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Sullivan-Quillen mixed type model for fibrations

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§1. Introduction.

D. Sullivan [10] associated a differential graded algebra (DGA, for brevity) M_X over the rationals Q to a topological space X, while D. Quillen [6] associated to X a differential graded Lie algebra (DGL, for brevity) L_X over Q. These algebraic objects contain all the informations about the rational homotopy type of X at least when X has the homotopy type of a simply connected CW complex of finite homology type. The DGA M_X (and respectively, DGL L_X) is called a Sullivan (and respectively, Quillen) type (algebraic) model for the space X.

These assignments are generalized to the case of a fibration $F \rightarrow E \rightarrow B$, where we can associate a sequence of homomorphisms $M_B \rightarrow M_B \underset{\tau}{\otimes} M_F \rightarrow M_F$ of DGA's, or $L_F \rightarrow L_F \underset{\phi}{\oplus} L_B \rightarrow L_B$ of DGL's [11] with $M_B \underset{\tau}{\otimes} M_F$ (and respectively, $L_F \underset{\phi}{\oplus} L_B$) a Sullivan (and respectively, Quillen) type model for the total space E of the fibration.

Mixing these two types, A. Haefliger [3] constructed a DGL $M_B \underset{\tau}{\otimes} L_F$ over M_B which is free as an M_B-module and whose quotient by the ideal (M_B^+) is L_F, in order to prove the Bott conjecture concerning the Gelfand-Fuks cohomology. An advantage of his mixed type model lies in the following facts; the Koszul cochain complex construction over M_B of $M_B \underset{\tau}{\otimes} L_F$ gives rise to a Sullivan type model for the total space E while the same construction over the ground field R gives a Sullivan type model for the space of cross-sections of the fibration. A technical disadvantage of the mixed type model is that the base is negatively graded while the fiber is positively graded and that we have to consider a projective system when both of the base and fiber contain nontrivial elements of arbitrary high (low) degrees.

In this note we present basic definitions concerning the Sullivan-Quillen mixed type and the Sullivan type algebraic fibrations in section 2, and prove the equivalence of the homotopy categories of mixed type and Sullivan type fibrations in section 5, using the generalized Koszul complex functor C_A^* defined in section 3 and the generalized Quillen L functor L_*^A defined in section 4. This generalizes Silveira's result [9] on the mixed type model for fibrations with a given cross-section. As a corollary, we see that the minimal model (=the Postnikov decomposition) of a Sullivan type fibration with its fiber having the rational homotopy type of a bouquet of spheres belongs to the image of the composite $C_A^* \circ L_*^A$ of the two functors above. Applying this result, we give in section 6, the explicit description of the minimal models for the DGA's $I_{(n)}=R[c_1, \cdots, c_n]/(\deg>2n)$ and $\hat{W}O_n=I_{(n)}\underset{\tau}{\otimes}E(h_{2i-1}; 2i-1\leqq n)$ considered by Hurder and Kamber [4], [5]. The latter one gives at the same time the complete solution to a computational problem of the Postnikov decomposition of the algebraic fibration $P_n\to P_n\underset{\tau}{\otimes}\hat{W}_n$ posed by Haefliger [2] in connection with the calculation of his model for the Gelfand-Fuks cohomology.

The main results of this note were announced in [8].

I would like to thank A. Haefliger for suggesting me to generalize the mixed type model theory for fibrations to the case of those without cross-sections. I owe a great deal to S. Hurder for his helpful suggestions on the computations of the minimal models for truncated Weil algebras. I also enjoyed valuable discussions with H. Shiga.

§2. Mixed type fibrations.

Throughout sections 2 to 5, A^* is a non-negatively graded, augmented DGA (i.e. $A^i=0$ for $i<0$, and $A^*\rightleftarrows R$) over a ground field R of characteristic zero with the differential d_A. *A graded Lie algebra* L_* *over* A^* is a graded A^*-module which is a graded Lie algebra with an A^*-bilinear Lie bracket. The grading in L_* is lowerly indexed and we have $\deg(a\cdot y)=\deg(y)-\deg(a)$ for every $a\in A^*$ and $y\in L_*$. So we may equivalently put $A_{-i}=A^i$ to have $\deg(\bar{a}\cdot y)=\deg(y)+\deg(\bar{a})$ for every $\bar{a}\in A_*$ and $y\in L_*$.

DEFINITION 2.1. A graded Lie algebra L_* over A_* is *an algebraic fibration of mixed type* (or simply, *a mixed type fibration*) *over* A_* if the following conditions hold.

(i) L_* is a free A_*-module.

(ii) $(R\underset{A_*}{\otimes}L_*)_i=0$ for every $i<0$, and $\dim_R(R\underset{A_*}{\otimes}L_*)_i<\infty$ for every $i\geqq 0$.

(iii) L_* is equipped with a pair (χ, D) where χ is an element of L_{-2} called *the Euler element* and $D: L_*\to L_{*-1}$ is an A_*-Lie derivation of degree -1, i.e.

$$D(a\cdot[y_1, y_2])=d_A(a)\cdot[y_1, y_2]+(-1)^{\deg(a)}a \cdot\{[D(y_1), y_2]+(-1)^{\deg(y_1)}[y_1, D(y_2)]\}, \tag{2.2}$$

called *the exterior covariant differentiation*, satisfying the following two formulas;

(2.3) ("the Bianchi identity") $D(\chi)=0$, and

(2.4) ("the curvature formula") $(D)^2(y)+[\chi, y]=0$ for every $y\in L_*$.

By condition (ii), the Euler element χ belongs to $A_-\cdot L_*$ $(=A^+\cdot L_*)$. So on the quotient Lie algebra $\bar{L}_*=R\underset{A_*}{\otimes}L_*$, the induced derivation $1\underset{A_*}{\otimes}D$ is a differential, i.e. $(1\underset{A_*}{\otimes}D)^2=0$ by virtue of (2.4). This quotient DGL

$$(\bar{L}_*=R\underset{A_*}{\otimes}L_*,\ 1\underset{A_*}{\otimes}D) \tag{2.5}$$

is called *the fiber* of the mixed type fibration (L_*, χ, D). By condition (i), it holds that $L_*\cong A_*\otimes\bar{L}_*$ as A_*-modules.

EXAMPLE 2.6. A mixed type model for the Hopf fibration

$$S^1 \longrightarrow S^3 \longrightarrow S^2 \tag{2.7}$$

is given by

$$R[u]/(u^2) \longrightarrow R[u]/(u^2)\otimes L(\sigma^{-1}v) \longrightarrow L(\sigma^{-1}v)\,, \tag{2.8}$$

where u and v are respectively of degree 2 and 1, $\sigma^{-1}v$ is the desuspension (the shift of degree by -1) of v, and $L(\sigma^{-1}v)$ is the free graded Lie algebra with one basis element $\sigma^{-1}v$. On the graded Lie algebra $R[u]/(u^2)\otimes L(\sigma^{-1}v)$ over $R[u]/(u^2)$, we have the exterior covariant differentiation $D\equiv 0$ and the Euler element $\chi=u\otimes\sigma^{-1}v$.

Now let $(\tilde{L}_*, \tilde{\chi}, \tilde{D})$ and (L_*, χ, D) be two mixed type fibrations over the same DGA A^*. And let $\psi:\tilde{L}_*\to L_*$ be a homomorphism of graded Lie algebras over A_*, i.e., a degree preserving A_*-module homomorphism commuting with the Lie brackets.

DEFINITION 2.9. Let $(\tilde{L}_*, \tilde{\chi}, \tilde{D})$, (L_*, χ, D) and ψ be as above. We define *the curvature* $\Omega=\Omega(\psi)$ *of the graded Lie algebra homomorphism* ψ as the difference of the Euler elements;

$$\Omega(\psi)=\chi-\psi(\tilde{\chi})\in L_{-2}\,. \tag{2.10}$$

Then the graded Lie algebra homomorphism ψ is called *a homomorphism of mixed type fibrations* if there is an element $\theta=\theta(\psi)$ of L_{-1}, called *the connection of the homomorphism* ψ, which satisfies the following two formulas;

(2.11) ("the structure equation") $D(\theta)+\dfrac{1}{2}[\theta, \theta]=\Omega(\psi)$, and

(2.12) ("the connection formula")

$$\psi\circ\tilde{D}(\tilde{y})-D\circ\psi(\tilde{y})=[\theta, \psi(\tilde{y})] \qquad \text{for every } \tilde{y}\in\tilde{L}_*.$$

Compared with the mixed type fibrations, the Sullivan type fibrations are defined as follows.

DEFINITION 2.13. A DGA (E^*, d) over A^* is *an algebraic fibration of the Sullivan type* (or simply, *a Sullivan type fibration*) *over* A^* if the following conditions hold.

(i) E^* is a free A^*-module.

(ii) $(R\underset{A^*}{\otimes}E^*)^i=0$ for every $i<0$, $(R\underset{A^*}{\otimes}E^*)^0\cong R$, and $\dim_R(R\underset{A^*}{\otimes}E^*)^i<\infty$ for every $i>0$.

(A^*, d_A) is called *the base* and the quotient DGA $(R\underset{A^*}{\otimes}E^*, 1\underset{A^*}{\otimes}d)$ is called *the fiber* of the Sullivan type fibration (E^*, d).

