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A singular quasilinear diffusion equation in L^1

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1. Introduction.

The initial and Dirichlet boundary value problem for the equation $u_t-\Delta\alpha(u)+\beta(u)\ni f$ has a unique generalized or integral solution in L^1 when α and β are maximal monotone graphs in $\boldsymbol{R}$, each containing the origin, and at each point of their common domain either one of α or β is single-valued. Weak Maximum and Comparison Principles follow from an L^∞ estimate on the solution and from an L^1 estimate on the difference of solutions, respectively. This L^1 integral solution is shown to satisfy the above partial differential equation in the sense of distributions when α is surjective (or the data is bounded) and β is continuous.

We shall consider the initial-boundary-value problem

(1.a) $u_t-\Delta v+w=f,\ v\in\alpha(u),\ w\in\beta(u)$ in Ω

(1.b) $v=0$ on $\partial G\times(0, T)$

(1.c) $u=u_0$ on $G\times\{0\}$

where G is a bounded domain in $\boldsymbol{R}^n$, $\Omega\equiv G\times(0, T)$, Δ is the Laplacian in $\boldsymbol{R}^n$, and α and β are maximal monotone graphs in $\boldsymbol{R}\times\boldsymbol{R}$, each containing the origin. The problem (1) will be regarded as an abstract Cauchy problem of the form

(2.a) $u'(t)+A(u(t))+B(u(t))\ni f(t)$, a.e. $t\in(0, T)$

(2.b) $u(0)=u_0$

in the Banach space $L^1(G)$. An *integral solution* of (2) in a Banach space X is a $u\in C(0, T; X)$ such that $u(0)=u_0$ and $u(t)\in\overline{\mathrm{dom}(A+B)}$,

$$\frac{1}{2}\|u(t)-x\|^2\leqq\frac{1}{2}\|u(s)-x\|^2+\int_s^t\langle f(\tau)-y,\ u(\tau)-x\rangle d\tau$$

for each $y\in(A+B)(x)$ and $0\leqq s\leqq t\leqq T$. The pairing in the integral is the semi-scalar-product

$$\langle y,\ x\rangle\equiv\sup\{(y,\ x^*): x^*\in X^*,\ x^*(x)=\|x\|=\|x^*\|\}$$

on the Banach space X. A multi-valued operator $A\subset X\times X$ is called *accretive* if

$$\|x_1-x_2\| \leqq \|(x_1-x_2)+\varepsilon(y_1-y_2)\|$$

for all $\varepsilon>0$, $[x_1, y_1]\in A$, $[x_2, y_2]\in A$. If, in addition, $\mathrm{Rg}(I+A)=X$, then A is called *m-accretive*. When $A+B$ is m-accretive we have the following important result [**3**; 1, p. 124]: if $u_0\in\overline{\mathrm{dom}(A+B)}$ and $f\in L^1(0, T; X)$ then (2) has a unique integral solution.

In Section 2 we show that the operator "$-\Delta\circ\alpha+\beta$" is m-accretive in $L^1(G)$, hence (1) has a unique integral solution if, for each $s\in\boldsymbol{R}$, either $\alpha(s)$ or $\beta(s)$ consists of at most one point. Certain special cases are well-known consequences of the general theory. For example, if β is a continuous monotone function which is *linearly bounded*,

$$|\beta(s)|\leqq C(1+|s|), \quad s\in\boldsymbol{R},$$

then the operator above is the sum of an m-accretive A and a continuous accretive B [**2, 18**]. Likewise, if β is a (not necessarily monotone) Lipschitz-continuous function, then the existence and uniqueness of a solution of (2) follow from either the observation that $A+B+\omega I$ is accretive for some $\omega>0$ or from a standard fixed-point construction [**7, 9**]. Next we show that bounded (or non-negative) data in (1) leads to a bounded (respectively, non-negative) integral solution. Also, if two sets of data are ordered (pointwise a.e.) then the corresponding L^1 integral solutions are similarly ordered. These Maximum and Comparison Principles follow from estimates obtained directly from the existence theory for (2) and corresponding estimates for the stationary or elliptic counterpart of (1) [**6**].

The L^1 integral solution of (2) is proposed as a "generalized solution" of the initial-boundary-value problem (1), even though the abstract notion of integral solution does not imply any differentiability. This occurs even in the case $f=0$, where the solution is given by the nonlinear semigroup generation theory [**1, 7, 13**]. Explicit examples (see below) show that du/dt need not exist in $L^1(G)$ at any time, even with $\beta=0$, so the sense in which the partial differential equation (1.a) is satisfied is an issue. If α is continuous and u is bounded, it follows that (1.a) holds in $\mathcal{D}^*(\Omega)$ [**5**]. If, in addition, α is strictly monotone then it is known that u is weak* continuously differentiable into $C_0(G)^*$ [**15**], hence, (1.a) holds in $\mathcal{D}^*(\Omega)$. If also α^{-1} is Lipschitz, then u is even (strongly) differentiable a.e. into $L^2(G)$ [**13**].

