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Invariant vector fields on a simple Lie algebra under the adjoint action

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Introduction.

The present study deals with the problem whether, given a function f (or a vector field v) on a closed subset Y of a manifold X, there exists a function F (or a vector field V) on the manifold X such that the restriction of F (or V) to Y coincides with f (or v) itself. In general, the function F (or the vector field V) may not be unique, and hence this problem requires an additional condition in order that F (or V) is unique.

An interesting example concerning this problem is the celebrated Chevalley isomorphism with respect to the complex simple Lie algebras. This turns out more clearly if we employ Kostant's identification: When $X=\mathfrak{g}$, a complex simple Lie algebra and $Y=\mathcal{S}_Z$, a transversal slice at a given regular nilpotent element Z of the G-orbit of Z (G denotes the adjoint group of $\mathfrak{g}$), there exists, for any polynomial f on $\mathcal{S}_Z$, a unique G-invariant polynomial F on $\mathfrak{g}$ such that the restriction of F to $\mathcal{S}_Z$ coincides with f. Here, according to Kostant [7], $\mathcal{S}_Z$ can be identified with the quotient space $\mathfrak{h}/W$ of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ under the action of the Weyl group W. Hence the above statement is readily paraphrased as the Chevalley isomorphism

$$I[\mathfrak{g}]\cong \boldsymbol{C}[\mathfrak{h}/W]\,.$$

Here $I[\mathfrak{g}]$ denotes the algebra of G-invariant polynomials on $\mathfrak{g}$ and $\boldsymbol{C}[\mathfrak{h}/W]$ denotes the coordinate ring of $\mathfrak{h}/W\cong\mathcal{S}_Z$. It should be noted that $\mathfrak{h}/W$ is isomorphic to the affine space $\boldsymbol{C}^l$ ($l=\text{rank}\ \mathfrak{g}$) and therefore $\boldsymbol{C}[\mathfrak{h}/W]$ becomes a polynomial ring.

The above example suggests us to inquire to what extent the isomorphism holds even if the functions are replaced by the vector fields.

In this paper, we shall show that it is possible to define the vector fields on $\mathfrak{g}$ such that 1) they are invariant under the adjoint action and 2) the restrictions of them to $\mathfrak{h}/W\cong\mathcal{S}_Z$ form a system of generators for the Lie algebra of vector fields on $\mathfrak{h}/W$ logarithmic along the discriminant set of $\mathfrak{h}/W$. As for the logarithmic vector fields on $\mathfrak{h}/W$, see for detail [13], where we were mainly concerned with the microlocal structure of a complex power of the discriminant polynomial on $\mathfrak{h}/W$ and examined such vector fields in greater detail. Such vector fields are also considered by several authors from several points of view: for example, Solomon [12], Arnol'd [1], Saito [10], Givental' [4].

The organization of this paper is as follows. In § 1, we give preliminaries on the simple Lie algebras and introduce a transversal slice of a regular nilpotent element along the line of Kostant [7]. In § 2, we discuss on the centralizer of any element of the transversal slice which is the main tool to obtain the main results stated in § 3. In § 4, we end with supplementary results and concluding remarks.

§ 1. A standard transversal slice of a regular nilpotent element.

(1.1) We begin with giving standard notations concerning the simple Lie algebras. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l and let $\mathfrak{g}_0$ be its normal real form. If $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, where $\mathfrak{k}_0$ is a maximal compact subalgebra of $\mathfrak{g}_0$, and $\mathfrak{k}=\mathfrak{k}_0\otimes_{\boldsymbol{R}}\boldsymbol{C}$ and $\mathfrak{p}=\mathfrak{p}_0\otimes_{\boldsymbol{R}}\boldsymbol{C}$, then $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is a direct sum decomposition of $\mathfrak{g}$. The adjoint group of $\mathfrak{g}$ is denoted by G and $B(\cdot, \cdot)$ denotes the Killing form of $\mathfrak{g}$. For any $X\in\mathfrak{g}$, let $Z_{\mathfrak{g}}(X)$ be the centralizer of X in $\mathfrak{g}$.

(1.2) Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ (this is actually possible because $\mathfrak{g}_0$ is a normal real form of $\mathfrak{g}$). Let Σ and W be the root system and the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, respectively. We fix an order on Σ.

(1.3) We denote by $\boldsymbol{C}[\mathfrak{g}]$ (resp. $\boldsymbol{C}[\mathfrak{h}]$) the algebra of polynomials on $\mathfrak{g}$ (resp. $\mathfrak{h}$) and by $\boldsymbol{C}[\mathfrak{g}]^G$ (resp. $\boldsymbol{C}[\mathfrak{h}]^W$) the subalgebra of $\boldsymbol{C}[\mathfrak{g}]$ (resp. $\boldsymbol{C}[\mathfrak{h}]$) consisting of G-invariant ones (resp. W-invariant ones). Then, as is known, $\boldsymbol{C}[\mathfrak{g}]^G$ is generated by l algebraically independent homogeneous polynomials $P_1, \cdots, P_l$, and the restrictions of these polynomials to $\mathfrak{h}$ are also algebraically independent and generate $\boldsymbol{C}[\mathfrak{h}]^W$. If $d_i=\deg P_i$ $(1\leqq i\leqq l)$, it is always possible to assume that $d_1=2<d_2\leqq\cdots\leqq d_{l-1}<d_l$ (cf. [3]). The maximal number d_l is often called the Coxeter number of W.