Suppose we are given an A^*-algebra homomorphism

$$e: E^* \longrightarrow A^* \tag{2.14}$$

such that

$$e\circ u=\mathrm{id}: A^* \longrightarrow A^* \tag{2.15}$$

where $u: A^*\to E^*$ is the unit map. Such a homomorphism will be called *a local section* to the Sullivan type fibration (E^*, d). If the local section e commutes with the differentials, i.e.

$$e\circ d=d_A\circ e\,, \tag{2.16}$$

then e is called *a cross-section* (or *a global section*).

§ 3. Generalized Koszul complex functor.

In this section we generalize the notion of the Koszul chain and cochain complex constructions for Lie algebras to those for mixed type fibrations.

DEFINITION 3.1. Let (L_*, χ, D) be an algebraic fibration of the mixed type over A_*. *The Koszul chain complex* $C_*^A(L_*, \chi, D)$ *over* A_* *of* (L_*, χ, D) is the DG coalgebra

$$(S_*^A(\sigma L_*),\ d=d_L+D+d_\chi)\,. \tag{3.2}$$

Here σL_* denotes the suspension of L_* (i.e. $(\sigma L)_i=L_{i-1}$) and $S_*^A(\sigma L_*)$ is the graded symmetric coalgebra of σL_* taken over A_* with the A_*-action on σL_* defined by

$$\bar{a}\cdot\sigma(y)=(-1)^{\deg(\bar{a})}\sigma(\bar{a}\cdot y) \qquad \text{for } \bar{a}\in A_* \text{ and } y\in L_*\,. \tag{3.3}$$

The differential d_L is the usual one arising from the Lie bracket of L_*, i.e.

$$d_L(\sigma y_1\cdots\sigma y_k)=\sum_{1\leq i<j\leq k}(-1)^{\eta(i,j)-1}\sigma[y_i,\ y_j]\sigma y_1\cdots\overset{i}{\check{}}\cdots\overset{j}{\check{}}\cdots\sigma y_k\,, \tag{3.4}$$

and the coderivation D is the one induced by the exterior covariant differentiation of L_* denoted by the same symbol, i.e.

$$D(\sigma y_1 \cdots \sigma y_k)=\sum_{1\leq i\leq k}(-1)^{\zeta(i)}\sigma y_1 \cdots \sigma(Dy_i)\cdots \sigma y_k\,. \tag{3.5}$$

The sign conventions $\eta(i, j)$ in (3.4) and $\zeta(i)$ in (3.5) are as in [**3**], p. 506. So we put on d_L the opposite sign to that of [**3**].

Finally the differential d_χ, called *the Euler differential,* is nothing but the multiplication by the suspension $\sigma\chi$ of the Euler element χ of L_{-2}, i.e.

$$d_\chi(\sigma y_1 \cdots \sigma y_k)=\sigma\chi\cdot\sigma y_1 \cdots \sigma y_k\,. \tag{3.6}$$

LEMMA 3.7. *In $C^A_*(L_*, \chi, D)$ the following formulas hold.*

$$(d_L)^2=0\,. \tag{3.8}$$

$$D\circ d_L+d_L\circ D=0\,. \tag{3.9}$$

$$D^2+d_\chi\circ d_L+d_L\circ d_\chi=0\,. \tag{3.10}$$

$$D\circ d_\chi+d_\chi\circ D=0\,. \tag{3.11}$$

$$(d_\chi)^2=0\,. \tag{3.12}$$

And consequently, we have

$$d^2=(d_L+D+d_\chi)^2=0\,. \tag{3.13}$$

PROOF. The formula (3.8) is a direct consequence of the Jacobi identity in L_*. And the sign conventions in (3.4) and (3.5) imply (3.9). The formula (3.10) is obtained from the curvature formula (2.4). The Bianchi identity (2.3) implies (3.11). The formula (3.12) is obvious from dimensional reasons. Q. E. D.

DEFINITION 3.14. *The Koszul cochain complex $C^*_A(L_*, \chi, D)$ over A^* of a mixed type fibration (L_*, χ, D) is the A_*-dual of its Koszul chain complex $C^A_*(L_*, \chi, D)$ over A_*;*

$$C^*_A(L_*, \chi, D)=\{\mathrm{Hom}^*_{A_*}(S^A_*(\sigma L_*), A_*), \mathrm{Hom}_{A_*}(d, 1)\}\,. \tag{3.15}$$

This is a DGA over A^*. Since L_* is assumed to be a free A_*-module of finite type and since the fiber $\bar{L}_*$ is non-negatively graded by 2.1 (ii) while the base A_* is non-positively graded, we have

$$\mathrm{Hom}^*_{A_*}(S^A_*(\sigma L_*), A_*)\cong A^*\otimes\mathrm{Hom}^*_R(S^R_*(\sigma\bar{L}_*), R)\,. \tag{3.16}$$

Therefore $C^*_A(L_*, \chi, D)$ is a Sullivan type fibration over A^*.

Notice that $C^*_A(L_*, \chi, D)$ has the natural local section

$$e=\mathrm{Hom}_{A_*}(\iota_{(0)}, 1)\ :\ \mathrm{Hom}^*_{A_*}(S^A_*(\sigma L_*), A_*)\longrightarrow \mathrm{Hom}^*_{A_*}(A_*, A_*)\cong A_*=A^{-*}\,, \tag{3.17}$$

where

$$\iota_{(0)} : A_* = S^A_{(0)}(\sigma L_*) \longrightarrow S^A_*(\sigma L_*) = \sum_{i \geq 0} S^A_{(i)}(\sigma L_*) \tag{3.18}$$

is the canonical inclusion. We remark that $\iota_{(0)}$ commutes with the differentials if and only if $\chi=0$. Therefore the local section (3.7) is a "global" section if and only if $\chi=0$.

EXAMPLE 3.19. Let $(R[u]/(u^2)\otimes L(\sigma^{-1}v),\ u\otimes\sigma^{-1}v,\ D\equiv 0)$ be the mixed type model for the Hopf fibration given in 2.6. Then its Koszul cochain complex over $R[u]/(u^2)$ is given by

$$R[u]/(u^2)\otimes E(v')\,;\quad d'(v')\,(=d'_\chi(v'))=u\,. \tag{3.20}$$

Now let us suppose, for the moment, that A^* satisfies the following condition;

(3.21) (finite degree condition) $A^i=0$ for every $i>N$ with respect to a certain sufficiently large N.

And let $\psi:(\tilde{L}_*, \tilde{\chi}, \tilde{D})\to(L_*, \chi, D)$ be a homomorphism of mixed type fibrations over A_* with connection $\theta=\theta(\psi)\in L_{-1}$.

DEFINITION 3.22. Under the condition (3.21) on A^*, let $(\tilde{L}_*, \tilde{\chi}, \tilde{D})$, (L_*, χ, D), ψ and θ be as above. We define an A_*-linear map

$$\psi_\theta\ :\ C^A_*(\tilde{L}_*, \tilde{\chi}, \tilde{D}) \longrightarrow C^A_*(L_*, \chi, D) \tag{3.23}$$

by

$$\psi_\theta(\sigma y_1 \cdots \sigma y_k)=\sigma(\psi(y_1))\cdots\sigma(\psi(y_k))\,\mathrm{Exp}(\sigma\theta)\,, \tag{3.24}$$

where $\mathrm{Exp}(x)$ denotes the formal power series $\sum_{h=0}^{\infty}(1/h!)x^h$. Notice that the connection θ belongs to $A_-\cdot L_*\ (=A^+\cdot L_*)$ by 2.1 (ii). So $(\sigma\theta)^h=0$ for sufficiently large h by hypothesis (3.21).

LEMMA 3.25. *The map ψ_θ is a homomorphism of DG coalgebras over A_*. Namely, denoting the comultiplication by μ_*, we have*

$$\mu_*\circ\psi_\theta=(\psi_\theta\otimes\psi_\theta)\circ\mu_* \quad \textit{and} \tag{3.26}$$

$$(d_L+D+d_\chi)\circ\psi_\theta=\psi_\theta\circ(d_{\tilde{L}}+\tilde{D}+d_{\tilde{\chi}})\,. \tag{3.27}$$

PROOF. The formula (3.26) is easy to check by definition. The formula (3.27) is also easy to verify by direct calculations using the structure equation (2.11) and the connection formula (2.12). Q. E. D.

Therefore a homomorphism $\psi:(\tilde{L}_*, \tilde{\chi}, \tilde{D})\to(L_*, \chi, D)$ of mixed type fibrations with connection θ induces a homomorphism of DGA's

$$\mathrm{Hom}_{A_*}(\psi_\theta, 1)\ :\ C^*_A(L_*, \chi, D)\longrightarrow C^*_A(\tilde{L}_*, \tilde{\chi}, \tilde{D}) \tag{3.28}$$

under the finite degree condition (3.21) on A^*.