In Section 3 we show that the L^1 integral solution of (2) satisfies the partial differential equation (1.a) in $\mathcal{D}^*(\Omega)$ whenever the (possibly multi-valued) maximal monotone α is surjective and β is a continuous, monotone and linearly bounded function. The surjectivity and boundedness hypotheses may be deleted when the data is bounded. The essential point is to show that the L^1 integral solution is also the H^{-1} strong solution [**4**]. Since the perturbation $\beta(u)$ prevents the direct application of the monotone existence theory in $H^{-1}(G)$, we first obtain the L^1 solution and then show it is the H^{-1} solution. These results are obtained more directly when β is Lipschitz continuous, though not necessarily monotone, and they are of interest even when $\beta\equiv 0$.

Examples of (1) are easily given to illustrate the extremely varied situations covered. The "maximal degenerate" case $\alpha\equiv 0$ gives the ordinary initial-value-problem

$$u'(t)+\beta(u(t))\ni f(t), \quad u(0)=u_0,$$

in which $x\in G$ occurs as a parameter, while the "maximal singular" case $\alpha(0)=\boldsymbol{R}$ (i.e., $\alpha^{-1}\equiv 0$) yields the elliptic boundary-value-problem

$$-\Delta v=f(x, t), \quad v|_{\partial G}=0,$$

in which $t\in(0, T)$ is a parameter. A considerably more interesting and important degenerate example is provided by $\alpha(u)=u^-+(u-1)^+$, where $x^+\equiv\max(x, 0)$ and $x^-\equiv\min(x, 0)$. Then (1) is the weak form of the Stefan free-boundary problem wherein u corresponds to enthalpy, v to the temperature, and $f, -\beta(u)$ are internally distributed heat sources [**9, 14, 17**]. A singular example of (1) arises as a model of diffusion in a partially-saturated medium with $\alpha^{-1}(v)=v^+-(v-1)^+$.

Recently, regularity results have been obtained for solutions of (1.a) in H^{-1}, and we mention these for completeness. If α is locally absolutely continuous, β is uniformly continuous and u_0, f are bounded, then the component v of the (possibly degenerate) problem (1) is continuous in Ω [**10, 16, 20**]. Similarly, with these hypotheses on α^{-1} (instead of α) the component u is continuous [**11**]. The first two (extreme) examples above show that in neither case should we expect both of u, v to be continuous. Of more interest is the observation that if α is permitted to be both singular and degenerate then *neither* u nor v need be continuous.

EXAMPLE. Let $\beta=0$, $f=0$ and $u_0(x)=1$ for $x\in G(0, 1)$. Let α be any maximal monotone graph with $\alpha(x)=\{1\}$ for $0<x<1$ and $\alpha(0)\supset[0, 1]$. The solution of (1) is given by $u(x, t)=1$ for $0\leqq t<1/8$, $\sqrt{2t}<x<1-\sqrt{2t}$, and $u(x, t)=0$ otherwise, and $v(x, t)=\min\{x/\sqrt{2t}, 1, (1-x)/\sqrt{2t}\}$ for $0<x<1$, $0<t<1/8$, and $v(x, t)=0$ otherwise. At the end of Section 3 we shall verify that this pair of functions is the solution of (1).

2. The integral solution.

The Sobolev space $W^{k,p}(G)$ is the Banach space of (equivalence classes of) functions in $L^p(G)$ whose derivatives to order k also belong to $L^p(G)$; $W_0^{k,p}(G)$ denotes the closure of $\mathcal{D}(G)$ in $W^{k,p}(G)$. The domain of the L^1-realization of the Dirichlet-Laplace operator Δ is given by $D(\Delta)\equiv\{v\in W_0^{1,1}(G):\Delta v\in L^1(G)\}$, so "$-\Delta v=f$ in $L^1(G)$" means that $v\in D(\Delta)$ and $-\Delta v=f$ in $\mathcal{D}^*(G)$. Denote by $\mathcal{M}$ the set of all maximal monotone graphs on $\boldsymbol{R}$ which contain the origin. Let $\alpha\in\mathcal{M}$ be given and define the operator A by $f\in A(u)$ iff u, $f\in L^1(G)$ and there is a (unique) $v\in D(\Delta)$ for which $-\Delta v=f$ in $L^1(G)$ and $v(x)\in\alpha(u(x))$, a.e. $x\in G$. Similarly, let $\beta\in\mathcal{M}$ be given and define the operator B by $f\in B(u)$ iff u, $f\in L^1(G)$ and $f(x)\in\beta(u(x))$, a.e. $x\in G$.

A consequence of the fundamental paper of H. Brezis and W. Strauss [**6**] is that A is m-accretive; this holds trivially for B. A crucial estimate from [**6**] is the following.