(1.4) We consider the characteristic polynomial of any element X of $\mathfrak{g}$:

$$P(\lambda, X)=\det(\lambda I_n-\mathrm{ad}_{\mathfrak{g}}(X)) \qquad (n=\dim\mathfrak{g}),$$

which is always divisible by λ^l, and hence $D(X)=[\lambda^{-l}P(\lambda, X)]_{\lambda=0}$ is a polynomial on $\mathfrak{g}$ and has the following properties:

(1.4.1) $$D(X)\in\boldsymbol{C}[\mathfrak{g}]^G,$$

(1.4.2) $$D(H)=\prod_{\alpha\in\Sigma}\alpha(H) \qquad \text{for any } H\in\mathfrak{h}.$$

It follows from (1.4.1) that $D(X)$ is written as

$$D(X)=F(P_1(X), \cdots, P_l(X))$$

with a polynomial $F(P_1, \cdots, P_l)$ of $P_1, \cdots, P_l$.

(1.5) Let X_0 be a regular nilpotent element of $\mathfrak{g}$ contained in $\mathfrak{p}$. Such an X_0 actually exists (cf. [8]). Then it follows from [8, Prop. 4] that there exist $H_0 \in \mathfrak{k}$ and $Y_0 \in \mathfrak{p}$ such that

$$(1.5.1) \qquad [H_0, X_0]=2X_0, \quad [H_0, Y_0]=-2Y_0, \quad [X_0, Y_0]=H_0.$$

The relation (1.5.1) implies that $\mathfrak{a}=\boldsymbol{C}X_0+\boldsymbol{C}H_0+\boldsymbol{C}Y_0$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$. Note that Y_0 is also a regular nilpotent element of $\mathfrak{g}$. The following lemma is also known and useful (cf. [8, Prop. 8]).

LEMMA (1.5.2). (1) $\dim Z_{\mathfrak{g}}(X_0)=\dim Z_{\mathfrak{g}}(Y_0)=l$.

(2) $Z_{\mathfrak{g}}(X_0)\subset\mathfrak{p}$, $Z_{\mathfrak{g}}(Y_0)\subset\mathfrak{p}$.

The adjoint action of H_0 leaves $Z_{\mathfrak{g}}(Y_0)$ invariant and the eigenvalues of $\mathrm{ad}(H_0)|Z_{\mathfrak{g}}(Y_0)$ are given by $-n_1, \cdots, -n_l$, where $n_1+1, \cdots, n_l+1$ are the dimensions of the irreducible components of $\mathfrak{g}$ as an $\mathfrak{a}$-module. If we assume $n_1 \leqq n_2 \leqq \cdots \leqq n_l$, then $d_i=\frac{1}{2}n_i+1$ $(1\leqq i\leqq l)$, where d_i is the degree of P_i (cf. (1.3)). In particular, each n_i is even. We choose a basis $u_1, \cdots, u_l$ of $Z_{\mathfrak{g}}(Y_0)$ in such a way that each u_i is an eigenvector of $\mathrm{ad}(H_0)$ with the eigenvalue $-n_i$. Put $v_i'=\mathrm{ad}(X_0)^{n_i}(u_i)$ $(1\leqq i\leqq l)$. Then $v_1', \cdots, v_l'$ span $Z_{\mathfrak{g}}(X_0)$. We can choose $u_1, \cdots, u_l$ so that $B(u_i, v_j')=0$ $(i\neq j)$ and $B(u_i, v_i')\neq 0$. Then $v_i=\frac{1}{B(u_i, v_i')}v_i'$ $(1\leqq i\leqq l)$ also form a basis of $Z_{\mathfrak{g}}(X_0)$ and $B(u_i, v_j)=\delta_{ij}$ (Kronecker's delta). We may take $u_1=Y_0$ and $v_1=\frac{1}{B(X_0, Y_0)}X_0$.

For the results of this paragraph, see [7].

(1.6) Put $\mathcal{S}_{X_0}=X_0+Z_{\mathfrak{g}}(Y_0)$. Then the following result is fundamental.

THEOREM ([7, Th. 0.10]). *Let $\mathfrak{g}_r$ be the totality of the regular elements of $\mathfrak{g}$. Then $\mathcal{S}_{X_0}$ is contained in $\mathfrak{g}_r$. Moreover, for any element X of $\mathfrak{g}_r$, the G-orbit of X intersects with $\mathcal{S}_{X_0}$ at one and only one point.*

We note that for any $X\in\mathcal{S}_{X_0}$, $Z_{\mathfrak{g}}(X)$ is abelian (cf. [6, Prop. 11]), and, in addition, if X is regular semisimple, $Z_{\mathfrak{g}}(X)$ is a Cartan subalgebra of $\mathfrak{g}$.