When A^* does not necessarily satisfy the finite degree condition (3.21), let $A^{[q]}$ $(q\geqq 0)$ be the quotient of A^* truncated by the ideal of the elements of degree $>q$. And let

$$\pi_q : A^* \longrightarrow A^{[q]}=A^*/(\deg>q)\,, \qquad \text{and} \tag{3.29}$$

$$\pi_{q,p} : A^{[q]} \longrightarrow A^{[p]} \qquad (q>p\geqq 0) \tag{3.29)*}$$

be the natural projections. By the change of base, a mixed type fibration (L_*, χ, D) over A_* gives rise to the ones $(A^{[q]}\underset{A^*}{\otimes} L_*, 1\underset{A^*}{\otimes}\chi, 1\underset{A^*}{\otimes}D)$ over $A^{[q]}$ for all $q\geqq 0$, which constitute a projective system of mixed type fibrations

$$\begin{aligned}\{\pi_{q,p} : (A^{[q]}\underset{A^*}{\otimes} L_*,\ 1\underset{A^*}{\otimes}\chi,\ 1\underset{A^*}{\otimes}D) \\ \longrightarrow (A^{[p]}\underset{A^*}{\otimes} L_*,\ 1\underset{A^*}{\otimes}\chi,\ 1\underset{A^*}{\otimes}D)\,;\ q>p\geqq 0\}\,.\end{aligned} \tag{3.30}$$

More generally, we define an admissible system of mixed type fibrations over A_* as follows.

DEFINITION 3.31. A set $\{(L_*^{[q]}, \chi^{[q]}, D^{[q]}),\ q\geqq 0\,;\ \Phi_{q,p},\ q>p\geqq 0\}$ of mixed type fibrations over $A^{[q]}$ and of isomorphisms

$$\Phi_{q,p} : (A^{[p]} \underset{A^{[q]}}{\otimes} L_*^{[q]},\ 1\otimes\chi^{[q]},\ 1\otimes D^{[q]}) \longrightarrow (L_*^{[p]},\ \chi^{[p]},\ D^{[p]}) \tag{3.32}$$

is said to be *an admissible system of mixed type fibrations over* A_* if

$$\Phi_{q,p}\circ(1 \underset{A^{[q]}}{\otimes} \Phi_{r,q})=\Phi_{r,p} \qquad (r>q>p\geqq 0)\,, \quad \text{and} \tag{3.33}$$

$$\theta(\Phi_{q,p})=0 \qquad (q>p\geqq 0)\,. \tag{3.34}$$

For example, every mixed type fibration (L_*, χ, D) naturally gives rise to an admissible system of mixed type fibrations $\{(L_*^{[q]}=A^{[q]}\underset{A^*}{\otimes} L_*, \chi^{[q]}=1\otimes\chi,\ D^{[q]}= 1\otimes D)\,;\ \Phi_{q,p}=\mathrm{id}\,;\ q>p\geqq 0\}$.

DEFINITION 3.35. We define *the Koszul cochain complex* $C_A^*(\{(L_*^{[q]},\chi^{[q]},D^{[q]}),\Phi_{q,p}\})$ *over* A^* *of an admissible system of mixed type fibrations* $\{(L_*^{[q]}, \chi^{[q]}, D^{[q]}),\ \Phi_{q,p}\}$ *over* A_* as the projective limit of the projective system

$$\{C^*_{A^{[q]}}(L_*^{[q]},\ \chi^{[q]},\ D^{[q]}),\ \pi'_{q,p}\,;\ q>p\geqq 0\}\,,$$

where $\pi'_{q,p}$ is the natural projection induced by that of the coefficients stated in (3.29)* and by $\Phi_{q,p}$.

Neglecting the differentials, we have the isomorphism

$$C_A^*(\{(L_*^{[q]},\ \chi^{[q]},\ D^{[q]}),\ \Phi_{q,p}\})\cong A^*\otimes(S_*^R(\sigma \bar{L}_*^{[0]}))'\,. \tag{3.36}$$

For an admissible system of mixed type fibration $\{(A^{[q]}\underset{A^*}{\otimes}L_*, 1\otimes\chi, 1\otimes D), \Phi_{q,p}=\mathrm{id}\}$ induced by a mixed type fibration (L_*, χ, D), the Koszul cochain complex $C_A^*(\{(A^{[q]}\underset{A^*}{\otimes}L_*, 1\otimes\chi, 1\otimes D), \Phi_{q,p}=\mathrm{id}\})$ of 3.35 agrees with $C_A^*(L_*, \chi, D)$ defined in 3.14.

DEFINITION 3.37. Let $\{(\tilde{L}_*^{[q]}, \tilde{\chi}^{[q]}, \tilde{D}^{[q]}), \tilde{\Phi}_{q,p}\}$ and $\{(L_*^{[q]}, \chi^{[q]}, D^{[q]}), \Phi_{q,p}\}$ be admissible systems of mixed type fibrations over A_*. A set of homomorphisms of mixed type fibrations $\{\phi^{[q]}:(\tilde{L}_*^{[q]}, \tilde{\chi}^{[q]}, \tilde{D}^{[q]})\to(L^{[q]}, \chi^{[q]}, D^{[q]})\}$ with connections $\theta(\phi^{[q]})\in L_{-1}^{[q]}$ $(q\geqq 0)$ is *a homomorphism of admissible systems of mixed type fibrations over* A_* if $\phi^{[p]}\circ\tilde{\Phi}_{q,p}=\Phi_{q,p}\circ(1\underset{A^{[q]}}{\otimes}\phi^{[q]})$ and $\Phi_{q,p}(1\underset{A^{[q]}}{\otimes}\theta(\phi^{[q]}))=\theta(\phi^{[p]})$.

A homomorphism of admissible systems of mixed type fibrations over A_* $\{\phi^{[q]}:(\tilde{L}_*^{[q]}, \tilde{\chi}^{[q]}, \tilde{D}^{[q]})\to(L_*^{[q]}, \chi^{[q]}, D^{[q]})\,;\, q\geqq 0\}$ with connections $\theta(q)=\theta(\phi^{[q]})$ induces a projective system of DGA homomorphisms

$$(3.38)\qquad \mathrm{Hom}_{A^{[q]}}(\phi_{\theta(q)}^{[q]}, 1)\ :\ \mathrm{Hom}_{A^{[q]}}^*(C_*^{A^{[q]}}(L_*^{[q]}, \chi^{[q]}, D^{[q]}), A^{[q]}) \longrightarrow \mathrm{Hom}_{A^{[q]}}^*(C_*^{A^{[q]}}(\tilde{L}_*^{[q]}, \tilde{\chi}^{[q]}, \tilde{D}^{[q]}), A^{[q]})$$

such that

$$(3.39)\qquad \tilde{\pi}_{q,p}'\circ\mathrm{Hom}_{A^{[q]}}(\phi_{\theta(q)}^{[q]}, 1)=\mathrm{Hom}_{A^{[p]}}(\phi_{\theta(p)}^{[p]}, 1)\circ\pi_{q,p}'\,.$$

Thus, taking the projective limit of (3.38), we obtain a DGA homomorphism over A^*;

$$(3.40)\qquad \phi_{\{\theta(q)\}}^*:C_A^*(\{(L_*^{[q]}, \chi^{[q]}, D^{[q]}), \Phi_{q,p}\})\longrightarrow C_A^*(\{(\tilde{L}_*^{[q]}, \tilde{\chi}^{[q]}, \tilde{D}^{[q]}), \tilde{\Phi}_{q,p}\})\,.$$

In particular, a homomorphism $\phi:(\tilde{L}_*, \tilde{\chi}, \tilde{D})\to(L_*, \chi, D)$ of mixed type fibrations over A_* with connection θ gives rise to that of admissible systems of mixed type fibrations $\{\phi^{[q]}:(A^{[q]}\underset{A^*}{\otimes}\tilde{L}_*, 1\underset{A^*}{\otimes}\tilde{\chi}, 1\underset{A^*}{\otimes}\tilde{D})\to(A^{[q]}\underset{A^*}{\otimes}L_*, 1\underset{A^*}{\otimes}\chi, 1\underset{A^*}{\otimes}D)\}$ with connections $\{\theta(q)=1\otimes\theta\in A^{[q]}\underset{A^*}{\otimes}L_*\}$, which induces a DGA homomorphism over A^*

$$(3.40)^*\qquad \phi_\theta^*:\ C_A^*(L_*, \chi, D)\longrightarrow C_A^*(\tilde{L}_*, \tilde{\chi}, \tilde{D})\,.$$

Let us denote respectively by $(\mathrm{DGL}/A_*)_{\mathrm{mix}}$ and $(\mathrm{DGL}/A_*)_{\mathrm{mix}}^{\wedge}$ the category of the mixed type fibrations over A_* and that of the admissible systems of mixed type fibrations over A_*. And denote also by $(\mathrm{DGA}/A^*)_{\mathrm{loc}}$ the category of Sullivan type fibrations over A^* with a given local section. The morphisms in $(\mathrm{DGA}/A^*)_{\mathrm{loc}}$ are simply the degree preserving A^*-algebra homomorphisms commuting with the differentials which are not required any compatibility with the given local sections.

