -3\sqrt{2}/2-\sqrt{3} & 3\sqrt{2}/2+\sqrt{3} \\ 1/\sqrt{2} & \sqrt{2}/2 & -3\sqrt{2}/2 \\ -1/\sqrt{2} & 3\sqrt{2}/2-\sqrt{3} & -3\sqrt{2}/2+\sqrt{3} \end{pmatrix},$$

$$A_{11}=\frac{1}{2}\begin{pmatrix} 1+\sqrt{6} & -3-\sqrt{6} & 9+3\sqrt{6} \\ -3 & 3 & -3 \\ -1+\sqrt{6} & 3-\sqrt{6} & -9+3\sqrt{6} \end{pmatrix},\quad A_{12}=\begin{pmatrix} & 1 & 3/2+\sqrt{6}/2 \\ -1 & 1 & -1/2 \\ & -1 & -3/2+\sqrt{6}/2 \end{pmatrix},$$

$$A_{13}=\frac{1}{2}\begin{pmatrix} -2\sqrt{2} & -3(\sqrt{3}+\sqrt{2}) & \sqrt{3}+\sqrt{2} \\ 2\sqrt{3} & \sqrt{3} & -\sqrt{3} \\ -2\sqrt{2} & 3(\sqrt{3}-\sqrt{2}) & -(\sqrt{3}-\sqrt{2}) \end{pmatrix},$$

$$D_1=D_2=D_3=D_4=D_5=E,\quad D_6=\begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix},\quad D_7=\begin{pmatrix} 1 & & \\ & 3 & \\ & & 3 \end{pmatrix},$$

$$D_8=\begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix},\quad D_9=\begin{pmatrix} 1 & & \\ & 3 & \\ & & 1 \end{pmatrix},\quad D_{10}=\begin{pmatrix} 1 & & \\ & 2 & \\ & & 6 \end{pmatrix},$$

$$D_{11}=\begin{pmatrix} 1 & & \\ & 3 & \\ & & 9 \end{pmatrix},\quad D_{12}=\begin{pmatrix} 1 & & \\ & 2 & \\ & & 1 \end{pmatrix},\quad D_{13}=\begin{pmatrix} 1 & & \\ & 3 & \\ & & 1 \end{pmatrix}.$$

Here A_ν and D_ν are matrices satisfying $C_\nu D_\nu={}^tA_\nu HA_\nu$.

In the sequel we shall fix the bases e_1, e_2, e_3 of V. The matrix representation of the fundamental reflection s_j under these bases is denoted by S_j.

Let $S(2, \boldsymbol{R})$ and $S(2, \boldsymbol{C})$ be the sets of 2×2 symmetric matrices over $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively. The group $GL(2, \boldsymbol{R})$ acts on $S(2, \boldsymbol{R})$ and $S(2, \boldsymbol{C})$ by

$$g:\ Z \longmapsto gZ{}^tg\,.$$

Let ρ be the isomorphism $V\to S(2, \boldsymbol{R})$ defined by

$$ue_1+we_2+ve_3 \longmapsto \begin{pmatrix} u & w \\ w & v \end{pmatrix}.$$

Notice that the cone $\mathring{I}$ is transformed by ρ onto the set of all positive definite matrices which will be denoted by $S^+(2, \boldsymbol{R})$. The isomorphism ρ induces the symmetric tensor representation ρ^* of $GL(2, \boldsymbol{R})$ into $GL(V)$, i.e.

$$\rho^*(g)\xi=\rho^{-1}g(\rho\xi){}^tg\,,\qquad \xi\in V\,,\quad g\in GL(2, \boldsymbol{R})\,.$$

REMARK 3.2. For $g\in GL(2, \boldsymbol{R})$, the transformation $\rho^*(g)\in GL(V)$ is a reflection if and only if $\operatorname{tr} g=0$ and $\det g=-1$.

This is easily proved if we notice that ρ^* is given by

$$\rho^*:\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & bc+ad & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

PROPOSITION 3.3. *Notations being as above, we have*

$$S_j=\rho^*\left(\frac{1}{\sqrt{d_j}}\begin{pmatrix} a_{2j} & -a_{1j} \\ a_{3j} & -a_{2j}\end{pmatrix}\right),$$

where $A=(a_{ij})$.