B-S LEMMA. *Let $\gamma\in\mathcal{M}$; if $v\in D(\Delta)$, $\sigma\in L^\infty(G)$ and $\sigma\in\gamma(v)$, then*

$$-\int_G \Delta v(x)\sigma(x)dx\geqq 0\,.$$

Whether an operator in $L^1(G)$ is accretive can be characterized by the L^1-L^∞ duality map involving the graph $\mathrm{sgn}\in\mathcal{M}$ given by $\mathrm{sgn}(x)=\{x/|x|\}$ for $x\neq 0$ and $\mathrm{sgn}(0)=[-1, 1]$. Thus, a general operator A is L^1-accretive if and only if for each selection $f_j\in A(u_j)$, $j=1, 2$, there exists a measurable selection $\sigma\in\mathrm{sgn}(u_1-u_2)$ for which $\int_G(f_1-f_2)\sigma\geqq 0$. The B-S Lemma shows a somewhat stronger condition holds for the operator $A=-\Delta\alpha$ above.

Our first result is that $A+B$ always satisfies the range condition and that it is m-accretive when an additional hypothesis holds.

THEOREM 1. *There is an m-accretive operator C on $L^1(G)$ with $C\subset A+B$; hence,* $\mathrm{Rg}(I+A+B)=L^1(G)$. *Let the pair α, β satisfy the* single-valuedness condition:

(SVC) *for each $s\in\boldsymbol{R}$ at least one of the two sets $\alpha(s)$, $\beta(s)$ consists of at most one point.*

Then $A+B$ is accretive; hence, $C=A+B$.

PROOF. First, let $\gamma\in\mathcal{M}$ with $\gamma\subset\alpha(I+\beta)^{-1}$. (For example, let α_0 denote the minimal section of α: $\alpha_0(x)=y$ where $y\in\alpha(x)$ and $|y|\leqq|z|$ for all $z\in\alpha(x)$. Then take γ to be an appropriate extension of the monotone function $\alpha_0(I+\beta)^{-1}$.) From [**6**] it follows that for each $f\in L^1(G)$ there exist $z_f\in L^1(G)$ and $v_f\in D(\Delta)$ such that $z_f-\Delta v_f=f$ and $v_f\in\gamma(z_f)\subset\alpha((I+\beta)^{-1}(z_f))$ a.e. in G. Define u_f

$=(I+\beta)^{-1}(z_f)$ and $w_f=z_f-u_f$ and observe that we have u_f, $w_f\in L^1(G)$, $v_f\in D(\Delta)$ satisfying $u_f-\Delta v_f+w_f=f$ in $L^1(G)$ and $v_f\in\alpha(u_f)$, $w_f\in\beta(u_f)$ a.e. in G. That is, $(I+A+B)(u_f)\ni f$, so we have $\mathrm{Rg}(I+A+B)=L^1(G)$. If we repeat the above for a second $g\in L^1(G)$ and obtain u_g, w_g, v_g as before, then since γ and β are in $\mathcal{M}$ we obtain the estimates

$$\|u_f-u_g\|_{L^1}\leqq\|u_f+w_f-u_g-w_g\|_{L^1}\leqq\|f-g\|_{L^1}.$$

Hence, the triple u_f, v_f, w_f depends uniquely on f.

Define the operator C on $L^1(G)$ by $C(u)=\{-\Delta v_f+w_f : u=u_f \text{ for some } f\in L^1\}$. Clearly we have $C\subset A+B$ and $\mathrm{Rg}(I+C)=L^1(G)$ from our construction. We shall show C is accretive. Thus let $-\Delta v_1+w_1\in C(u_1)$ and $-\Delta v_2+w_2\in C(u_2)$ in the preceding notation and define $\sigma\in L^\infty(G)$ by

$$\sigma(x)=\mathrm{sgn}_0(u_1(x)-u_2(x)+v_1(x)-v_2(x)+w_1(x)-w_2(x)).$$

To see C is accretive, we need to check that

$$\int_G(-\Delta(v_1-v_2)+(w_1-w_2))\sigma\,dx\geqq 0.$$

The first term is non-negative by the B-S Lemma since $\sigma\in\mathrm{sgn}(v_1-v_2)$; the second is non-negative since $\sigma\in\mathrm{sgn}(w_1-w_2)$. Since $\sigma\in\mathrm{sgn}(u_1-u_2)$ this shows C is accretive.

It is easy to check that SVC is equivalent to requiring that $\alpha(I+\beta)^{-1}$ is accretive, hence, belongs to $\mathcal{M}$. To show $A+B$ is L^1-accretive, it suffices to check that $(I+\varepsilon(A+B))^{-1}$ is a contraction for each $\varepsilon>0$. But $\varepsilon\alpha(\cdot)$ and $\varepsilon\beta(\cdot)$ belong to $\mathcal{M}$, so we need only check for $\varepsilon=1$, and this follows from the first part of the proof, with $\gamma=\alpha(I+\beta)^{-1}$, where we obtained $\|u_f-u_g\|_{L^1(G)}\leqq\|f-g\|_{L^1(G)}$.