With these notations, we can summarize our results above as saying that we have constructed the following diagram of functors

$$
\begin{tikzcd}
(\mathrm{DGL}/A_*)_{\mathrm{mix}} \arrow[rr, "C_A^*"] \arrow[dr, "\{A_{[q]}\underset{A_*}{\otimes}(-)\,;\,q\geqq 0\}"'] & & (\mathrm{DGA}/A^*)_{\mathrm{loc}} \\
& (\mathrm{DGL}/A_*)^{\wedge}_{\mathrm{mix}} \arrow[ur, "C_A^*"'] &
\end{tikzcd}
\tag{3.41}
$$

We call $C_A^*(-)$ *the generalized Koszul* (*cochain*) *complex functor*.

Now we explain about the relation between the Euler element χ and the existence of a cross-section.

PROPOSITION 3.42. *Suppose the finite degree condition* (3.21) *holds for* A^*. *The Koszul cochain complex* $C_A^*(L_*, \chi, D)$ *over* A^* *of a mixed type fibration* (L_*, χ, D) *over* A_* *admits a cross-section if and only if*

(3.43) ("the structure equation")

$$D(\theta)+\frac{1}{2}[\theta, \theta]=\chi$$

for the Euler element χ *holds for some element* $\theta\in L_{-1}$. *When* A^* *does not satisfy* (3.21), *the same result as above holds if we replace a mixed type fibration by an admissible system of mixed type fibrations.*

PROOF. We only give the proof for the case where A^* satisfies the condition (3.21), from which the proof for the other case is easy to deduce.

Suppose the structure equation (3.43) holds for some element $\theta\in L_{-1}$. Then it is easy to see that the triple $(L_*, \tilde{\chi}=0, \tilde{D}=D+[\theta, -])$ is a mixed type fibration, and that there is the natural isomorphism

$$\text{"id"}: (L_*, \tilde{\chi}=0, \tilde{D}=D+[\theta, -]) \longrightarrow (L_*, \chi, D) \tag{3.44}$$

with connection $\theta(\text{"id"})=\theta$. So by (3.40)*, we have the isomorphism

$$(\text{"id}_\theta\text{"})^*: C_A^*(L_*, \chi, D) \longrightarrow C_A^*(L_*, \tilde{\chi}=0, \tilde{D})$$

over A^*. The Sullivan type fibration $C_A^*(L_*, \tilde{\chi}=0, \tilde{D})$ has the cross-section $\tilde{\varepsilon}$ stated in (3.17) and (3.18). Therefore $C_A^*(L_*, \chi, D)$ has the cross-section;

$$\tilde{\varepsilon}\circ(\text{"id}_\theta\text{"})^*: C_A^*(L_*, \chi, D) \longrightarrow C_A^*(L_*, \tilde{\chi}=0, \tilde{D}) \longrightarrow A^*. \tag{3.45}$$

Conversely, suppose $C_A^*(L_*, \chi, D)$ admits a cross-section

$$\varepsilon: C_A^*(L_*, \chi, D) \longrightarrow A^* \tag{3.46}$$

commuting with the differentials. Remark that

$$\begin{aligned}\varepsilon\in \mathrm{Hom}_{A^*}(C_A^*(L_*, \chi, D), A^*)&=\mathrm{Hom}_{A^*}(\mathrm{Hom}_{A_*}(S_*^A(\sigma L_*), A_*), A^*)\\ &\cong S_*^A(\sigma L_*).\end{aligned}$$

So ε can be considered as an element of $S^A_*(\sigma L_*)$ by identifying the biduals via the canonical isomorphism.

Let $p_1: S^A_*(\sigma L_*)\to S^A_{(1)}(\sigma L_*)=\sigma L_*$ be the canonical projection. Then it can be shown that

$$-\frac{1}{2}[\sigma^{-1}p_1\varepsilon,\ \sigma^{-1}p_1\varepsilon]-D(\sigma^{-1}p_1\varepsilon)+\chi=0 \tag{3.47}$$

in L_*. Therefore, putting $\theta=\sigma^{-1}p_1\varepsilon$, we obtain the structure equation (3.43). This completes the proof of Proposition.

§4. Generalized Quillen L functor.

Let (E^*, d) be a Sullivan type algebraic fibration over A^* with a given local section $e: E^*\to A^*$. The obstruction for e to be a global section will be denoted by

$$\sigma\chi=d_A\circ e-e\circ d\in \mathrm{Hom}^{+1}_{A^*}(E^*,\ A^*)\,. \tag{4.1}$$

And we use the usual notation;

$$\bar{E}^*=\mathrm{Ker}\,e,\quad \text{i.e.}\quad E^*=A^*\oplus\bar{E}^*\,. \tag{4.2}$$

Now we again suppose, for the moment, the finite degree condition (3.21) on A^*.

Consider the A^*-dual $\{\mathrm{Hom}^*_{A^*}(E^*, A^*),\ d'=\mathrm{Hom}_{A^*}(d, 1)\}$ of (E^*, d). And consider the free graded Lie algebra over A_*

$$L^A_*(\sigma^{-1}\mathrm{Hom}^*_{A^*}(\bar{E}^*,\ A^*)) \tag{4.3}$$

generated by the desuspension of $\mathrm{Hom}^*_{A^*}(\bar{E}^*, A^*)$.

The homomorphism $\sigma\chi$ of (4.1) restricts to $\bar{E}^*$, and we define *the Euler element* $\chi\in L^A_{-2}(\sigma^{-1}\mathrm{Hom}_{A^*}(\bar{E}^*, A^*))$ by

$$\begin{aligned}\chi=\sigma^{-1}(d_A\circ e-e\circ d)=-\sigma^{-1}(e\circ d)&\in\sigma^{-1}\mathrm{Hom}^{+1}_{A^*}(\bar{E}^*,\ A^*)\\ &\subset L^A_{-2}(\sigma^{-1}\mathrm{Hom}^*_{A^*}(\bar{E}^*,\ A^*))\,.\end{aligned} \tag{4.4}$$

The dual differential $d'=\mathrm{Hom}_{A^*}(d, 1)$ and the comultiplication $\mu'=\mathrm{Hom}_{A^*}(\mu, 1)$ restrict to $\mathrm{Hom}^*_{A^*}(\bar{E}^*, A^*)$, and we define *the exterior covariant differentiation D* in $L^A_*(\sigma^{-1}\mathrm{Hom}^*_{A^*}(\bar{E}^*, A^*))$ as an A_*-Lie derivation defined by

$$D(\sigma^{-1}f)=D_1(\sigma^{-1}f)+D_2(\sigma^{-1}f) \tag{4.5}$$

where

$$D_1(\sigma^{-1}f)=-\sigma^{-1}(d-e\circ d)'(f)\,,\qquad \text{and} \tag{4.6}$$

(4.7) $$D_2(\sigma^{-1}f)=-\frac{1}{2}\sum_i(-1)^{\deg h_i}[\sigma^{-1}h_i, \sigma^{-1}g_i]$$

if

(4.8) $$\mu'(f)=f\otimes 1+1\otimes f+\sum_i h_i \underset{A^*}{\otimes} g_i .$$

LEMMA 4.9. *The triple* $(L_*^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\bar{E}^*, A^*)), \chi, D)$ *as defined above is an algebraic fibration of the mixed type as defined in Definition* 2.1, *and will be denoted by* $L_*^A(E^*, e)$.

PROOF. The conditions (i) and (ii) of 2.1 are obviously satisfied by virtue of Definition 2.13. By the definitions of the Euler element χ and the exterior covariant differentiation D, the Bianchi identity (2.3) and the curvature formula (2.4) are easily verified by direct calculations. Q. E. D.

Next consider the homomorphism of Sullivan type fibrations over A^*

(4.10) $$\phi:(E^*, d)\longrightarrow(\tilde{E}^*, \tilde{d}).$$

Let e and $\tilde{e}$ be the respective local sections of E^* and $\tilde{E}^*$.

Taking A^*-duals, the homomorphism ϕ induces a homomorphism of reduced coalgebras over A_*

(4.11) $$\phi' : \mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,\tilde{e}, A^*)\longrightarrow \mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,e, A^*)$$

defined by

(4.12) $$\langle\phi'(\tilde{f}), x\rangle_A=\langle\tilde{f}, (\phi-\tilde{e}\circ\phi)(x)\rangle_A$$

for every $\tilde{f}\in\mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,\tilde{e}, A^*)$ and $x\in\mathrm{Ker}\,e$.

The homomorphism $\sigma^{-1}\phi'$ naturally extends to a free graded Lie algebra homomorphism

(4.13) $$L(\phi') : L_*^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,\tilde{e}, A^*))\longrightarrow L_*^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,e, A^*)).$$

DEFINITION 4.14. Let e, $\tilde{e}$, ϕ and $L(\phi')$ be as stated above. We first define an A^*-algebra homomorphism $\sigma\theta$ of degree zero by

$$\sigma\theta=\tilde{e}\circ(\phi\,|\,\mathrm{Ker}\,e)\in\mathrm{Hom}_{A^*}^0(\mathrm{Ker}\,e, A^*).$$

And then we define *the connection* $\theta=\theta(L(\phi'))$ *of the homomorphism* $L(\phi')$ as the canonical image in $L_*^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,e, A^*))$ of the desuspension of $\sigma\theta$;

(4.15) $$\theta=\sigma^{-1}(\tilde{e}\circ\phi\,|\,\mathrm{Ker}\,e)\in\sigma^{-1}\mathrm{Hom}_{A^*}^0(\mathrm{Ker}\,e, A^*)$$
$$\subset L_{-1}^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,e, A^*)).$$

LEMMA 4.16. *The connection* $\theta=\theta(L(\phi'))$ *defined above satisfies the structure equation* (2.11) *and the connection formula* (2.12) *with respect to the graded Lie* *algebra homomorphism* $L(\phi')$.