PROOF. Let s_j be the matrix representation of s_j with respect to the bases α_1, α_2, α_3. Then we have

$$s_j=E-\delta_j{}^t\delta_j{}^tC,$$

where $\delta_j={}^t(0, \cdots, \overset{j}{\check{1}}, \cdots, 0)$ and E is the unit matrix. Thus we have:

$$\begin{aligned} S_j&=As_jA^{-1}\\ &=E-A\delta_j{}^t\delta_j{}^tCA^{-1}\\ &=E-A\delta_j{}^t\delta_jD^{-1}{}^tAH\\ &=E-\begin{pmatrix} a_{1j}\\ a_{2j}\\ a_{3j}\end{pmatrix}(-a_{3j}\ 2a_{2j}\ -a_{1j})\Big/ d_j\\ &=\begin{pmatrix} 1+a_{1j}a_{3j}/d_j & -2a_{1j}a_{2j}/d_j & a_{1j}^2/d_j\\ a_{2j}a_{3j}/d_j & 1-2a_{2j}^2/d_j & a_{1j}a_{2j}/d_j\\ a_{3j}/d_j & -2a_{2j}a_{3j}/d_j & 1+a_{1j}a_{3j}/d_j\end{pmatrix}. \end{aligned}$$

On the other hand, $CD={}^tAHA$ and $C_{jj}=2$ implies

$$\frac{1}{d_j}(a_{2j}^2-a_{1j}a_{3j})=1.$$

These and the representation of ρ^* given above prove the proposition.

Q.E.D.

Put

$$S_j^*=\frac{1}{\sqrt{d_j}}\begin{pmatrix} a_{2j} & -a_{1j}\\ a_{3j} & -a_{2j}\end{pmatrix}\qquad (j=1, 2, 3).$$

We shall denote by W^* the subgroup of GL(2, $\boldsymbol{R}$) generated by three reflections S_1^*, S_2^* and S_3^*, and by W_ν^* the group W^* for $C=C_\nu$. Notice that the homomorphism ρ^* gives the isomorphism between W^* and W.

§4. Arithmetic triangle groups in SL(2, $\boldsymbol{R}$).

In the previous section, we defined the reflection group $W^*\subset$GL(2, $\boldsymbol{R}$) for C. In this section we shall study the group $\overline{W}^*=\langle W^*, -1\rangle\cap$SL(2, $\boldsymbol{R}$), where $\langle\alpha, \beta, \cdots\rangle$ denotes the group generated by α, β, $\cdots$. Put

$$\overline{W}_\nu^* = \langle W_\nu^*, -1\rangle \cap \mathrm{SL}(2, \boldsymbol{R}), \qquad 1 \leqq \nu \leqq 13.$$

PROPOSITION 4.1. *If we regard the group* $\overline{W}_\nu^*$ *as a subgroup of* PSL(2, $\boldsymbol{R}$) *operating on the upper half plane* $H = \{\tau \in \boldsymbol{C} \mid \mathrm{Im}\,\tau > 0\}$, *then the group* $\overline{W}_\nu^*$ *is a triangle group and the signature is given by the orders of* S_1S_2, S_2S_3 *and* S_3S_1: $\mathrm{sign}(\overline{W}_1^*) = (2, 3, \infty)$, $\mathrm{sign}(\overline{W}_2^*) = (3, 3, \infty)$, $\mathrm{sign}(\overline{W}_3^*) = (2, \infty, \infty)$, $\mathrm{sign}(\overline{W}_4^*) = (3, \infty, \infty)$, $\mathrm{sign}(\overline{W}_5^*) = (\infty, \infty, \infty)$, $\mathrm{sign}(\overline{W}_6^*) = (2, 4, \infty)$, $\mathrm{sign}(\overline{W}_7^*) = (2, 6, \infty)$, $\mathrm{sign}(\overline{W}_8^*) = (4, 4, \infty)$, $\mathrm{sign}(\overline{W}_9^*) = (6, 6, \infty)$, $\mathrm{sign}(\overline{W}_{10}^*) = (2, 4, 6)$, $\mathrm{sign}(\overline{W}_{11}^*) = (2, 6, 6)$, $\mathrm{sign}(\overline{W}_{12}^*) = (3, 4, 4)$ *and* $\mathrm{sign}(\overline{W}_{13}^*) = (3, 6, 6)$.