COROLLARY 1. *If the pair* α, $\beta\in\mathcal{M}$ *satisfies the* SVC, *then for each* $u_0\in\overline{\mathrm{dom}(A+B)}$ *and* $f\in L^1(\Omega)$ *there exists a unique integral solution of the Cauchy problem* (2) *in the Banach space* $L^1(G)$.

REMARKS. 1°. When SVC does not hold it is easy to construct examples to show $A(I+B)^{-1}\cong-\Delta\alpha(I+\beta)^{-1}$ is not necessarily accretive. If, for example, α, $\beta\in\mathcal{M}$ satisfy $\alpha(0)\supset[0, 1/8]$ and $\beta(0)\supset[0, 1]$, the problem

$$w-\Delta v=1 \quad \text{in} \quad L^1(G),\quad v\in\alpha((I+\beta)^{-1}(w)),\ G=(0, 1)$$

has many solutions, two of which are given by $w_1=1$, $v_1=0$ and $w_2=0$, $v_2(x)=(1/2)(x-x^2)$. Note that these two solutions correspond to monotone restrictions γ_1, γ_2 of $\alpha(I+\beta)^{-1}$ which satisfy $\gamma_1(x)=0$ for $0\leqq x\leqq 1$ and $\gamma_2(0)\supset[0, 1/8]$, respectively. Moreover, for each such $\gamma\in\mathcal{M}$ with $\gamma\subset\alpha(I+\beta)^{-1}$, there is a unique solution of the problem. Finally, note that for any solution w, v of this problem we have $u\equiv(I+\beta)^{-1}(w)=0$ so the above does not imply that $A+B$ is not accretive.

2°. Let $\alpha, \beta, \gamma\in\mathcal{M}$ be given and consider the abstract Cauchy problem in $L^1(G)$ for the equation

$$\frac{d}{dt}\gamma(w(t))-\Delta\alpha(w(t))+\beta(w(t))\ni f(t)\,.$$

This is equivalent to the Cauchy problem for the equation

$$\frac{d}{dt}u(t)-\Delta\alpha(\gamma^{-1}(u(t)))+\beta(\gamma^{-1}(u(t)))\ni f(t)$$

and we note that $\alpha\circ\gamma^{-1}$ and $\beta\circ\gamma^{-1}$ belong to $\mathcal{M}$ when both pairs α, γ and β, γ satisfy the SVC. In that case, the pair $\alpha\circ\gamma^{-1}$, $\beta\circ\gamma^{-1}$ satisfies SVC if it holds for α, β on dom(γ). Thus the Corollary 1 applies in this situation if for each $s\in\boldsymbol{R}$ at least two of the three sets $\alpha(s)$, $\beta(s)$, $\gamma(s)$ consist of at most one point.

We turn now to L^∞ estimates on integral solutions of (1) and L^1 estimates on the difference of two solutions. These comprise a Maximum Principle and an Order Principle, respectively, and will be obtained from corresponding results for the stationary problem. The stationary results are obtained exactly as in [**6**]; the SVC permits the perturbation β.

LEMMA 2. *If the pair $\alpha, \beta\in\mathcal{M}$ satisfies the* SVC *and $u=(I+A+B)^{-1}(f)$, then $\|u^+\|_{L^\infty}\leqq\|f^+\|_{L^\infty}$, $\|u^-\|_{L^\infty}\leqq\|f^-\|_{L^\infty}$, and $\|u\|_{L^\infty}\leqq\|f\|_{L^\infty}$.*

PROOF. It suffices to prove the first estimate. Let $k\geqq0$ and define $\sigma(x)=H_0(u(x)+v(x)+w(x)-k-\alpha_0(k)-\beta_0(k))$ where v, w are given as in the proof of Theorem 1 and "subscript-0" denotes the minimal section of each graph in $\mathcal{M}$. Then $\sigma(x)\in H(u(x)-k)$ is immediate; from the SVC we obtain $\sigma(x)\in H(v(x)-\alpha_0(k))$ and $\sigma(x)\in H(w(x)-\beta_0(k))$. Thus, multiply the equation

$$(u-k)-\Delta v+(w-\beta_0(k))+\beta_0(k)=(f-k) \qquad \text{in} \quad L^1(G)$$

by $\sigma\in L^\infty$, integrate over G, and deduce from the B-S Lemma that

$$\int_G(u^+-k)^+ + \int_G(w^+-\beta_0(k))^+\leqq\int_G(f^+-k)^+\,.$$

Choosing $k=\|f^+\|_{L^\infty}$ gives the desired result.