PROOF. The lemma follows easily by direct calculations using the definitions of χ and D.

By Lemmas 4.9 and 4.16, $L_*^A(-)$ is a functor from $(\mathrm{DGA}/A^*)_{\mathrm{loc}}$ to $(\mathrm{DGL}/A_*)_{\mathrm{mix}}$ under the finite degree condition (3.21) on A^*. We call it *the generalized Quillen L functor.*

Let us consider the case when (E^*, d) is the Koszul cochain complex $C_A^*(L_*, \chi, D)$ over A^* of a mixed type fibration (L_*, χ, D) over A_*.

Identifying the A^*-biduals by the canonical isomorphisms, we have, by definition,

$$L_*^A(C_A^*(L_*, \chi, D), e)=(L_*^A(\sigma^{-1}S_{(+)}^A(\sigma L_*)), \chi, D_1+D_2), \tag{4.17}$$

where χ is the canonical image of the Euler element of L_* denoted by the same symbol by the natural inclusion

$$\begin{aligned} L_*=\sigma^{-1}(\sigma L_*)=\sigma^{-1}S_{(1)}^A(\sigma L_*) &\longrightarrow \sigma^{-1}S_{(+)}^A(\sigma L_*) \\ &\longrightarrow L_*^A(\sigma^{-1}S_{(+)}^A(\sigma L_*)), \end{aligned} \tag{4.18}$$

and the A_*-Lie derivation D_1 is characterized by

$$D_1(\sigma^{-1}x)=-\sigma^{-1}(d_L+D+d_\chi)(x) \tag{4.19}$$

for every $x\in S_{(+)}^A(\sigma L_*)$, and the second A_*-Lie derivation D_2 is determined by the graded symmetric coalgebra structure of $S_{(+)}^A(\sigma L_*)$.

Let

$$\sigma^{-1}(p_1) : \sigma^{-1}S_{(+)}^A(\sigma L_*) \longrightarrow \sigma^{-1}S_{(1)}^A(\sigma L_*)=\sigma^{-1}(\sigma L_*)=L_* \tag{4.20}$$

be the natural projection. We define a graded Lie algebra homomorphism over A_*

$$\alpha : L_*^A(\sigma^{-1}S_{(+)}^A(\sigma L_*)) \longrightarrow L_* \tag{4.21}$$

as the A_*-Lie algebra extension of the map $\sigma^{-1}(p_1)$. Then it can be verified that α is a homomorphism of mixed type fibrations over A_* with connection $\theta(\alpha)=0$.

If we start from a Sullivan type fibration (E^*, d) over A^* with a given local section e, we obtain the other adjunction map

$$\beta : C_A^*(L_*^A(E^*, e)) \longrightarrow E^*, \tag{4.22}$$

which is the A^*-algebra extension of the natural projection

$$\begin{aligned} \mathrm{Hom}_{A_*}(\sigma\iota_1, 1) : \mathrm{Hom}_{A_*}^*(\sigma L_*^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\bar{E}^*, A^*)), A_*) \\ \longrightarrow \mathrm{Hom}_{A_*}^*(\sigma(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\bar{E}^*, A^*)), A_*)\cong\bar{E}^*. \end{aligned} \tag{4.23}$$

Again it can be verified that the homomorphism β is compatible with the differentials in view of the sign conventions in (3.4) and (3.5).

In case A^* does not satisfy the finite degree condition (3.21), we obtain an admissible system of mixed type fibrations over A_*

$$\{L_*^{A^{[q]}}(A^{[q]}\underset{A^*}{\otimes}E^*,\ 1\underset{A^*}{\otimes}e),\ q\geqq 0\,;\ \Phi_{q,p}=\mathrm{id},\ q>p\geqq 0\} \tag{4.24}$$

as the image of the functor $L_*^A(-):(\mathrm{DGA}/A^*)_{\mathrm{loc}}\to(\mathrm{DGL}/A_*)\hat{}_{\mathrm{mix}}$. And the same type of results as above are easily extended to the admissible system of mixed type fibrations. We omit the details.

§ 5. Weak equivalences.

A homomorphism of Sullivan type fibrations over A^* is called *a weak equivalence* (or *a quasi-isomorphism*) if it induces an isomorphism in cohomology.

Let $\phi:(\tilde{L}_*,\tilde{\chi},\tilde{D})\to(L_*,\chi,D)$ be a homomorphism of mixed type fibrations over A_* with connection θ. Then ϕ induces a homomorphism of DGL's over R on the fiber;

$$\bar{\phi}\ :\ (\bar{\tilde{L}}_*=R\underset{A_*}{\otimes}\tilde{L}_*,\ 1\underset{A_*}{\otimes}\tilde{D})\longrightarrow(\bar{L}_*=R\underset{A_*}{\otimes}L_*,\ 1\underset{A_*}{\otimes}D)\,. \tag{5.1}$$

Definition 5.2. A homomorphism ϕ of mixed type fibrations over A_* is *a weak equivalence* (or *a quasi-isomorphism*) if the quotient homomorphism $\bar{\phi}$ on the fiber induces an isomorphism in homology. And a homomorphism of admissible systems of mixed type fibrations over A_*

$$\{\phi^{[q]}:(\tilde{L}_*^{[q]},\ \tilde{\chi}^{[q]},\ \tilde{D}^{[q]})\longrightarrow(L_*^{[q]},\ \chi^{[q]},\ D^{[q]})\,;\ q\geqq 0\} \tag{5.3}$$

is *a weak equivalence* (or *a quasi-isomorphism*) if $\bar{\phi}^{[0]}$ induces an isomorphism in homology on the fiber.

Let us denote by (DGA/A^*), (DGA_1/A^*) and $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$ the respective categories of Sullivan type fibrations over A^*, of those with a 1-connected fiber, and of those with a 1-connected fiber and a local section. And let us denote respectively by $(\mathrm{DGL}_0/A_*)_{\mathrm{mix}}$ and $(\mathrm{DGL}_0/A_*)\hat{}_{\mathrm{mix}}$ the category of mixed type fibrations over A_* with a connected fiber (i.e. $\bar{L}_i=0$ if $i\leqq 0$) and that of admissible systems of mixed type fibrations over A_* with a connected fiber.

For simplicity's sake, we will assume till the end of this section that A^* is 1-connected, i.e.

$$A^i=0\quad(i<0\ \text{or}\ i=1)\quad\text{and}\quad A^0\cong R\,. \tag{5.4}$$

In fact this assumption can be weakened (cf. [**9**]).

Theorem 5.5. (i) *If* $\phi:(\tilde{L}_*,\tilde{\chi},\tilde{D})\to(L_*,\chi,D)$ *is a weak equivalence in*

$(\mathrm{DGL}_0/A_*)_{\mathrm{mix}}$ *with connection* θ, *then*

$$\psi_\theta^* : C_A^*(L_*, \chi, D) \longrightarrow C_A^*(\tilde{L}_*, \tilde{\chi}, \tilde{D})$$

is a weak equivalence in $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$. *Slightly generally, if*

$$\{\psi^{[q]} : \{(\tilde{L}_*^{[q]}, \tilde{\chi}^{[q]}, \tilde{D}^{[q]}), \tilde{\Phi}_{q,p}\} \longrightarrow \{(L_*^{[q]}, \chi^{[q]}, D^{[q]}), \Phi_{q,p}\} ; q \geqq 0\}$$

is a weak equivalence in $(\mathrm{DGL}_0/A_*)\hat{}_{\mathrm{mix}}$ *with connection* $\theta = \{\theta(q) ; q \geqq 0\}$, *then*

$$\psi_\theta^* (= \mathrm{proj\,lim}\, (\psi_{\theta(q)}^{[q]})^*) \ : \ C_A^*(\{(L_*^{[q]}, \chi^{[q]}, D^{[q]}), \Phi_{q,p}\}) \longrightarrow C_A^*(\{(\tilde{L}_*^{[q]}, \tilde{\chi}^{[q]}, \tilde{D}^{[q]}), \tilde{\Phi}_{q,p}\})$$

is a weak equivalence in $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$.

(ii) *If* $\phi : (E^*, d) \to (\tilde{E}^*, \tilde{d})$ *is a weak equivalence in* $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$, *then*

$$\{L_*^{[q]}(\phi') : L_*^{A^{[q]}}(\sigma^{-1}\mathrm{Hom}_{A^{[q]}}^*(A^{[q]} \underset{A^*}{\otimes} \mathrm{Ker}\,\tilde{e}, A^{[q]})) \longrightarrow L_*^{A^{[q]}}(\sigma^{-1}\mathrm{Hom}_{A^{[q]}}^*(A^{[q]} \underset{A^*}{\otimes} \mathrm{Ker}\,e, A^{[q]})) ; q \geqq 0\}$$

is a weak equivalence in $(\mathrm{DGL}_0/A_*)\hat{}_{\mathrm{mix}}$.