PROOF. The subgroup $\overline{W}^*$ of W^* is the set of even products of the generators S_1^*, S_2^* and S_3^* of W^*. Since S_1^*, S_2^* and S_3^* are of order two, the group $\overline{W}^*$ is generated by three elements $S_1^*S_2^*$, $S_2^*S_3^*$ and $S_3^*S_1^*$. The order of $S_i^*S_j^*$ is equal to that of S_iS_j and is obtained by the Dynkin diagram. Q.E.D.

COROLLARY 4.2 (Takeuchi [7]). *The groups* $\overline{W}_\nu^*$ ($\nu=1, \cdots, 13$) *are arithmetic triangle groups, and* $\{\overline{W}_1^*, \cdots, \overline{W}_9^*\}$ *and* $\{\overline{W}_{10}^*, \cdots, \overline{W}_{13}^*\}$ *are complete members, up to conjugacy, of commensurability classes of type I and II* (*classification in* [7]).

PROPOSITION 4.3 (cf. [7]). *The groups* $\overline{W}_\nu^*$ ($\nu=1, \cdots, 5$) *are congruence subgroups of* SL(2, $\boldsymbol{Z}$): $\overline{W}_1^* = \mathrm{SL}(2, \boldsymbol{Z})$, $\overline{W}_2^* =$ *the unique subgroup of* SL(2, $\boldsymbol{Z}$) *of index* 2, $\overline{W}_3^* = \Gamma_0(2)$, $\overline{W}_4^* = \Gamma_0(3)$, $\overline{W}_5^* = \Gamma(2)$. *The groups* $\overline{W}_\nu^*$ ($\nu=6, \cdots, 9$) *are commensurable with* SL(2, $\boldsymbol{Z}$).

Here we used the familiar notations:

$$\Gamma(p) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{Z}) \,\middle|\, a \equiv d \equiv 1,\ b \equiv c \equiv 0 \bmod p\right\},$$

$$\Gamma_0(p) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{Z}) \,\middle|\, c \equiv 0 \bmod p\right\}.$$

PROOF. Notice that a real matrix of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad a^2 + bc = 1$$

operates on the upper half plane H with Poincaré metric as an isometric reflection which fixes a geodesic curve $c|\tau|^2 - 2a\,\mathrm{Re}\,\tau = b$. Thus the element S_j^* fixes a curve l_j: $a_{3j}|\tau|^2 - 2a_{2j}\mathrm{Re}\,\tau = -a_{1j}$. The triangle T in H which is bounded by l_j ($j=1, 2, 3$) gives a fundamental domain of W^*. If we draw a picture of T, the statement of the proposition is obtained immediately. Q.E.D.

Let B be an indefinite quaternion algebra over $\boldsymbol{Q}$ defined by

$$B = \boldsymbol{Q} + \boldsymbol{Q}\alpha + \boldsymbol{Q}\beta + \boldsymbol{Q}2\alpha\beta,$$

$$\alpha^2 = \beta^2 = 6, \qquad \alpha\beta + \beta\alpha = 0,$$

and $\mathcal{O}$ a maximal order defined by

$$\mathcal{O}=\boldsymbol{Z}+\boldsymbol{Z}\frac{\alpha+\beta}{2}+\boldsymbol{Z}\Big(\frac{1}{2}+\frac{\alpha}{2}+\frac{\alpha\beta}{12}\Big)+\boldsymbol{Z}\Big(\frac{1}{2}+\frac{\beta}{2}+\frac{\alpha\beta}{12}\Big),$$

and ϕ a representation of B given by

$$x=x_0+x_1\alpha+x_2\beta+x_3\alpha\beta \longmapsto \begin{pmatrix} x_0+\sqrt{6}\,x_1 & \sqrt{6}\,(x_2+\sqrt{6}\,x_3) \\ \sqrt{6}\,(x_2-\sqrt{6}\,x_3) & x_0-\sqrt{6}\,x_1 \end{pmatrix}.$$

We define a Fuchsian group

$$\Gamma(1, \mathcal{O})=\{\phi(x)\in \mathrm{SL}(2, \boldsymbol{R}) \mid x\in\mathcal{O},\ N(x)=1\},$$

where $N(x)$ is the reduced norm of $x\in B$.