(1.7) Consider the invariant morphism $\chi: \mathfrak{g}\to\mathfrak{h}/W$ defined through $\chi(X)=(P_1(X), \cdots, P_l(X))$ (cf. (1.3)). It follows then from (1.6) that the restriction of χ to $\mathcal{S}_{X_0}$ is a bijection between $\mathcal{S}_{X_0}$ and $\mathfrak{h}/W$. This tells us that we may take $P_1, \cdots, P_l$ such that

$$(1.7.1) \qquad P_i\Big(X_0+\sum_{j=1}^{l}\lambda_j u_j\Big)=\lambda_i \qquad (1\leqq i\leqq l)$$

for any $(\lambda_1, \cdots, \lambda_l)\in \boldsymbol{C}^l$. Accordingly, $\lambda=(\lambda_1, \cdots, \lambda_l)$ is regarded as a coordinate system of both $\mathcal{S}_{X_0}$ and $\mathfrak{h}/W$. Hereafter we use the notation:

$$X(\lambda)=X(\lambda_1, \cdots, \lambda_l)=X_0+\sum_{i=1}^{l}\lambda_i u_i .$$

Then, for instance, the discriminant polynomial $D(X)$ on $\mathfrak{g}$ is written on $\mathcal{S}_{X_0}$ as $D(X(\lambda))=F(\lambda)$ (cf. (1.4)).

§2. The centralizer of an element of $\mathcal{S}_{X_0}$.

In this section, by regarding $\lambda_1, \cdots, \lambda_l$ as indeterminates commuting with any element of $\mathfrak{g}$, we look for the elements of $\mathfrak{g}\otimes \boldsymbol{C}[\lambda]$ which commute with $X(\lambda)$. This kind of centralizers will play a basic role in the next section.

(2.1) Let us define an affine ring $\mathfrak{R}$ by $\mathfrak{R}=\boldsymbol{C}[\lambda_1, \cdots, \lambda_l]$ and a vector field E by $E=\sum_{i=1}^{l} d_i\lambda_i\dfrac{\partial}{\partial\lambda_i}$. For any integer d, let $\mathfrak{R}_d=\{f\in\mathfrak{R}\,; Ef=df\}$. Then $\mathfrak{R}=\sum_{d\geq 0}\mathfrak{R}_d$.

By assuming that each λ_i commutes with $\mathfrak{g}$, we extend $\mathfrak{g}$ to a Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes_{\boldsymbol{C}}\mathfrak{R}$ over the ring $\mathfrak{R}$. Since $\mathrm{ad}(H_0)$ is naturally extended to an endomorphism of $\tilde{\mathfrak{g}}$, we decompose $\tilde{\mathfrak{g}}$ into eigenspaces of $\mathrm{ad}(H_0)$ such that $\tilde{\mathfrak{g}}=\bigoplus_k\tilde{\mathfrak{g}}(k)$ with $\tilde{\mathfrak{g}}(k)=\{Z(\lambda)\in\tilde{\mathfrak{g}} : \mathrm{ad}(H_0)Z(\lambda)=kZ(\lambda)\}$.

We now turn to a determination of $Z_{\tilde{\mathfrak{g}}}(X(\lambda))=\{Z(\lambda)\in\tilde{\mathfrak{g}} : [X(\lambda), Z(\lambda)]=0\}$.

PROPOSITION (2.2). (1) *For any* i $(1\leqq i\leqq l)$, *there exists a unique element* $X_i(\lambda)$ *of* $Z_{\tilde{\mathfrak{g}}}(X(\lambda))$ *satisfying the condition* $B(X_i(\lambda), u_j)=\delta_{ij}$.

(2) *The element* $X_i(\lambda)$ *has the following properties.*

$$\left(\frac{1}{2}\mathrm{ad}(H_0)+E\right)X_i(\lambda)=(d_i-1)X_i(\lambda)\,, \tag{2.2.1}$$

$$B(X_1(\lambda), X_i(\lambda))=\frac{1}{B(X_0, Y_0)}d_i\lambda_i\,, \tag{2.2.2}$$

$$X_i(\lambda)\in\mathfrak{p}\otimes_{\boldsymbol{C}}\mathfrak{R}\,. \tag{2.2.3}$$

(3) *The Lie algebra* $Z_{\tilde{\mathfrak{g}}}(X(\lambda))$ *is abelian and is spanned by* $X_1(\lambda), \cdots, X_l(\lambda)$ *over* $\mathfrak{R}$.

PROOF. (1) Let $T_j(\lambda)\in\tilde{\mathfrak{g}}(j)$ for any j, and consider an element $X_i(\lambda)$ of the form

$$X_i(\lambda)=v_i+\sum_{j<n_i}T_j(\lambda)$$

with the proviso that

$$[X(\lambda), X_i(\lambda)]=0\,, \tag{2.2.4}$$

$$T_j(\lambda)\in[Y_0, \tilde{\mathfrak{g}}] \quad \text{for any} \quad j. \tag{2.2.5}$$

We first show that such an element $X_i(\lambda)$ actually exists. Since $[v_1, v_i]=0$, the equation (2.2.4) is written as

$$\sum_{j<n_i}[X_0, T_j(\lambda)]+\sum_{k=1}^{l}\lambda_k[u_k, v_i]+\sum_{k=1}^{l}\sum_{j<n_i}\lambda_k[u_k, T_j(\lambda)]=0.$$

Inspecting the $\tilde{\mathfrak{g}}(j+2)$-components of the equation, we readily have

$$[X_0, T_j(\lambda)]=\sum_{n_i-n_k=j+2}\lambda_k[v_j, u_i]-\sum_{m-n_k=j+2}\lambda_k[u_k, T_m(\lambda)]. \tag{2.2.6}$$

Suppose that for a fixed j, $T_k(\lambda)$ $(k>j)$ have already been given. Then since $\mathrm{ad}(H_0):[X_0, \tilde{\mathfrak{g}}]\to[Y_0, \tilde{\mathfrak{g}}]$ is bijective, it follows from (2.2.5) and (2.2.6) that $T_j(\lambda)$ for the fixed j is determined uniquely. Thus all of $T_j(\lambda)$ are determined uniquely by induction. Accordingly, X_i commutes with $X(\lambda)$. By definition, we have $T_j(\lambda)=0$ for any j so that $X_i(0)=v_i$. Then, because of the choice of $u_1, \cdots, u_l$ and $v_1, \cdots, v_l$ in (1.5), we have $B(X_i(\lambda), u_j)=B(v_i, u_j)=\delta_{ij}$. The uniqueness of $X_i(\lambda)$ is nearly obvious. We note that $X_1(\lambda)=\dfrac{1}{B(X_0, Y_0)}X(\lambda)$.