When A^* *satisfies the finite degree condition* (3.21),

$$L(\phi') : L_*^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,\tilde{e}, A^*)) \longrightarrow L_*^A(\sigma^{-1}\mathrm{Hom}_{A^*}^*(\mathrm{Ker}\,e, A^*))$$

is a weak equivalence in $(\mathrm{DGL}_0/A_*)_{\mathrm{mix}}$.

(iii) *The adjunction map*

$$\{\alpha^{[q]}\} \ : L_*^A \circ C_A^*(\{(L_*^{[q]}, \chi^{[q]}, D^{[q]}), \Phi_{q,p}\}) \longrightarrow \{(L_*^{[q]}, \chi^{[q]}, D^{[q]}), \Phi_{q,p}\}$$

is a weak equivalence in $(\mathrm{DGL}_0/A_*)\hat{}_{\mathrm{mix}}$.

When A^* *satisfies* (3.21), *the adjunction map*

$$\alpha : L_*^A \circ C_A^*(L_*, \chi, D) \longrightarrow (L_*, \chi, D)$$

is a weak equivalence in $(\mathrm{DGL}_0/A_*)_{\mathrm{mix}}$.

(iv) *The adjunction map*

$$\beta : C_A^* \circ L_*^A(E^*, e) \longrightarrow (E^*, d)$$

is a weak equivalence in $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$.

PROOF. The theorem being true for $A^* = R$ ([**6**], Appendix B), the general case follows from Moore's comparison theorem for spectral sequences (cf. [**3**], p. 509 and [**9**], 5.1). Q.E.D.

Let $\mathrm{ho}(\mathrm{DGA}_1/A^*)$ be the category obtained from (DGA_1/A^*) by the localization with respect to the weak equivalences [**6**], i.e. by adding formally an inverse to each weak equivalence. And let $\mathrm{ho}(\mathrm{DGL}_0/A_*)\hat{}_{\mathrm{mix}}$ and $\mathrm{ho}(\mathrm{DGL}_0/A_*)_{\mathrm{mix}}$ be the categories obtained respectively from $(\mathrm{DGL}_0/A_*)\hat{}_{\mathrm{mix}}$ and $(\mathrm{DGL}_0/A_*)_{\mathrm{mix}}$ by the localization with respect to the weak equivalences.

COROLLARY 5.6. *We keep the assumption* (5.4) *on the* 1*-connectedness of* A^*. *The adjoint functors*

$$(\mathrm{DGA}/A^*)_{\mathrm{loc}} \underset{C_A^*}{\overset{L_*^A}{\rightleftarrows}} (\mathrm{DGL}/A_*)^{\wedge}_{\mathrm{mix}}$$

induce equivalence of categories

$$\mathrm{ho}(\mathrm{DGA}_1/A^*) \cong \mathrm{ho}(\mathrm{DGL}_0/A_*)^{\wedge}_{\mathrm{mix}}\,. \tag{5.7}$$

When the finite degree condition (3.21) *for* A^* *holds, the right hand side above is reduced to* $\mathrm{ho}(\mathrm{DGL}_0/A_*)_{\mathrm{mix}}$.

PROOF. Each quivalence class in $\mathrm{ho}(\mathrm{DGA}_1/A^*)$ has a representative belonging to $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$ by virtue of the existence of a minimal model (=a Postnikov decomposition) ([**3**], p. 508). This fact, together with (ii) of the preceding theorem, implies that the functor $L_*^A : \mathrm{ho}(\mathrm{DGA}_1/A^*) \to \mathrm{ho}(\mathrm{DGL}_0/A_*)^{\wedge}_{\mathrm{mix}}$ is well-defined. The rest of the assertions follow immediately from the preceding theorem. Q.E.D.

COROLLARY 5.8 (Generalization of Hilton's theorem, cf. [**3**], p. 507). *Let* (E^*, d) *belong to* $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$. *Suppose the fiber* $(\bar{F}, \bar{d}) = (R \underset{A^*}{\otimes} E^*, 1 \underset{A^*}{\otimes} d)$ *satisfies the following two trivialities.*

(i) *The multiplication in* $\bar{F}$ *is trivial, i.e.* $\bar{F}^+ \cdot \bar{F}^+ = 0$.

(ii) *The differential in* $\bar{F}$ *is trivial, i.e.* $\bar{d} \equiv 0$.

Then the adjunction map

$$\beta : C_A^* \circ L_*^A(E^*, e) \longrightarrow (E^*, d)$$

is the minimal model (=*the Postnikov decomposition*) *of the unit map* $u : A^* \to E^*$.

PROOF. By definition, $C_A^* \circ L_*^A(E^*, e)$ is a free graded A^*-algebra, and the adjunction map β is a weak equivalence over A^* by Theorem 5.5 (iv). The triviality assumptions (i) and (ii) of the corollary imply the triviality of $D = D_1 + D_2$ modulo (A^+) in $L_*^A(E^*, e)$. This, in turn, implies the decomposability of the differential $d' = (d_L + D + d_\chi)'$ in $C_A^*(L_*^A(E^*, e))$, for d_L' and d_χ' are evidently decomposable by definition. Therefore the homomorphism β is the minimal model. Q. E. D.

The corollary above gives a more conceptual proof of Lemma 2.1 of [**7**] although under the restriction (5.4) on the 1-connectedness of A^*.

COROLLARY 5.9 (Lemma 2.1 of [**7**]). *Let* (E^*, d) *belong to* $(\mathrm{DGA}_1/A^*)_{\mathrm{loc}}$, *and suppose the fiber* $(\bar{F}, \bar{d})$ *of* (E^*, d) *is isomorphic as a* DGA *to the Koszul cochain complex* $C_R^*(L(V_*))$ *over* R *of a free graded Lie algebra* $L(V_*)$ *over a positively graded vector space* V_* (*i.e.* $V_i = 0$ *for* $i \leq 0$). *Then* (E^*, d) *is isomorphic over* A^* *to* $C_A^* \circ L_*^A(A^* \underset{\tau}{\otimes} \hat{V}^*, e)$, *where* $\hat{V}^* = R \oplus (\sigma V_*)'$, *the differential and the* *multiplication in* $A^*\underset{\tau}{\otimes}\hat{V}^*$ *are induced from those of* (E^*, d) *and the local section* e *is the one naturally induced from the projection* $\hat{V}^*\to R$.

PROOF. By hypothesis, there is an isomorphism of free graded A^*-modules

$$\Phi \ : \ E^*\cong A^*\otimes \mathrm{Hom}_R^*(S_*^R(\sigma L(V_*)), R). \tag{5.10}$$

The A^*-algebra structure and the differential in E^* induce the quotient ones in its retract $A^*\otimes\hat{V}^*$ via the projection and inclusion

$$\begin{aligned} E^* \underset{\cong}{\xrightarrow{\Phi}} & A^*\otimes\mathrm{Hom}_R^*(S_*^R(\sigma L(V_*)), R) \\ & \rightleftarrows A^*\otimes\mathrm{Hom}_R^*(R\oplus\sigma(V_*), R)=A^*\otimes\hat{V}^*. \end{aligned} \tag{5.11}$$

Since the projection $\mathrm{Hom}_R^*(S_*^R(\sigma L(V_*)), R)\to\hat{V}^*$ is a weak equivalence ([**3**], p. 507), the Moore comparison theorem of spectral sequences asserts that the projection in (5.11) is also a weak equivalence. So the projection in (5.11) is a minimal model for the fibration $A^*\to A^*\underset{\tau}{\otimes}\hat{V}^*$. On the other hand, $\beta : C_A^*\circ L_*^A(A^*\underset{\tau}{\otimes}\hat{V}^*, e)\to A^*\underset{\tau}{\otimes}\hat{V}^*$ is also a minimal model for the same fibration by the preceding corollary. Therefore the uniqueness property of the minimal model insists that there is an isomorphism over A^* between E^* and $C_A^*\circ L_*^A(A^*\underset{\tau}{\otimes}V^*, e)$. This completes the proof of Corollary.

§ 6. Applications.

Let $EU_n^{(2n)}\to BU_n^{(2n)}$ be the universal U_n-bundle restricted over the (homotopical) $2n$-skeleton of the base space and

$$\gamma_n \ : \ EU_n^{(2n)} \longrightarrow EU_n^{(2n)}\underset{U_n}{\times}EU_n \longrightarrow BU_n \tag{6.1}$$

be the associated fiber bundle over BU_n with fiber $EU_n^{(2n)}$. The minimal model (in the sense of Sullivan) for the base space BU_n is given by

$$I^n=R[\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_n]; \quad \deg\bar{c}_i=2i, \ d(\bar{c}_i)=0. \tag{6.2}$$

A Sullivan type algebraic model for the fiber $EU_n^{(2n)}$ is given by

$$\hat{W}_n=(R[c_1, c_2, \cdots, c_n]/(\deg>2n))\underset{\tau}{\otimes}E(h_1, h_2, \cdots, h_n) \tag{6.3}$$

with $\deg h_i=2i-1$, $\deg c_i=2i$, $d(h_i)=c_i$, $d(c_i)=0$.