PROPOSITION 4.4 (cf. [7]). *The groups* $\overline{W}{}^*_{10}$, $\overline{W}{}^*_{11}$ *and* $\overline{W}{}^*_{12}$ *are extensions of the group* $\Gamma(1, \mathcal{O})$ *with indices* 4, 2 *and* 2 *respectively. The group* $\overline{W}{}^*_{13}$ *is a subgroup of* $\overline{W}{}^*_{11}$ *of index* 2.

PROOF. Put

$$R_1=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix},\quad R_2=\begin{pmatrix} 1/2 & 3/2+\sqrt{6}/2 \\ 3/2-\sqrt{6}/2 & -1/2 \end{pmatrix},$$

$$R_3=\begin{pmatrix} -\sqrt{3}/2 & -(\sqrt{3}+\sqrt{2})/2 \\ -(\sqrt{3}-\sqrt{2})/2 & \sqrt{3}/2 \end{pmatrix},\quad R_4=\begin{pmatrix} -1 & \\ & 1 \end{pmatrix},$$

$$R_5=\begin{pmatrix} \sqrt{3} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{3} \end{pmatrix},\quad R_6=\begin{pmatrix} -3/2 & -1/2-\sqrt{6}/2 \\ -1/2+\sqrt{6}/2 & 3/2 \end{pmatrix}.$$

Then we have

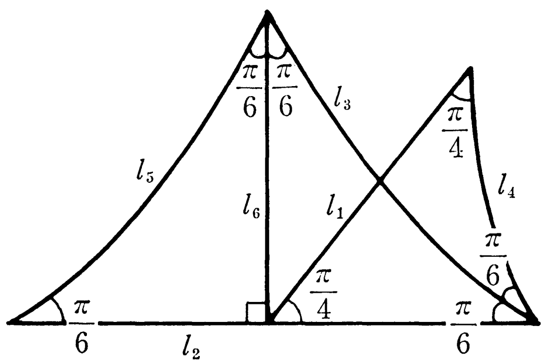
$$W^*_{10}=\langle S^*_1=R_1,\ S^*_2=R_2,\ S^*_3=R_3\rangle,$$

$$W^*_{11}=\langle S^*_1=R_6,\ S^*_2=-R_3,\ S^*_3=-R_2\rangle,$$

$$W^*_{12}=\langle S^*_1=R_4,\ S^*_2=R_1,\ S^*_3=R_2\rangle,$$

$$W^*_{13}=\langle S^*_1=R_5,\ S^*_2=R_2,\ S^*_3=R_3\rangle.$$

Fundamental domains of W^*_ν ($\nu=10, \cdots, 13$) are geodesic triangles bounded by three of the curves l_j ($j=1, \cdots, 6$) which is fixed by R_j:



This picture of fundamental domains implies that $\overline{W}_{10}^*, \cdots, \overline{W}_{13}^*$ are commensurable and deduces the following inclusion relations of index two:

$$\overline{W}_{13}^* \subset \overline{W}_{11}^* \subset \overline{W}_{10}^*, \qquad \overline{W}_{12}^* \subset \overline{W}_{10}^*$$

and

$$\overline{W}_{10}^* = \langle \overline{W}_{11}^*, \overline{W}_{12}^* \rangle .$$