COROLLARY 2. *If $\varepsilon>0$ and $u+\varepsilon(A(u)+B(u)-g)\ni f$, then*

$$\|u^+\|_{L^\infty}\leqq\|f^+\|_{L^\infty}+\varepsilon\|g^+\|_{L^\infty}\,.$$

LEMMA 3. *If the pair $\alpha, \beta\in\mathcal{M}$ satisfies the* SVC *and for each $j=1, 2$ we have $u_j=(I+A+B)^{-1}(f_j)$, then $\|[u_1-u_2]^+\|_{L^1}+\|[w_1-w_2]^+\|_{L^1}\leqq\|[f_1-f_2]^+\|_{L^1}$.*

PROOF. After introducing v_j, w_j for $j=1, 2$ as before, we define $\sigma(x) = H_0(u_1(x)+v_1(x)+w_1(x)-u_2(x)-v_2(x)-w_2(x))$. Then $\sigma(x) \in H(u_1(x)-u_2(x))$ is immediate and the SVC implies $\sigma(x) \in H(v_1(x)-v_2(x))$ and $\sigma(x) \in H(w_1(x)-w_2(x))$. We subtract the two equations, multiply by σ, and obtain the desired estimate from the B-S Lemma.

COROLLARY 3. *If* $\varepsilon>0$ *and* $u_j+\varepsilon(A(u_j)+B(u_j)-g_j) \ni f_j$ *for* $j=1, 2$, *then*

$$\|[u_1-u_2]^+\|_{L^1} \leqq \|[f_1-f_2]^+\|_{L^1} + \varepsilon \|[g_1-g_2]^+\|_{L^1}.$$

PROPOSITION 1. *Let* $\alpha, \beta \in \mathcal{M}$ *satisfy the* SVC. *Then for each* $f \in L^1(0, T; L^1(G))$ *and* $u_0 \in \overline{\mathrm{dom}(A+B)}$ *the integral solution of* (2) *satisfies*

$$\|u(t)^+\|_{L^\infty} \leqq \|u_0^+\|_{L^\infty} + \int_0^t \|f(s)^+\|_{L^\infty} ds, \qquad 0 \leqq t \leqq T.$$

PROOF. We may assume $f^+ \in L^1(0, T; L^\infty)$ and let $\{f_n\}$ be a sequence in $L^1(0, T; L^1)$ converging to f and with $\lim_{n\to\infty} f_n^+ = f^+$ in $L^1(0, T; L^\infty)$. If u_n is the solution of (1) with data f_n, then $\lim_{n\to\infty} u_n = u$ in $C(0, T; L^1)$ so the desired estimate will follow if we can establish it for the special case of a step function f as above.

Consider such a step function given by $f(t)=g_i$, $t_{i-1} \leqq t < t_i$, where $\{t_i : 0 \leqq i \leqq n\}$ is a partition of $[0, T]$. Let S_i be the non-linear semigroup generated by $-A^i$, where $A^i(v) \equiv A(v)+B(v)-g_i$, $1 \leqq i \leqq n$. Thus we have [7] $S_i(t)(v) = \lim_{j\to\infty} \left[I+\frac{t}{j}A^i\right]^{-j}(v)$. From Corollary 2 we obtain

$$\|S_i(t)(v)^+\|_{L^\infty} \leqq \|v^+\|_{L^\infty} + t\|g_i^+\|_{L^\infty}, \qquad t \geqq 0.$$

Since the integral solution is given by $u(t) = S_i(t-t_{i-1})u(t_{i-1})$, $t_{i-1} \leqq t \leqq t_i$, we thereby obtain

$$\begin{aligned} \|u(t)^+\|_{L^\infty} &\leqq \|u(t_{i-1})^+\|_{L^\infty} + (t-t_{i-1})\|g_i^+\|_{L^\infty} \\ &\leqq \|u(t_0)^+\|_{L^\infty} + (t_1-t_0)\|g_1^+\|_{L^\infty} + (t_2-t_1)\|g_2^+\|_{L^\infty} + \cdots + (t-t_{i-1})\|g_i^+\|_{L^\infty}, \end{aligned}$$

and this is the desired estimate.

COROLLARY. *The integral solution satisfies the estimates*

$$\|u(t)^-\|_{L^\infty} \leqq \|u_0^-\|_{L^\infty} + \int_0^t \|f(s)^-\|_{L^\infty} ds,$$

$$\|u(t)\|_{L^\infty} \leqq \|u_0\|_{L^\infty} + \int_0^t \|f(s)\|_{L^\infty} ds, \qquad 0 \leqq t \leqq T.$$

In a similar manner we may use Corollary 3 to prove the following.

PROPOSITION 2. *Let $\alpha, \beta \in \mathcal{M}$ satisfy the* SVC. *For $j=1, 2$, let $f^j \in L^1(0, T; L^1(G))$, $u_0^j \in \overline{\mathrm{dom}(A+B)}$ and denote the corresponding integral solution of* (1) *by u_j. Then*

$$\|[u_1(t)-u_2(t)]^+\|_{L^1} \leqq \|[u_0^1-u_0^2]^+\|_{L^1}+\int_0^t \|[f^1(s)-f^2(s)]^+\|_{L^1} ds\,, \qquad 0 \leqq t \leqq T\,.$$

Finally, we give an L^∞ order estimate which is of independent interest though not of use for the evolution equation.