A model for the total space of (6.1) is given by

$$I^n\underset{\tau}{\otimes}\hat{W}_n; \quad d(h_i)=c_i-\bar{c}_i, \ d(c_i)=d(\bar{c}_i)=0. \tag{6.4}$$

If we take the projection onto the left factor of $EU_n^{(2n)} \underset{U_n}{\times} EU_n$, we obtain

$$EU_n \longrightarrow EU_n^{(2n)} \underset{U_n}{\times} EU_n \longrightarrow BU_n^{(2n)} \tag{6.5}$$

with contractible fiber EU_n. So the total space of (6.1) is homotopy equivalent to $BU_n^{(2n)}$. A model for $BU_n^{(2n)}$ is given by

$$I_{(n)} = R[c_1, c_2, \cdots, c_n]/(\deg > 2n); \quad \deg c_i = 2i, \ d(c_i) = 0. \tag{6.6}$$

Therefore the minimal model for the Sullivan type fibration $I^n \to I^n \underset{\tau}{\otimes} \hat{W}_n$ is isomorphic to that for the DGA $I_{(n)}$ of (6.6). This observation was communicated to me by S. Hurder. In fact, $I_{(n)}$ can be regarded as the $\mathrm{Ad}_{GL(n,R)}$-invariant polynomial algebra on $gl(n, R)$ truncated by the relation given by the Bott vanishing theorem. Hurder and Kamber [**4**], [**5**] studied the minimal model for $I_{(n)}$ in order to compute the dual homotopy invariants of codimension n foliations.

We are going to give an explicit description of this minimal model using Corollary 5.8 of the last section. In order to do this, we should first know the explicit structure of $H^*(\hat{W}_n)$.

We denote the monomial $h_{i_1} h_{i_2} \cdots h_{i_m} \otimes c_{j_1} c_{j_2} \cdots c_{j_q}$ in $\hat{W}_n$ by $v(I; J)$ or $h(I)c(J)$, where $I = (i_1, i_2, \cdots, i_m)$ and $J = (j_1, j_2, \cdots, j_q)$ are partitions by positive integers $1 \leqq i_1 < i_2 < \cdots < i_m \leqq n$ and $1 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_q \leqq n$.

THEOREM 6.7 (Vey [**1**]). *An additive basis for the reduced cohomology* $\tilde{H}^*(\hat{W}_n)$ *is given by the classes of cocycles* $v(I; J)$ *satisfying the following conditions*

(1) $\|J\| = j_1 + j_2 + \cdots + j_q \leqq n$.

(2) $i_1 + \|J\| > n$.

(3) $i_1 \leqq j_1$.

COROLLARY 6.8. (i) $\tilde{H}^q(\hat{W}_n) = 0$ *for* $q \leqq 2n$.

(ii) *The multiplicative structure of* $\tilde{H}^*(\hat{W}_n)$ *is trivial, i.e.* $[v(I; J)] \cdot [v(I'; J')] = 0$.

(iii) *The homomorphism* $V: H^*(\hat{W}_n) \to \hat{W}_n$ *defined by* $V([v(I; J)]) = v(I; J)$ *is a weak equivalence of* DGA's, *where* $H^*(\hat{W}_n)$ *is considered as a* DGA *with the trivial differential.*

Using these results of Vey, we define an I^n-algebra structure in $I^n \otimes H^*(\hat{W}_n)$ by

$$\begin{aligned} &[v(I; J)] \cdot [v(I'; J')] \\ &= \sum \mathrm{sign}\Big[\Big\{\prod_{2 \leqq \mu, \nu} (i'_\nu - i_\mu)\Big\} \Big\{\prod_{2 \leqq \mu} (j'^{(2)}_1 - i_\mu)\Big\} \Big\{\prod_{2 \leqq \nu} (i'_\nu - j^{(2)}_1)\Big\} (j'^{(2)}_1 - j^{(2)}_1)\Big] \\ &\quad \cdot \bar{c}(J(1)) \bar{c}(J'(1)) \otimes [v(I - i_1 + j^{(2)}_1 + I' - i'_1 + j'^{(2)}_1; J(2) - j^{(2)}_1 + J'(2) - j'^{(2)}_1)] \end{aligned} \tag{6.9}$$

where the summation runs over all the subdivisions $i_1 + J = J(1) + J(2)$ and $i'_1 + J'$

$=J'(1)+J'(2)$ such that $\max J(1) \leqq j_1^{(2)} (=\min J(2)) < i_2$ and $\max J'(1) \leqq j_1'^{(2)} (=\min J'(2)) < i_2'$ and that $[v(I-i_1+j_1^{(2)}+I'-i_1'+j_1'^{(2)}\,;\,J(2)-j_1^{(2)}+J'(2)-j_1'^{(2)})]$ is a Vey basis element given in Theorem 6.7. (In fact the subdivision $J'(1)+J'(2)$ satisfying these conditions is uniquely determined by the subdivision $J(1)+J(2)$.) Here $I-i_1+j_1^{(2)}+I'-i_1'+j_1'^{(2)}$ means the disjoint union $(I-\{i_1\})\cup\{j_1^{(2)}\}\cup(I'-\{i_1'\})\cup\{j_1'^{(2)}\}$ as a set, and so on. In the above, we allow the case $J(1)=\emptyset$ (and respectively, $J'(1)=\emptyset$), in which case we put $j_1^{(2)}=i_1$ (and respectively, $j_1'^{(2)}=i_1'$).

LEMMA 6.10. *The product in $I^n\otimes H^*(\hat{W}_n)$ defined above is graded commutative and associative.*

PROOF. Easy to check by definition.

Next we define an I^n-linear map of degree $+1$

$$d\,:\,I^n\otimes H^*(\hat{W}_n)\longrightarrow I^n\otimes H^*(\hat{W}_n) \tag{6.11}$$

by

$$\begin{aligned}(6.12)\qquad d([v(I\,;\,J)])=&\sum_{2\leq\mu\leq m}(-1)^{\mu}\bar{c}_{i_\mu}\otimes[v(I-i_\mu\,;\,J)]\\&-\delta(i_2\leqq j_{r(2)+1})\bar{c}(i_1,\,j_1,\,\cdots,\,j_{r(2)})\otimes[v(I-i_1\,;\,J-\{j_1,\,\cdots,\,j_{r(2)}\})]\\&-\sum_{2\leq\mu\leq m}(-1)^{\mu}\delta(i_2>j_{r(\mu)+1})\bar{c}(i_1,\,j_1,\,\cdots,\,j_{r(\mu)})\\&\qquad\otimes[v(I-\{i_1,\,i_\mu\}+j_{r(\mu)+1}\,;\,J-\{j_1,\,\cdots,\,j_{r(\mu)+1}\}+i_\mu)]\end{aligned}$$

for every $(I\,;\,J)=(i_1,\,i_2,\,\cdots,\,i_m\,;\,j_1,\,j_2,\,\cdots,\,j_q)$ with $m\geqq 2$ satisfying the conditions (1), (2) and (3) of Theorem 6.7. Here the symbol $\delta(i_2\leqq j_{r(\mu)+1})$ implies either 1 (if $i_2\leqq j_{r(\mu)+1}$) or 0 (otherwise), and so on. And $\bar{c}(i_1,\,j_1,\,\cdots,\,j_{r(\mu)})$ means $\bar{c}_{i_1}\bar{c}_{j_1}\cdots\bar{c}_{j_{r(\mu)}}$, and $r(\mu)$ is the integer $(0\leqq r(\mu)\leqq q-1)$ defined by

$$i_\mu+\|J\|-j_1-\cdots-j_{r(\mu)}>n\,,\quad\text{and} \tag{6.13}$$

$$i_\mu+\|J\|-j_1-\cdots-j_{r(\mu)}-j_{r(\mu)+1}\leqq n\,. \tag*{(6.13)*}$$

For $m=1$, we put

$$d([v(i_1\,;\,J)])=-\bar{c}(i_1,\,j_1,\,\cdots,\,j_q)\,. \tag*{(6.12)*}$$

LEMMA 6.14. *Let d be as above. Then $(I^n\otimes H^*(\hat{W}_n),\,d)$ is a Sullivan type fibration over I^n. Namely,*

(i) *d is a derivation, i.e.*

$$\begin{aligned}(6.15)\qquad &d([v(I^{(1)}\,;\,J^{(1)})]\cdot[v(I^{(2)}\,;\,J^{(2)})]\cdot\cdots\cdot[v(I^{(k)}\,;\,J^{(k)})])\\&=\sum_{1\leq h\leq k}(-1)^{|I^{(1)}|+|I^{(2)}|+\cdots+|I^{(h-1)}|}[v(I^{(1)}\,;\,J^{(1)})]\\&\qquad\cdots d([v(I^{(h)}\,;\,J^{(h)})])\cdots[v(I^{(k)}\,;\,J^{(k)})]\,,\end{aligned}$$

where $|I^{(h)}|$ *denotes the number of elements in the partition* $I^{(h)}$, *which agrees modulo* 2 *with the degree of* $[v(I^{(h)}; J^{(h)})]$.

(ii) d *is a differential, i.e.* $d^2 \equiv 0$.

PROOF. Part (i) can first be proved for the case $k=2$ by explicitly computing right and left hand sides using definitions (6.9) and (6.12). Then the general case is proved by induction on k by virtue of the associativity of the multiplication in $I^n \otimes H^*(\hat{W}_n)$. Part (ii) is easily checked by direct calculations. Q.E.D.