Put $R_7 = R_1 R_4 R_1$ and let l_7 be the curve fixed by R_7. By direct computation one can see that

$$R_4R_7,\ R_7R_6,\ R_6R_2,\ R_2R_4 \in \Gamma(1, \mathcal{O}) .$$

The area of the fundamental domain of the group $\langle R_4R_7, R_7R_6, R_6R_2, R_2R_4 \rangle$ which is equal to the twice of the area of the polygon bounded by l_4, l_7, l_6 and l_2 is, by Gauss-Bonnet formula, $2\pi/3$. On the other hand since the indefinite quaternion algebra B over $\boldsymbol{Q}$ ramifies only at 2 and 3, it is known that $\mathrm{sign}(\Gamma(1, \mathcal{O})) = (2, 2, 3, 3)$ and the area of H modulo $\Gamma(1, \mathcal{O})$ is equal to $2\pi/3$. This implies $\langle R_4R_7, R_7R_6, R_6R_2, R_2R_4 \rangle = \Gamma(1, \mathcal{O})$, the inclusion relations of index two: $\Gamma(1, \mathcal{O}) \subset \overline{W}_{11}^*$, $\Gamma(1, \mathcal{O}) \subset \overline{W}_{12}^*$, and so $\Gamma(1, \mathcal{O}) = \overline{W}_{11}^* \cap \overline{W}_{12}^*$. These prove the proposition. Q.E.D.

§ 5. Representation of $\widetilde{W}$ in a parabolic subgroup of Sp(2, $\boldsymbol{R}$).

The siegel upper half space $\mathfrak{S}_2$ of degree 2 is the domain

$$\mathrm{S}(2, \boldsymbol{R}) + \sqrt{-1}\,\mathrm{S}^+(2, \boldsymbol{R})$$

in S(2, $\boldsymbol{C}$), and the real symplectic group Sp(2, $\boldsymbol{R}$) of degree 2 is the group

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(4, \boldsymbol{R}) \,\middle|\, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix} = E \right\} .$$

We consider a maximal parabolic subgroup

$$\mathfrak{P} = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2, \boldsymbol{R}) \,\middle|\, C = 0 \right\}$$

of Sp(2, $\boldsymbol{R}$) corresponding to a zero dimensional boundary component P. We have the following exact sequence:

$$0 \longrightarrow \mathrm{S}(2, \boldsymbol{R}) \xrightarrow{\iota} \mathcal{P} \longrightarrow \mathrm{GL}(2, \boldsymbol{R}) \longrightarrow 1 .$$

Notice that $\mathcal{P}$ is the semi-direct product $\mathrm{GL}(2, \boldsymbol{R}) \ltimes \mathrm{S}(2, \boldsymbol{R})$.

Let C be a symmetrizable hyperbolic G. C. M. of rank 3. For the root lattice $Q=\boldsymbol{Z}\alpha_1+\boldsymbol{Z}\alpha_2+\boldsymbol{Z}\alpha_3 \subset V$ of C, we denote by Q^* the lattice $\rho(Q)$ of S(2, $\boldsymbol{R}$) and Q_ν^{*j} the lattice Q^* for $C=C_\nu^j$. The semi-direct product $W^* \ltimes Q^*$ is a discrete subgroup of $\mathcal{P}$, which we shall denote by $\widetilde{W}^*$, and $\widetilde{W}_\nu^{*j}$ the group $\widetilde{W}^*$ for $C=C_\nu^j$. By the isomorphism $\rho: V \to \mathrm{S}(2, \boldsymbol{R})$, the action of the group $\widetilde{W}$ on $V+\sqrt{-1}\mathring{I}$ is equivalent to that of $\widetilde{W}^*$ on $\mathfrak{S}_2$.

Notice that the lattice Q_ν^{*j} is isogenus to the lattice Q_ν^{*1}, and that for any non-zero real number a, the group $W^* \ltimes aQ^*$ is conjugate in $\mathcal{P}$ to the group $\widetilde{W}^*=W^* \ltimes Q$.