(2) Let $\tilde{E}=\dfrac{1}{2}\mathrm{ad}(H_0)+E-1$. If we decompose $X_i(\lambda)$ into the sum of the eigenvectors of $\tilde{E}$:

$$X_i(\lambda)=\sum_k X_i^k(\lambda) \qquad \text{such that} \quad \tilde{E}X_i^k(\lambda)=kX_i^k(\lambda),$$

then it follows from $\tilde{E}X(\lambda)=0$ (cf. (1.7.1)) that $\tilde{E}([X(\lambda), X_i^k(\lambda)])=k[X(\lambda), X_i^k(\lambda)]$ and therefore $[X(\lambda), X_i^k(\lambda)]=0$ for any k. Since $\tilde{E}v_i=(d_i-2)v_i$ (cf. (1.5)), we have $X_i^{d_i-2}(\lambda)\neq 0$. Then the uniqueness of $X_i(\lambda)$ implies that $X_i(\lambda)=X_i^{d_i-2}(\lambda)$. By the same token, we can justify (2.2.3) by noting that $\tilde{\mathfrak{g}}=\mathfrak{k}\otimes_C\mathfrak{R}+\mathfrak{p}\otimes_C\mathfrak{R}$ is a direct sum decomposition.

Next we go to a calculation of $B(X_1(\lambda), X_i(\lambda))$. We first write $X_i(\lambda)$ as

$$X_i(\lambda)=v_i+\sum_{k=1}^{l}f_{ik}(\lambda)u_k+X_i'(\lambda) \tag{2.2.7}$$

with $f_{ik}(\lambda)\in\mathfrak{R}$, $X_i'(\lambda)\in[X_0, \tilde{\mathfrak{g}}]\cap[Y_0, \tilde{\mathfrak{g}}]$. Then the condition $[X_1(\lambda), X_i(\lambda)]=0$ is paraphrased as

$$\sum_{k=1}^{l}f_{ik}[v_1, u_k]+\sum_{j=1}^{l}f_{1j}(\lambda)[u_j, v_i]+[v_1, X_i'(\lambda)] + \sum_{j=1}^{l}f_{1j}(\lambda)[u_j, X_i'(\lambda)]=0. \tag{2.2.8}$$

On the other hand, by direct calculation, it is easily seen that

$$B([v_i, u_j], H_0)=\delta_{ij}n_i\,,$$
(2.2.9)
$$B([v_1, X_i'(\lambda)], H_0)=0\,,$$
$$B([u_j, X_i'(\lambda)], H_0)=0\,,$$

which, together with (2.2.8), shows that

(2.2.10)
$$n_1 f_{i1}(\lambda)-n_i f_{1i}(\lambda)=0\,.$$

Using this and the expression (2.2.7), we can eventually obtain

$$\begin{aligned}B(X_1(\lambda), X_i(\lambda)) &= \sum_{j=1}^{l} f_{ij}(\lambda)B(v_1, u_j)+\sum_{k=1}^{l} f_{1k}(\lambda)B(u_k, v_i)\\ &= f_{i1}(\lambda)+f_{1i}(\lambda)\\ &= \frac{1}{B(X_0, Y_0)}d_i\lambda_i\,.\end{aligned}$$

Here we have made use of the relations $f_{1i}(\lambda)=\dfrac{1}{B(X_0, Y_0)}\lambda_i$, $n_1=2$ and $d_i=\dfrac{1}{2}n_i+1$.

(3) For any given $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, we consider

$$Z'(\lambda)=Z(\lambda)-\sum_{i=1}^{l}B(Z(\lambda), u_i)X_i(\lambda)\,.$$

Then $Z'(\lambda)$ is contained in $[Y_0, \tilde{\mathfrak{g}}]$ which is the orthogonal complement of $Z_{\mathfrak{g}}(X_0)$ with respect to the Killing form B. On the other hand, it is possible to write

$$Z'(\lambda)=\sum_j Z_j'(\lambda)$$

with $Z_j'(\lambda)\in\tilde{\mathfrak{g}}(j)$. It follows from the above definition that each $Z_j'(\lambda)$ is contained in $[Y_0, \tilde{\mathfrak{g}}]$. Suppose now that k is the number such that $Z_j'(\lambda)=0$ for $j>k$ and $Z_k'(\lambda)\neq 0$. Then from the condition $Z'(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, we must have $[Z_k(\lambda), X_0]=0$. This contradicts the assumption. Therefore $Z_{\mathfrak{g}}(X(\lambda))$ is spanned by $X_1(\lambda), \cdots, X_l(\lambda)$ over $\mathfrak{R}$. Lastly, $Z_{\mathfrak{g}}(X(\lambda))$ is obviously abelian (cf. (1.6)). Q.E.D.