PROPOSITION 3. *Assume α is Lipschitz continuous: $|\alpha(x)-\alpha(y)| \leqq K|x-y|$ for $x, y \in \boldsymbol{R}$. If for $j=1, 2$ we have $u_j=(I+A+B)^{-1}(f_j)$, then*

$$\|[\alpha(u_1)-\alpha(u_2)]^+\|_{L^\infty} \leqq K\|[f_1-f_2]^+\|_{L^\infty}\,.$$

PROOF. With $w_j \in \beta(u_j)$ as above, we have

$$(u_1-u_2)-\Delta(\alpha(u_1)-\alpha(u_2))+(w_1-w_2)=f_1-f_2\,.$$

Multiply by $H_0(\alpha(u_1)-\alpha(u_2)-k)$ for any $k \geqq 0$ and integrate to obtain

$$\int_G (u_1-u_2)H_0(\alpha(u_1)-\alpha(u_2)-k) \leqq \int_G (f_1-f_2)H(\alpha(u_1)-\alpha(u_2)-k)\,.$$

The left side of this inequality is equal to

$$\int_G |u_1-u_2|H_0(\alpha(u_1)-\alpha(u_2)-k) \geqq (1/K)\int_G |\alpha(u_1)-\alpha(u_2)|H_0(\alpha(u_1)-\alpha(u_2)-k)\,.$$

This leads to

$$\int_G [[\alpha(u_1)-\alpha(u_2)]^+-k]^+ \leqq \int_G (K[f_1-f_2]^+-k)$$

and the result follows by choosing $k=K\|[f_1-f_2]^+\|_{L^\infty}$.

Note that if in addition α is strongly monotone: $\alpha(x)-\alpha(y) \geqq k(x-y)$ for $x, y \in \boldsymbol{R}$, where $k>0$, then we obtain

$$\|[u_1-u_2]^+\|_{L^\infty} \leqq (K/k)\|[f_1-f_2]^+\|_{L^\infty}\,.$$

Since $-\Delta\alpha$ is not accretive in $L^p(G)$ for $p>1$, unless α is linear so $K=k$, we can not expect much more.

3. The strong solution.

The Laplace operator $-\Delta$ is the Riesz isomorphism of the Hilbert space $H_0^1(G) \equiv W_0^{1,2}(G)$ onto its dual space $H^{-1}(G)$. We shall give sufficient conditions for the Cauchy problem (2) to have a unique strong solution in $H^{-1}(G)$, even though B is not accretive on $H^{-1}(G)$, and this strong solution is the integral solution in $L^1(G)$. Thus, (2) is consistent on L^1 and H^{-1}; recall that $L^1 \subset H^{-1}$ only if $n \leqq 2$. The objective is to show the L^1 integral solution of (2) does satisfy the original partial differential equation (1.a) in the sense of distributions.

A *strong solution* of (2) in the Hilbert space H^{-1} is an absolutely continuous $u : [0, T] \to H^{-1}(G)$, therefore differentiable at a.e. $t \in [0, T]$, which verifies $u(t) \in \operatorname{dom}(A+B)$ and (2.a) at a.e. $t \in [0, T]$.

THEOREM 2. *Let α, β be given in $\mathcal{M}$ with* $\operatorname{Rg}(\alpha) = \boldsymbol{R}$ *and β a continuous linearly-bounded function. Define $D \equiv \{w \in L^1 \cap H^{-1} :$ there is a $v \in H_0^1$ with $\Delta v \in L^1$ and $v(x) \in \alpha(w(x))$ a.e. $x \in G\}$ and let u_0 be given in the closure of D in $L^1 \cap H^{-1}$. Let $f \in L^1(\Omega)$ and assume the L^1-integral solution u of* (2) *satisfies $f - B(u) \in L^2(0, T; H^{-1}(G))$. Then u is a strong solution in $H^{-1}(G)$ of*

$$u'(t) + A(u(t)) \ni f(t) - B(u(t)), \qquad a.e.\ t \in (0, T), \tag{3}$$

and thereby satisfies (1.a) *in $\mathcal{D}^*(\Omega)$.*

REMARKS. 3°. Suppose $u_0 \in L^\infty(G)$ and $f \in L^1(0, T; L^\infty(G))$. Then we may delete the hypotheses that α is surjective and that β is linearly bounded. That is, Proposition 1 shows $k \equiv \|u\|_{L^\infty(\Omega)}$ is finite, so by the standard device of altering α and β off the interval $[-k, k]$ these hypotheses are obtained automatically. The condition on $f - B(u)$ then holds if $f \in L^2(0, T; H^{-1}(G))$.

LEMMA 4. *Assume A is m-accretive in the Banach space X, B is continuous and accretive on X, $f \in L^1(0, T; X)$ and $u_0 \in \overline{\operatorname{dom}(A)}$. Then there exists a unique integral solution of* (3) *with $u(0) = u_0$. That is,*

$$\frac{1}{2}\|u(t)-x\|^2 \leqq \frac{1}{2}\|u(s)-x\|^2 + \int_s^t \langle f(\tau) - B(u(\tau)) - y,\ u(\tau) - x \rangle d\tau$$

for each $y \in A(x)$ and $0 \leqq s \leqq t \leqq T$.