For the order $\mathcal{O}$ of the quaternion algebra B, we define a lattice $\Gamma(2, \mathcal{O})$ of Sp(2, $\boldsymbol{R}$) by

$$\Gamma(2, \mathcal{O})=K\left\{\begin{pmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{pmatrix} \in \mathrm{GL}(4, \boldsymbol{R}) \;\middle|\; a, b, c, d \in \mathcal{O},\right.$$
$$\left.\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}=\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right\} K^{-1},$$

where $a \mapsto a'$ is the canonical involution of B and

$$K=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

PROPOSITION 5.1. *The groups* $\widetilde{W}_1^{*j}, \cdots, \widetilde{W}_5^{*j}$, $W_6^* \ltimes \sqrt{2}\,Q_6^{*j}$, $W_7^* \ltimes \sqrt{3}\,Q_7^{*j}$, $\widetilde{W}_8^{*j}$ *and* $\widetilde{W}_9^{*j}$ *are commensurable with* $\mathrm{Sp}(2, \boldsymbol{Z}) \cap \mathcal{P}$, *and* $W_{10}^* \ltimes \sqrt{3}\,Q_{10}^{*j}$, $W_{11}^* \ltimes \sqrt{6}\,Q_{11}^{*j}$, $W_{12}^* \ltimes \sqrt{6}\,Q_{12}^{*j}$ *and* $W_{13}^* \ltimes \sqrt{2}\,Q_{13}^{*j}$ *are commensurable with* $\Gamma(2, \mathcal{O}) \cap \mathcal{P}$.

PROOF. As we noticed above, the group $\widetilde{W}_\nu^{*j}$ is commensurable with the group $\widetilde{W}_\nu^*$. Thus we assume that $j=1$. By definition, we have

$$Q_\nu=\sum_{j=1}^{3} \boldsymbol{Z}\begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix},$$

where $(a_{ij})=A_\nu$, and so,

$$Q_\nu^*=\sum_{j=1}^{3} \boldsymbol{Z}\begin{pmatrix} a_{1j} & a_{2j} \\ a_{2j} & a_{3j} \end{pmatrix}.$$

We identify the linear space S(2, $\boldsymbol{R}$) and the image of the inclusion homomorphism $\iota: S(2, \boldsymbol{R})\to\mathcal{P}$. Since we have

$$Sp(2, \boldsymbol{Z})\cap S(2, \boldsymbol{R})=S(2, \boldsymbol{Z}),$$

the assertion for the groups $\widetilde{W}_1^*, \cdots, \widetilde{W}_9^*$ is clear. We want to know the lattice $\Gamma(2, \mathcal{O})\cap S(2, \boldsymbol{R})$. Put $J=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$. Then by the definition of $\Gamma(2, \mathcal{O})$ and $\mathcal{P}$, we have

$$\Gamma(2, \mathcal{O})\cap\mathcal{P}=\left\{\begin{pmatrix}\phi(a) & \phi(b)J^{-1}\\ 0 & J\phi(d)J^{-1}\end{pmatrix} \;\middle|\; a, b, d\in\mathcal{O},\right.$$

$$\left. ba'+ab'=0,\ ad'=da'=1\right\},$$

and so,

$$\Gamma(2, \mathcal{O})\cap S(2, \boldsymbol{R})=\{\phi(b)J^{-1} \mid b+b'=0,\ b\in\mathcal{O}\}$$

$$=\left\{\begin{pmatrix}-\sqrt{6}(x_2^2+\sqrt{6}x_3) & \sqrt{6}x_1\\ \sqrt{6}x_1 & \sqrt{6}(x_2-\sqrt{6}x_3)\end{pmatrix} \;\middle|\; x_1\alpha+x_2\beta+x_3\alpha\beta\in\mathcal{O}\right\}.$$

For $\nu=10, \cdots, 13$, we can calculate and check that a suitable multiple of the lattice Q_ν^* is contained in $\Gamma(2, \mathcal{O})\cap S(2, \boldsymbol{R})$. Q.E.D.