PROPOSITION (2.3). *For any $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, we can assert that*

(1) *There is a unique element $U(\lambda)$ of $\mathfrak{k}\otimes_C\mathfrak{R}$ such that*

(2.3.1)
$$Z(\lambda)=\sum_{i=1}^{l}B(Z(\lambda), X_i(\lambda))u_i+[X(\lambda), U(\lambda)]\,.$$

(2) *If $B(Z(\lambda), X_i(\lambda))=0$ for $i=1, \cdots, l$, then $Z(\lambda)=0$.*

PROOF. (1) Due to Proposition (2.2), $X_1(\lambda), \cdots, X_l(\lambda)$ span $Z_{\mathfrak{g}}(X(\lambda))$ over $\mathfrak{R}$, and hence it suffices to show (2.3.1) for each $X_i(\lambda)$ $(1\leqq i\leqq l)$. Let us take an element $U_i(\lambda)$ of $\mathfrak{k}\otimes_C\mathfrak{R}$ of the form

$$U_i(\lambda)=\frac{1}{n_i}[Y_0, v_i]+\sum_{j<n_i-2}Q_j(\lambda)$$

with $Q_j(\lambda)\in\tilde{\mathfrak{g}}(j)\cap\mathfrak{k}\underset{C}{\otimes}\mathfrak{R}$, and consider the equation

$$X_i(\lambda)=\sum_{j=1}^{l}h_{ij}(\lambda)u_j+[X(\lambda),\ U_i(\lambda)]$$

with $h_{ij}(\lambda)\in\mathfrak{R}$. By an argument similar to the one used in the proof of Proposition (2.2) (1), we can determine $U_i(\lambda)$ and $h_{ij}(\lambda)$ uniquely. We shall next show that $h_{ij}(\lambda)=B(X_i(\lambda),\ X_j(\lambda))$. Since $B(u_k,\ X_j(\lambda))=\delta_{kj}$ (cf. Prop. (2.2)) and since

$$B([X(\lambda),\ U_i(\lambda)],\ X_j(\lambda))=-B(U_i(\lambda),\ [X(\lambda),\ X_j(\lambda)])=0\,,$$

we have

$$\begin{aligned}B(X_i(\lambda),\ X_j(\lambda))&=\sum_{k=1}^{l}h_{ik}(\lambda)B(u_k,\ X_j(\lambda))+B([X(\lambda),\ U_i(\lambda)],\ X_j(\lambda))\\&=h_{ij}(\lambda)\,.\end{aligned}$$

(2) It is sufficient to show that if $U(\lambda)$ is an element of $\mathfrak{k}\underset{C}{\otimes}\mathfrak{R}$ such that $[X(\lambda),\ U(\lambda)]\in Z_{\mathfrak{g}}(X(\lambda))$, then $U(\lambda)=0$. We write $U(\lambda)$ as $U(\lambda)=\sum_j U_j(\lambda)$ in terms of $U_j(\lambda)\in\tilde{\mathfrak{g}}(j)\cap\mathfrak{k}\underset{C}{\otimes}\mathfrak{R}$ for all j. We now assume that for some k, $U_j(\lambda)=0$ $(j>k)$, and $U_k(\lambda)\neq 0$. Then the equation

$$[X(\lambda),\ [X(\lambda),\ U(\lambda)]]=0$$

implies that

$$[X_0,\ [X_0,\ U_k(\lambda)]]=0\,.$$

Then $U_k(\lambda)$ is contained in $Z_{\mathfrak{g}}(X_0)$. Since $\mathrm{ad}(X_0):[\tilde{\mathfrak{g}},\ Y_0]\rightarrow[\tilde{\mathfrak{g}},\ X_0]$ is bijective and $U_k(\lambda)\in[\tilde{\mathfrak{g}},\ Y_0]$, we have

$$U_k(\lambda)=\mathrm{ad}(Y_0)\Big(\sum_{n_j-2=k}f_j(\lambda)X_j(\lambda)\Big)$$

for some $f_j(\lambda)\in\mathfrak{R}$. Then

$$U'(\lambda)=U(\lambda)-\mathrm{ad}(Y_0)\Big(\sum_{n_j-2=k}f_j(\lambda)X_j(\lambda)\Big)$$

is contained in $\mathfrak{k}\underset{C}{\otimes}\mathfrak{R}$, and if we write $U'(\lambda)=\sum_j U'_j(\lambda)$ with $U_j(\lambda)\in\tilde{\mathfrak{g}}\cap\mathfrak{k}\underset{C}{\otimes}\mathfrak{R}$, the assumption implies that $U'_j(\lambda)=0$ $(j\geqq k)$. Continuing this process, we finally obtain that $U(\lambda)=\mathrm{ad}(Y_0)Z(\lambda)$ for some $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, and hence

$$\begin{aligned}[X(\lambda),\ U(\lambda)]&=[[X(\lambda),\ Y_0],\ Z(\lambda)]\\&=[H_0,\ Z(\lambda)]\,.\end{aligned}$$

Since for each i $(1\leqq i\leqq l)$, $[H_0,\ X_i(\lambda)]-n_iX_i(\lambda)$ is not zero and is contained in $[Y_0,\ \tilde{\mathfrak{g}}]$, Proposition (2.2) (3) implies that $[H_0,\ Z(\lambda)]$ is not contained in $Z_{\mathfrak{g}}(X(\lambda))$. This contradicts the assumption. Hence the claim. Q.E.D.