The proof of Lemma 4 follows by a modification of the main result of [**2**]; also see [**1**, pp. 152-156]. It is necessary only to include f. Since B is accretive it is immediate that this integral solution of (3) is the integral solution of (2).

Define the operator A_1 on $H^{-1}(G)$ by $f \in A_1(u)$ iff $f \in H^{-1}$, $u \in H^{-1} \cap L^1$ and there exists a (unique) $v \in H_0^1$ such that $v \in \alpha(u)$ and $-\Delta v = f$. Then A_1 is m-accretive, in fact, a subgradient on the Hilbert space H^{-1} [**4**] and $\operatorname{dom}(A_1) = D$.

LEMMA 5. *For each $\varepsilon > 0$, $(I + \varepsilon A)^{-1} = (I + \varepsilon A_1)^{-1}$ on $L^1 \cap H^{-1}$.*

PROOF. Observe first that A_1 restricted to L^1 is just $A\cap A_1$. That is, if $A_1(u)\ni g\in L^1\cap H^{-1}$, then $g\in A(u)$ follows since $H_0^1\subset W_0^{1,1}$. Thus, if $g\in L^1\cap H^{-1}$ and $u_0=(I+\varepsilon A_1)^{-1}g$, then we have $u_0=(I+\varepsilon A)^{-1}g$.

In order to prove Theorem 2 we let u be the integral solution of (2) in L^1. Since B is continuous, Lemma 4 implies that u is the integral solution in L^1 of (3). As such, it is obtained as the limit in $C(0, T; L^1(G))$ of the strong solutions $\{u_\varepsilon\}$ of

$$u_\varepsilon'(t)+A_\varepsilon(u_\varepsilon(t))=f(t)-B(u(t)), \qquad \text{a.e. } t\in[0, T] \tag{4.a}$$

$$u_\varepsilon(0)=u_0 \tag{4.b}$$

where the Lipschitz function $A_\varepsilon=(1/\varepsilon)[I-(I+\varepsilon A)^{-1}]$ is the Yosida approximation of A for each $\varepsilon>0$. Lemma 5 shows that A_ε is also the Yosida approximation of A_1 in H^{-1}. Furthermore, since A_1 is a subgradient it is known [**4**] that the Cauchy problem for (3) has a unique strong solution which is obtained as the limit in $C(0, T; H^{-1}(G))$ of the solutions $\{u_\varepsilon\}$ of (4). By the uniqueness of limits, u is that strong solution.

COROLLARY. *If α, β, u_0 and f are given as above, then there exists at most one strong solution u of* (3) *in H^{-1} with $u(0)=u_0$, $f-B(u)\in L^2(0, T; H^{-1})$ and $B(u)\in L^1(\Omega)$.*

PROOF. If u is such a strong solution then it is the limit in $C(0, T; H^{-1})$ of solutions u_ε of (4). Since $f-B(u)\in L^1(\Omega)$ we have $u=\lim_{\varepsilon\to 0} u_\varepsilon$ also in $C(0, T; L^1(G))$ and u is the L^1-integral solution of (3) with $u(0)=u_0$. Since B is L^1-accretive, u is the integral solution of (2).

REMARKS. 4°. Theorem 2 holds when β is Lipschitz but not necessarily monotone. By a standard fixed-point construction in $C(0, T; L^1(G))$ there exists a unique integral solution of (3) with $u(0)=u_0$. Then as above it follows that u is a strong H^{-1} solution. The uniqueness of the strong H^{-1} solution follows as in the Corollary.

5°. Let K be the Lipschitz constant for β, so $\beta+KI$ is monotone. Thus there is a unique L^1 integral solution of the Cauchy problem for

$$u'(t)+(A+B+K)u(t)\ni f(t)+Ku(t), \qquad 0\leqq t\leqq T.$$

Let $\tilde{u}$ be the integral solution of the Cauchy problem for

$$\tilde{u}'(t)+A(\tilde{u}(t))\ni f(t)-B(u(t)), \qquad 0\leqq t\leqq T,$$

and therefore, by accretiveness of $B+KI$, for

$$\tilde{u}'(t)+(A+B+K)\tilde{u}(t)\ni f(t)+(B+K)(\tilde{u}(t))-B(u(t)), \qquad 0\leqq t\leqq T.$$

Thus we obtain the estimate

$$\|u(t)-\tilde{u}(t)\|_{L^1}^2 \leqq 2\int_0^t \langle (B+K)u(\tau)-(B+K)\tilde{u}(\tau),\ u(\tau)-\tilde{u}(\tau)\rangle d\tau$$

which implies $u=\tilde{u}$, so u is an integral solution of (3). This argument with $K=0$ in the general setting of Lemma 4 shows when B is accretive *and* Lipschitz that integral solutions of (2.a) and (3) are equivalent.