For some C, we shall give an example of an arithmetic subgroup Λ of $Sp(2, \boldsymbol{R})$ such that the group $\widetilde{W}^*$ is conjugate in $\mathcal{P}$ to the group $\Lambda\cap\mathcal{P}$. Put

$\Lambda_1=Sp(2, \boldsymbol{Z})$,

$\Lambda_2=$the subgroup of $Sp(2, \boldsymbol{Z})$ generated by the principal congruence subgroup $\boldsymbol{\Gamma}(2)$ and $\widetilde{W}_2^*$,

$$\Lambda_3=\left\{X\in Sp(2, \boldsymbol{Z}) \;\middle|\; X\equiv\begin{pmatrix}* & * & * & *\\ 0 & * & * & 0\\ & & * & 0\\ & 0 & * & *\end{pmatrix} \bmod 2\right\},$$

$$\Lambda_4=\left\{X\in Sp(2, \boldsymbol{Z}) \;\middle|\; X\equiv\begin{pmatrix}* & * & * & *\\ 0 & * & * & 0\\ & & * & 0\\ & 0 & * & *\end{pmatrix} \bmod 3\right\},$$

$$\Lambda_5=\left\{X=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(2, \boldsymbol{Z}) \;\middle|\; \begin{matrix}X\equiv E \bmod 2,\ \text{diagonal of}\\ {}^tAC \text{ and } {}^tBD\equiv 0 \bmod 4\end{matrix}\right\}.$$

This group is denoted by $\Gamma(2, 4)$ in [2] and $\Lambda(2, 2)$ in [1]. Then we have

$$\Lambda_\nu\cap\mathcal{P}=\widetilde{W}_\nu^{*1} \qquad \nu=1, 2, 3, 4,$$

and

$$\Lambda_5 \cap \mathcal{P} = W_5^* \ltimes 2Q_5^{*1} .$$

§ 6. Uniformizability of Satake compactification.

In this section, we shall show that Satake's (partial) compactification of the factor space $\mathfrak{S}_2/\widetilde{W}^*$ coincides with Looijenga's (partial) compactification, and so it is uniformizable.

Let π denote the projection: $S^+(2, \boldsymbol{R}) \to S^+(2, \boldsymbol{R})/\boldsymbol{R}^+$, where $\boldsymbol{R}^+$ stands for the multiplicative group of positive numbers. We introduce an isomorphism $\sigma : S^+(2, \boldsymbol{R})/\boldsymbol{R}^+ \to H = \{\tau \in \boldsymbol{C} \mid \mathrm{Im}\,\tau > 0\}$ by

$$\sigma : \quad \pi\begin{pmatrix} u & w \\ w & v \end{pmatrix} \longmapsto \tau = \frac{w}{v} + i\frac{\sqrt{uv-w^2}}{v} .$$

We have

$$\sigma^{-1}(\tau) = \pi\begin{pmatrix} |\tau|^2 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & 1 \end{pmatrix},$$

and

$$\sigma^{-1}\left(\frac{a\tau+b}{c\tau+d}\right) = \pi\,{}^t\!\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} |\tau|^2 & \mathrm{Re}\,\tau \\ \mathrm{Re}\,\tau & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{R}) .$$

This implies that, by the isomorphism σ, the action of $\overline{W}^*$ on $S^+(2, \boldsymbol{R})/\boldsymbol{R}^+$ is equivalent to that of $\overline{W}^*$ on H, where the action of $\overline{W}^* \subset \mathrm{SL}(2, \boldsymbol{R})$ on H is the linear fractional one:

$$\tau \longmapsto \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{R}) .$$

Let $\mathcal{F}$ be the set of every 1-dimensional rational boundary component F of $\mathfrak{S}_2$ such that $P \in \overline{F}$ (the closure of F). Then the set $\mathcal{F}$ corresponds, in a one to one way, to the set of every real line, defined over $\boldsymbol{Q}$, on $\partial S^+(2, \boldsymbol{R}) \subset S(2, \boldsymbol{R})$, and so, by the mapping $\sigma \circ \pi$, to the rational boundary $\boldsymbol{Q} \cup \{\infty\}$ of H.

By the arguments above, we conclude that, for each matrix $\mathcal{C}$, the set $\mathcal{F}$ of W^*-equivalence classes of F corresponds, in a one to one way, to the set $\overline{\mathcal{F}}'$ of $\overline{W}^*$-equivalence classes of $\boldsymbol{Q} \cup \{\infty\}$, i.e. the cusps of $\overline{W}^*$.