§ 3. Main results.

In this section, we shall show that there is a natural bijection between the centralizer $Z_{\mathfrak{g}}(X(\lambda))$ of $X(\lambda)\in\mathcal{S}_{X_0}$ and the Lie algebra of the vector fields on $\mathfrak{h}/W$ logarithmic along the discriminant set of $\mathfrak{h}/W$.

(3.1) As in § 1, we regard $\lambda=(\lambda_1, \cdots, \lambda_l)$ as a coordinate system on the quotient space $S=\mathfrak{h}/W$ of $\mathfrak{h}$ by the Weyl group W. If $f(X)$ is a G-invariant polynomial on $\mathfrak{g}$, we put $\tilde{f}(\lambda)=f(X(\lambda))$ for any $X(\lambda)\in\mathcal{S}_{X_0}$. Identifying $\mathcal{S}_{X_0}$ with S (cf. (1.7)), we regard $\tilde{f}(\lambda)$ as a polynomial on S. In particular $\tilde{D}(\lambda)=D(X(\lambda))=F(\lambda)$ (cf. (1.7)).

(3.2) We denote by $\mathcal{G}_F$ the totality of the vector fields L on S with polynomial coefficients such that $LF(\lambda)\in\mathfrak{R}F(\lambda)$. Then the following facts are available (cf. [12, 11]):

(3.2.1) For each i $(1\leqq i\leqq l)$, there exists a vector field $L'_i\in\mathcal{G}_F$ of the form

$$L'_i=d_i\lambda_i\frac{\partial}{\partial\lambda_i}+\sum_{j=2}^{l}m_{ij}(\lambda)\frac{\partial}{\partial\lambda_j},$$

where $m_{ij}(\lambda)\in\mathfrak{R}$.

(3.2.2) The Lie algebra $\mathcal{G}_F$ is a free $\mathfrak{R}$-module of rank l, spanned by $L'_1, \cdots, L'_l$ over $\mathfrak{R}$.

(3.3) Let $Z(\lambda)$ be an element of $Z_{\mathfrak{g}}(X(\lambda))$. Then we associate with $Z(\lambda)$ a vector field

$$L_Z=\sum_{i=1}^{l}B(Z(\lambda),\ X_i(\lambda))\frac{\partial}{\partial\lambda_i}$$

on S. For the sake of simplicity, we put $L_i=L_{X_i}$ $(1\leqq i\leqq l)$. Note that $B(X_0, Y_0)L_1=\sum_{i=1}^{l}d_i\lambda_i\frac{\partial}{\partial\lambda_i}=E$ (cf. (2.1), (2.2)). One of our objectives is then to show that $L_1, \cdots, L_l$ span $\mathcal{G}_F$ over $\mathfrak{R}$.

THEOREM (3.4). *Let $f(X)$ be a G-invariant function on $\mathfrak{g}$. Then for any $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, we have*

$$\frac{d}{dt}f(X(\lambda)+tZ(\lambda))\Big|_{t=0}=(L_Zf)(\lambda). \tag{3.4.1}$$

PROOF. It is easy to show that if $f(X)$ is contained in $\boldsymbol{C}[\mathfrak{g}]^G$, we have

$$\frac{d}{dt}f(X+t[Y,\ X])\Big|_{t=0}=0 \tag{3.4.2}$$

for any $X, Y\in\mathfrak{g}$. Then this together with Proposition (2.3) justifies the following calculation:

$$(L_Zf)(\lambda)=\frac{d}{dt}f(X(\lambda)+t\sum_{i=1}^{l}B(Z(\lambda),\ X_i(\lambda))u_i)\Big|_{t=0}$$

$$=\frac{d}{dt}f(X(\lambda)+t(Z(\lambda)-[X(\lambda),\ U(\lambda)]))\Big|_{t=0}$$

$$=\frac{d}{dt}f(X(\lambda)+tZ(\lambda))\Big|_{t=0}. \qquad \text{Q.E.D.}$$

LEMMA (3.5). *For any $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, there exists a polynomial $C_Z(\lambda)\in\mathcal{R}$ such that*

$$D(Z(\lambda))=C_Z(\lambda)F(\lambda). \tag{3.5.1}$$

PROOF. We recall that $X(\lambda)\in\mathcal{S}_{X_0}$ is regular semisimple if and only if $F(\lambda)\neq 0$. Therefore, if $F(\lambda)=0$, then $X(\lambda)$ is not semisimple and $Z_{\mathfrak{g}}(X(\lambda))$ is not a Cartan subalgebra of $\mathfrak{g}$. Since $\dim Z_{\mathfrak{g}}(X(\lambda))=l$, this means that not every element of $Z_{\mathfrak{g}}(X(\lambda))$ is semisimple. Let $Z(\lambda)=Z(\lambda)_s+Z(\lambda)_n$ be the Jordan decomposition of $Z(\lambda)$ $(\in Z_{\mathfrak{g}}(X(\lambda)))$, that is, $Z(\lambda)_s$ (resp. $Z(\lambda)_n$) is the semisimple (resp. nilpotent) part of $Z(\lambda)$. Then in this case, $Z(\lambda)_s$ is not regular. Since $D(X)$ is G-invariant, we have $D(Z(\lambda))=D(Z(\lambda)_s)=0$. Here we used the following fact: for any $X\in\mathfrak{g}$ and $f\in \boldsymbol{C}[\mathfrak{g}]^G$, $f(X)=f(X_s)$ (X_s is the semisimple part of X). We have thus proved that if $F(\lambda)=0$, then $D(Z(\lambda))=0$. Since $F(\lambda)$ is a reduced polynomial and $D(Z(\lambda))$ is also a polynomial, $D(Z(\lambda))$ is divisible by $F(\lambda)$ and the lemma is established.