Now we define an I^n-module homomorphism

$$\phi_V : I^n \otimes H^*(\hat{W}_n) \longrightarrow I^n \otimes \hat{W}_n \tag{6.16}$$

by

$$\phi_V([v(I; J)]) = \sum_{0 \leq \lambda \leq \xi(2)} \bar{c}(i_1, j_1, \cdots, j_{\lambda-1}) \otimes h_{j_\lambda} h_{I-i_1} c(J - \{j_1, \cdots, j_\lambda\}), \tag{6.17}$$

where $\xi(2)$ is the integer defined by

$$j_{\xi(2)} < i_2 \leqq j_{\xi(2)+1} \quad \text{if } |I| \geqq 2, \quad \text{or} \tag{6.18}$$

$$j_{\xi(2)} = \max J \quad \text{if } I = \{i_1\}. \tag{6.18*}$$

In the definition above, we used the convention that $j_0 = i_1$.

LEMMA 6.19. *The homomorphism* ϕ_V *defined above is a weak equivalence in* $(\mathrm{DGA}/I^n)_{\mathrm{loc}}$. *Namely,*

(i) ϕ_V *is multiplicative,*

(ii) ϕ_V *commutes with the differentials, and*

(iii) ϕ_V *induces an isomorphism in cohomology.*

PROOF. As in the proof of Lemma 6.14, part (i) can first be proved in the case for products of two basis elements $[v(I; J)] \cdot [v(I'; J')]$ by explicit calculations. The general case is proved by induction using the associativity of the multiplication. Part (ii) is easy to verify by explicit calculations.

For part (iii), remark that the quotient map

$$\bar{\phi}_V : H^*(\hat{W}_n) \longrightarrow \hat{W}_n \tag{6.20}$$

is nothing but the Vey map V stated in Corollary 6.8 (iii). So $\bar{\phi}_V$ induces an isomorphism in cohomology, and the Moore comparison theorem of spectral sequences asserts that ϕ_V also induces an isomorphism in cohomology. This completes the proof of Lemma.

Combining Corollary 5.8 and Lemma 6.19, we obtain the following result.

THEOREM 6.21. *The composite homomorphism*

$$(6.22) \qquad \phi_V \circ \beta \ : \ C^*_{I^n} \circ L^{I^n}_*(I^n \underset{\tau}{\otimes} H^*(\hat{W}_n),\ e) \longrightarrow I^n \underset{\tau}{\otimes} H^*(\hat{W}_n) \longrightarrow I^n \underset{\tau}{\otimes} \hat{W}_n$$

is the minimal model for the homomorphism $I^n \to I^n \underset{\tau}{\otimes} \hat{W}_n$ *in the sense of Sullivan, where* β *is the adjunction map of the functors given in* (4.22), ϕ_V *is the extended Vey homomorphism defined in* (6.17), *and* e *is the natural augmentation induced from that of* $H^*(\hat{W}_n)$.

The explicit description of $L^{I^n}_*(I^n \underset{\tau}{\otimes} H^*(\hat{W}_n),\ e)$ was given in [8].

Now that the minimal model construction is preserved by a change of base (again by the Moore comparison theorem of spectral sequences, cf. [3], p. 509), we obtain the minimal model of the algebraic fibration $P_n \to P_n \underset{\tau}{\otimes} \hat{W}_n$ (P_n denotes the universal Pontrjagin algebra $\cong I^n/(\bar{c}_{\text{odd}})$) by applying the tensor product functor $I^n/(\bar{c}_{\text{odd}}) \underset{I^n}{\otimes} (-)$ to each term of (6.22) and identifying P_n with $I^n/(\bar{c}_{\text{odd}})$. So we have the following result.

COROLLARY 6.23. *The minimal model for the Sullivan type algebraic fibration* $P_n \underset{\tau}{\otimes} \hat{W}_n$ *over* P_n *is given by the composite homomorphism*

$$(6.24) \qquad \hat{\phi}_V \circ \beta \ : \ C^*_{P_n} \circ L^{P_n}_*(P_n \underset{\tau}{\otimes} H^*(\hat{W}_n),\ e) \longrightarrow P_n \underset{\tau}{\otimes} H^*(\hat{W}_n) \longrightarrow P_n \underset{\tau}{\otimes} \hat{W}_n$$

where P_n *is identified with* $I^n/(\bar{c}_{\text{odd}})$ *and* $\hat{\phi}_V$ *is the image of the extended Vey map* ϕ_V *by the tensor product functor* $I^n/(\bar{c}_{\text{odd}}) \underset{I^n}{\otimes} (-)$, *and* β *is the adjunction map of the functors.*

This is a reasonable answer to the problem of explicit description of the minimal model for the algebraic fibration $P_n \to P_n \underset{\tau}{\otimes} \hat{W}_n$ posed by Haefliger in [2].

Now that

$$(6.25) \qquad P_n \underset{\tau}{\otimes} \hat{W}_n = (I_{(n)} \underset{\tau}{\otimes} E(h_{2i-1}\,;\, 2i-1 \leqq n)) \underset{\tau}{\otimes} (P_n \underset{\tau}{\otimes} E(h_{2i}\,;\, 2i \leqq n)),$$

and that the fiber $P_n \underset{\tau}{\otimes} E(h_{2i}\,;\, 2i \leqq n)$ is contractible, it holds that the natural inclusion $\hat{W}O_n = I_{(n)} \underset{\tau}{\otimes} E(h_{2i-1}\,;\, 2i-1 \leqq n) \to P_n \underset{\tau}{\otimes} \hat{W}_n$ is a weak equivalence. Therefore the minimal model of $P_n \to P_n \underset{\tau}{\otimes} \hat{W}_n$ given in (6.24) also serves as the minimal model for the DGA $\hat{W}O_n$, which has the same rational homotopy type as the truncated Weil algebra $W(gl(n, R), O(n))_n$.

Let M be a paracompact Hausdorff smooth manifold of dimension $n \geqq 1$ and τ_C be the complexification of the tangent bundle τ of M classified by a map $f^C_M : M \to BU_n$. Then f^C_M induces a bundle $(f^C_M)^* \hat{\gamma}_n$ over M with fiber $EU^{(2n)}_n$, where $\hat{\gamma}_n$ is defined in (6.1).

An algebraic model for an induced bundle is obtained by the base change. Namely, let $\Omega^*(M)$ be the de Rham algebra of M and p_i be a differential form representing the i-th Pontrjagin class of M. The homomorphism $I^n \to \Omega^*(M)$ defined by $\bar{c}_{2i} \to p_i$ and $\bar{c}_{2i-1} \to 0$ makes $\Omega^*(M)$ an I^n-algebra. A model for $(f_M^C)^*\hat{\gamma}_n$ is obtained from that of $\hat{\gamma}_n$ given in (6.4) by the base change;

(6.26) $$\Omega^*(M) \underset{I^n}{\otimes} (I^n \underset{\tau}{\otimes} \hat{W}_n) \cong \Omega^*(M) \underset{\tau}{\otimes} \hat{W}_n .$$

Its mixed type model is also obtained by the base change;

(6.27) $$L_*^{\Omega^*(M)}(\Omega^*(M) \underset{\tau}{\otimes} H^*(\hat{W}_n),\ e) .$$

Notice that the fiber bundle $(f_M^C)^*\hat{\gamma}_n$ has a unique homotopy class of cross-sections since its fiber $EU_n^{(2n)}$ is $2n$-connected. The Koszul cochain complex *over the ground field R*

(6.28) $$C_R^*(L_*^{\Omega^*(M)}(\Omega^*(M) \underset{\tau}{\otimes} H^*(\hat{W}_n),\ e))$$

is a Sullivan type model for the space of cross-sections $\Gamma((f_M^C)^*\hat{\gamma}_n)$ (Haefliger, cf. Silveira [9]), and this is *the Haefliger model for the Gelfand-Fuks cohomology.*

REMARK 6.29. If the stable tangent bundle $\tau_M \oplus R^n$ admits a complex vector bundle structure classified by a map $f_M^s : M \to BU_n$, we may consider the induced bundle $(f_M^s)^*\hat{\gamma}_n$, whose mixed type model is obtained from that of $\hat{\gamma}_n$ by the change of base via the homomorphism $I^n \to \Omega^*(M)$ defined by $\bar{c}_i \to \tilde{c}_i$, where $\tilde{c}_i \in \Omega^{2i}(M)$ is a representing form of the i-th Chern class of $\tau_M \oplus R^n$. The cochain complex

(6.30) $$C_R^*(L_*^{\Omega^*(M)}(\Omega^*(M) \underset{\tilde{\tau}}{\otimes} H^*(\hat{W}_n),\ e))$$

is a Sullivan type model for the cross-section space $\Gamma((f_M^s)^*\hat{\gamma}_n)$, and we call it *a complex analogue of the Haefliger model* for the weakly almost complex manifold M. For example, the complex analogue of the Haefliger model for $CP(1)$ is evidently different from the original Haefliger model since $c_1 \neq 0$ while $p_1 = 0$.

Does the complex analogue of the Haefliger model have any vector field interpretation?

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