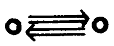
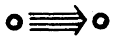
On the other hand, let $\mathcal{S}$ be the set of subsets $\{\alpha_i, \alpha_j\}$ of the roots $\{\alpha_1, \alpha_2, \alpha_3\}$ such that the order of $s_i s_j \in \mathrm{GL}(V)$ is infinite, i.e. two vertices in the Dynkin diagram of $\mathcal{C}$ connected by

$$\circ \rightleftarrows \circ \quad \text{or} \quad \circ \Rrightarrow \circ .$$

Looijenga's (partial) compactification consists of $\mathfrak{S}_2/\widetilde{W}^*$, 1-dimensional boundary components corresponding to each elements of $\mathcal{S}$ and the 0-dimensional boundary component P.

We have shown in § 4 that there exists a one to one correspondence between the sets $\mathfrak{F}'$ and $\mathcal{S}$. On the other hand, by the "minimality" of Satake's compactification (cf. [6]), there exists a birational morphism from Looijenga's one onto Satake's one. Since both compactifications are obtained by adding one dimensional analytic sets, the morphism is one to one. Thus Zariski main theorem implies that Satake's (partial) compactification of $\mathfrak{S}_2/\widetilde{W}^*$ is isomorphic to Looijenga's one. Since Looijenga's (partial) compactification is always uniformizable, we have the following theorem.

THEOREM. *Let C be a symmetrizable hyperbolic generalized Cartan matrix of rank* 3 *and* $\widetilde{W}^*$ *the discrete subgroup of* $\mathcal{P}$ *defined in* § 5. *Then Satake's (partial) compactification of* $\mathfrak{S}_2/\widetilde{W}^*$ *is uniformizable. Every* 1-*dimensional cusp corresponds to one of the subdiagrams*

 *or* 

of the Dynkin diagram of C.

We conclude this paper by giving a conjecture.

CONJECTURE. For an arithmetic subgroup Λ of Sp(2, $\boldsymbol{R}$), Satake compactification of $\mathfrak{S}_2/\Lambda$ is uniformizable at a 0-dimensional cusp P if and only if $\Lambda \cap \mathcal{P}$ is conjugate to the group $\widetilde{W}^*$ for some symmetrizable hyperbolic G.C.M. C of rank 3, where $\mathcal{P}$ is the parabolic subgroup of Sp(2, $\boldsymbol{R}$) corresponding to P.

Appendix.

Let $\mathcal{D}$ be an irreducible classical bounded symmetric domain, P a 0-dimensional boundary component of $\mathcal{D}$ and $\mathcal{P}$ the maximal parabolic subgroup of the analytic automorphism group Aut($\mathcal{D}$) corresponding to P. It is known that the group Aut($\mathcal{D}$) contains a (quasi-) reflection if and only if $\mathcal{D}=B_n=\{(z_1, \cdots, z_n) \in \boldsymbol{C}^n \mid |z_1|^2+\cdots+|z_n|^2<1\}$ $(n\geqq 2)$, $\mathcal{D}=\mathfrak{S}_2$ or $\mathcal{D}$ is a bounded symmetric domain L_n $(n\geqq 4)$ of type IV. Here the domain L_n is defined as follows:

$$L_n=\left\{(z_1, \cdots, z_n)\in \boldsymbol{C}^n \,\middle|\, |z_1|^2+\cdots+|z_n|^2<\frac{1}{2}\{1+|z_1^2+\cdots+z_n^2|^2\}<1\right\}.$$

In the case $\mathcal{D}=B_n$, we studied in [8] some relations between discrete subgroups of $\mathcal{P}$ and the G.C.M. of euclidean type. In the appendix we shall state the theorem, for the domain $\mathcal{D}=L_n$, corresponding to the theorem in § 6, without giving a proof. Detailed proof will be published elsewhere.

THEOREM. *Let C be a symmetrizable hyperbolic G.C.M. of rank n. Then the group $\widetilde{W}$ can be considered as a discrete subgroup of* $\mathcal{P}\subset \mathrm{Aut}(L_n)$. *Satake's* *(partial) compactification of the factor space* $L_n/\widetilde{W}$ *is uniformizable. The set of 1-dimensional cusps corresponds bijectively to the set of euclidean subdiagrams of the Dynkin diagram of* C.

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