(3.6) We are now in a position to state and prove the main theorem.

THEOREM (3.6.1). *For any $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, the corresponding vector field L_Z is contained in $\mathcal{G}_F$ and the map $Z(\lambda)\to L_Z$ is a bijection between $Z_{\mathfrak{g}}(X(\lambda))$ and $\mathcal{G}_F$. In particular, $L_1, \cdots, L_l$ span $\mathcal{G}_F$ over $\mathcal{R}$.*

PROOF. For any $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, $X(\lambda)+tZ(\lambda)$ also belongs to $Z_{\mathfrak{g}}(X(\lambda))$. Then by use of the notation in Lemma 3.5, $C_{X+tZ}(\lambda)$ is contained in $\mathcal{R}$ and is also a polynomial of t. Define

$$c_Z(\lambda)=\frac{d}{dt}C_{X+tZ}(\lambda)\Big|_{t=0}.$$

Then it follows from Lemma (3.5) and Theorem (3.4) that

$$\begin{aligned}(L_ZF)(\lambda)&=\frac{d}{dt}D(X(\lambda)+tZ(\lambda))\Big|_{t=0}\\&=\frac{d}{dt}(C_{X+tZ}(\lambda)F(\lambda))\Big|_{t=0}\\&=c_Z(\lambda)F(\lambda).\end{aligned}$$

This readily implies that for any $Z(\lambda)\in Z_{\mathfrak{g}}(X(\lambda))$, L_Z is contained in $\mathcal{G}_F$.

Proposition (2.3) proves that the map $Z(\lambda)\to L_Z$ is bijective. On the other hand, L_i defined in (3.3) can be written in the form

$$L_i = \frac{1}{B(X_0, Y_0)} d_i \lambda_i \frac{\partial}{\partial \lambda_i} + \sum_{j=2}^{l} B(X_i(\lambda), X_j(\lambda)) \frac{\partial}{\partial \lambda_i} \qquad (1 \leqq i \leqq l)$$

in view of Proposition (2.2), so that $L_1, \cdots, L_l$ span $\mathcal{G}_F$ over $\mathcal{R}$ as justified from (3.2). The theorem is thus proved.

REMARK (3.7). The Lie algebra $\mathcal{G}_F$ of vector fields does not depend on the type of the corresponding simple Lie algebra but only on the associated Weyl group. Since the Weyl group of type C_l is isomorphic to that of type B_l, this remark implies that the Lie algebra of the logarithmic vector fields for the simple Lie algebra of type C_l is isomorphic to one for the simple Lie algebra of type B_l.

(3.8) We finally show a connection between the invariant vector fields on $\mathfrak{g}$ under the adjoint action and the logarithmic vector fields $L_1, \cdots, L_l$ defined above.

Since the map $\phi: G \times \mathcal{S}_{X_0} \to \mathfrak{g}$ defined by $\phi(g, X(\lambda)) = \mathrm{Ad}(g)(X(\lambda))$ is regular and its image coincides with $\mathfrak{g}_r$ (cf. (1.6)), we can define the basis $\{M_1(X), \cdots, M_l(X)\}$ of $Z_{\mathfrak{g}}(X)$ for any $X \in \mathfrak{g}_r$ in such a way that $M_i(\mathrm{Ad}(g)(X(\lambda))) = \mathrm{Ad}(g)(X_i(\lambda))$ $(1 \leqq i \leqq l)$. Noting that $\mathrm{codim}_{\mathfrak{g}}(\mathfrak{g} - \mathfrak{g}_r) \geqq 2$, we conclude that $M_1(X), \cdots, M_l(X)$ are uniquely extended to the whole space $\mathfrak{g}$ as holomorphic functions of X. In terms of $M_i(X)$, we introduce the vector field M_i on $\mathfrak{g}$ by taking the tangent vector of M_i at X as $M_i(X)$. Then the following theorem is a direct consequence of the definition of M_i as well as Theorem (3.6.1).

THEOREM (3.8.1). (1) *The vector fields $M_1, \cdots, M_l$ are invariant under the adjoint action and are linearly independent over $\boldsymbol{C}[\mathfrak{g}]$. Here $\boldsymbol{C}[\mathfrak{g}]$ denotes the algebra of polynomials on $\mathfrak{g}$.*

(2) *Defining $c_i(X) = c_{X_i}(P_1(X), \cdots, P_l(X))$* (cf. the proof of Theorem (3.6.1)), *we have*

$$M_i D(X) = c_i(X) D(X) \qquad (1 \leqq i \leqq l).$$

§4. Supplementary results and concluding remarks.

We here collect miscellaneous remarks, some of which are related with the results in the main text and some with further outlook.