6°. Let $\alpha, \beta, \gamma \in \mathcal{M}$ and consider the Dirichlet-Cauchy problem for the equation

$$\frac{d}{dt}\gamma(w(t))-\Delta\alpha(w(t))+\beta(w(t))\ni f(t).$$

We have shown it is well-posed in L^1 if all three pairs (α, β), (α, γ) and (β, γ) satisfy the SVC. It has a strong solution in H^{-1} if, in addition, $\mathrm{Rg}(\alpha\circ\gamma^{-1})=\boldsymbol{R}$ and $\beta\circ\gamma^{-1}$ is a continuous linearly-bounded function.

Such problems were resolved in [**12**] in the form

$$\frac{d}{dt}\gamma(\alpha^{-1}(u(t)))-\Delta u+\beta(\alpha^{-1}(u(t)))\ni f(t)$$

with operator coefficients being monotone in the H_0^1-H^{-1} duality. To apply these results we assume the pairs (α, β) and (α, γ) satisfy the SVC and that the compositions $\gamma\circ\alpha^{-1}$ and $\beta\circ\alpha^{-1}$ are linearly-bounded, hence their domains are all of $\boldsymbol{R}$. These hypotheses are not comparable, even with $\gamma=$identity, but they are very similar.

EXAMPLE. We consider the Cauchy-Dirichlet problem (1) with $\beta=0$, $f=0$, $u_0(x)\equiv 1$ for $0<x<1$, and α given by $\alpha(0)=(-\infty, 1]$, $\alpha(x)=\{1\}$ for $x>0$. The existence of a unique integral (or semigroup) solution u in L^1 is immediate from [**6, 7**] as well as Corollary 1. From Proposition 1 we have $0\leqq u(x, t)\leqq 1$ a.e. in Ω, and from Theorem 2 (see Remark 3) it follows that the pair u, v satisfies (1) in $\mathcal{D}^*(\Omega)$. It is easy to verify that the pair given for $0<t<1/8$ by

$$u(x, t)=H(x-\sqrt{2t})-H(x-1+\sqrt{2t})$$

$$v(x, t)=\min\{x/\sqrt{2t},\ 1,\ (1-x)/\sqrt{2t}\}$$

and $u=v=0$ for $t>1/8$ is the strong H^{-1} solution, hence, the L^1 solution.

For this example, we shall compute the solution directly from the semigroup formula $u(t)=\lim_{n\to\infty} u_n(t)$, $u_n(t)=(I+(t/n)A)^{-n}u_0$. Note first that the resolvent identity, $u=(I+\varepsilon A)^{-1}w$ for $\varepsilon>0$, is characterized by

$$u-\varepsilon v_{xx}=w\,, \qquad v\in\alpha(u)\,, \quad v\in W_0^{1,1}\,.$$

The inclusion of $[u, v]$ in α is equivalent to

$$1-v\geqq 0, \quad u\geqq 0 \quad \text{and} \quad u(1-v)=0,$$

so the resolvent formula is an elliptic variational inequality for v:

$$v\in W_0^{1,1}, \quad 1-v\geqq 0, \quad w+\varepsilon v_{xx}\geqq 0 \quad \text{and} \quad (1-v)(w+\varepsilon v_{xx})=0.$$

When w is of the special form $w(x)=H(x-x_{n-1})H(1-x_{n-1}-x)$ for some $0\leqq x_{n-1}\leqq 1/2$ we compute directly the solution v_n as that function symmetric on (0, 1) given by

$$v_n(x)=\begin{cases} 2x/(\sqrt{x_{n-1}^2+2\varepsilon}+x_{n-1}), & 0\leqq x\leqq x_{n-1}, \\ 1-(x-x_n)^2/2\varepsilon, & x_{n-1}\leqq x\leqq x_n, \\ 1 & x_n\leqq x\leqq 1/2, \end{cases}$$

where $x_n=\sqrt{x_{n-1}^2+2\varepsilon}=\sqrt{2n\varepsilon}$. From the variational inequality we obtain $u(x)=H(x-x_n)H(1-x_n-x)$. Thus, the semigroup approximation $u_n(t)$ is *exactly* the solution u, the free boundary of the approximating problem is exactly that of (1), and the corresponding approximation v_n of v is smoother. The corresponding finite-time-difference approximation leads to a sequence of elliptic variational inequalities and provides a very efficient numerical procedure for (1).

The Yosida approximation (4) converges in $C(0, T; L^1(G)\cap H^{-1}(G))$ to the solution of (1). The corresponding solutions are characterized by the integral equation

$$u_\varepsilon(t)=e^{-t/\varepsilon}+(1/\varepsilon)\int_0^t e^{(s-t)/\varepsilon}(I+\varepsilon A)^{-1}(u(s))ds.$$

The second and major term is just $(1-e^{-t/\varepsilon})$ times a weighted average of $\{(I+\varepsilon A)^{-1}(u(s)) : 0\leqq s\leqq t\}$, heavily weighted on $t-\varepsilon<s<t$. We shall discuss elsewhere the characterization of (4) as a free-boundary problem for a "pseudo-parabolic" partial differential equation.

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