(4.1) First we note how $M_1, \cdots, M_l$ given by (3.8) are realized in the case of $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{C})$. Let us denote by $M_1(X), \cdots, M_l(X)$ the centralizers of $X \in \mathfrak{g}$ ($l = n-1$ in this case). We also define $N_i(X) = X^i - \frac{1}{n}(\mathrm{tr}\, X^i) I_n$ $(1 \leqq i \leqq l)$. If X is regular, then the set $\{N_1(X), \cdots, N_l(X)\}$ is a basis of $Z_{\mathfrak{g}}(X)$. It easily follows from this and (3.8) that there are G-invariant polynomials $p_{ij}(X)$ and $q_{ij}(X)$ such that

$$M_i(X) = \sum_{j=1}^{n-1} p_{ij}(X) N_j(X),$$

$$N_i(X)=\sum_{j=1}^{n-1} q_{ij}(X)M_j(X).$$

It seems to be interesting to determine the polynomials $p_{ij}(X)$ and $q_{ij}(X)$, but this remains unsolved.

(4.2) We here give a remark concerning the system of generators for $\boldsymbol{C}[\mathfrak{h}]^W$. The subject of [**11**] was to define uniquely a system of generators for $\boldsymbol{C}[\mathfrak{h}]^W$ satisfying an appropriate condition. The condition adopted there is connected with the theory of the primitive integrals developed by K. Saito (cf. [**10**]). On the other hand, if $\lambda_1, \cdots, \lambda_l$ are as in (1.7), then under the identification $\mathcal{S}_{X_0}\cong\mathfrak{h}/W$, they also form a system of generators for $\boldsymbol{C}[\mathfrak{h}]^W$. We must note here that the system of generators for $\boldsymbol{C}[\mathfrak{h}]^W$ in [**10**] and the system $\{\lambda_1, \cdots, \lambda_l\}$ in this paper are essentially different. But it is interesting to ask whether there is a significant relation between these two systems of generators. In addition, what kind of the structure on $\mathfrak{g}$ corresponds to the linear structure on the quotient space $\mathfrak{h}/W$? We should mention that to look for a breakthrough of these problems has been one of motivations of this study.

(4.3) We give here a comment on the discriminant polynomial $F(\lambda)$ defined at (1.4). If $F(\lambda)$ is regarded as a polynomial of λ, we have

$$F(\lambda)=a_0\lambda_l^l+a_1(\lambda')\lambda_l^{l-1}+\cdots+a_l(\lambda'), \tag{4.3.1}$$

where $\lambda'=(\lambda_1, \cdots, \lambda_{l-1})$ and a_0 is a constant. It was shown in [**10**] and [**11**] (also in [**2**]) that $a_0\neq 0$. This fact is inevitable in introducing the linear structure on $\mathfrak{h}/W$ mentioned in (4.2). The proof in [**10**] and [**11**] employed an important property of the eigenvalues of a Coxeter transformation of the Weyl group. We remark however that the fact $a_0\neq 0$ can alternatively be proved in the case of the Weyl group by use of a result of Kostant [**7**]:

PROPOSITION (4.3.2). *If $\lambda_l\neq 0$, then $F(0, \cdots, 0, \lambda_l)\neq 0$.*

PROOF. It is shown in [**7**], by introducing the notion of a cyclic element of $\mathfrak{g}$, that an element X of $\mathfrak{g}$ is cyclic if and only if $P_i(X)=0$ $(1\leqq i\leqq l-1)$ and $P_l(X)\neq 0$. Here $P_1, \cdots, P_l$ are the fundamental invariant polynomials on $\mathfrak{g}$ (cf. (1.3)). Then, by definition, $X(0, \cdots, 0, \lambda_l)$ is cyclic. On the other hand, if X is cyclic, then X is regular semisimple and in particular $D(X)\neq 0$ (cf. [**7**]). Therefore $F(0, \cdots, 0, \lambda_l)=D(X(0, \cdots, 0, \lambda_l))\neq 0$. Q.E.D.

(4.4) Finally a possible interpretation of Theorems (3.6.1) and (3.8.1) is added. In his famous study on invariant eigendistributions, Harish-Chandra [**5**] established a method of separation of variables on differential operators on an open subset of $\mathfrak{g}$. To explain briefly the idea in [**5**], let X be a general nilpotent element of $\mathfrak{g}$ and let $\mathcal{S}_X$ be a transversal slice at X of the G-orbit of X. Then for any differential operator $P(X, \partial_X)$ on an open subset Ω of $\mathfrak{g}$ containing X, we obtain a certain differential operator on $\mathcal{S}_X\cap\Omega$, which is called the radial component of

$P(X, \partial_X)$. In particular, Harish-Chandra obtained an explicit expression of the radial component of the Euler operator on $\mathfrak{g}$. One thing we should like to stress here is that since $X_1(\lambda)=\frac{1}{B(X_0, Y_0)}X(\lambda)$, $B(X_0, Y_0)M_1$ is nothing but the Euler operator. Then using the notion of the radial component, we are led to restate Theorem (3.6.1) (and (3.8.1)) as follows.

(4.4.1) Let X_0 be regular nilpotent with $\mathcal{S}_{X_0}$ the transversal slice defined in (1.6). Then the logarithmic vector fields $L_1, \cdots, L_l$ in (3.3) are the radial components on $\mathcal{S}_{X_0}$ of the G-invariant vector fields $M_1, \cdots, M_l$ in (3.8).

It should be noted that the radial components $L_1, \cdots, L_l$ here are just the one we termed the restriction of the vector fields at the beginning of the Introduction.

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