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Extension of modifications of ample divisors on fourfolds

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Introduction.

In this article we consider the following problem: Let A be an ample divisor on a connected four dimensional projective manifold X. Assume that the Kodaira dimension of X is non-negative. Suppose that A is the blow up of a projective manifold A' with center R_g where R_g is a smooth curve of genus $\geqq 1$ which is contained in A'.

Does there exist a four dimensional manifold X' such that A' lies on X' as a divisor and such that X is the blow up of X' with center R_g?

The answer to this question turned out to be positive. In fact following Sommese's idea, see [**13**], we construct a divisor D on X with the following properties:

1) $D\cap A=Y$, where Y is the exceptional divisor on A over R_g

2) the natural projection $Y\rightarrow R_g$ can be extended to a surjective holomorphic map $\tilde{p}: D\rightarrow R_g$

3) $\tilde{p}$ makes D a $\boldsymbol{P}^2$-bundle over R_g where $\dim A'-\dim R_g=2$. Moreover, each fibre f' of Y over $x\in R_g$ is a hyperplane on $F=\tilde{p}^{-1}(x)\cong\boldsymbol{P}^2$.

4) $[D]_F=\mathcal{O}_{\boldsymbol{P}^2}(-1)$.

The above is enough to ensure the existence of X' such that A' is a divisor on X' and X is the blow up of X' with center R_g, see [8].

The above problem, in a more general setting, was already considered by Sommese in [**14**] and by Fujita in [**3**]. In fact they set up the problem for a projective manifold X of any dimension and without any assumption on the Kodaira dimension of X. Sommese in [**14**] showed that when $\operatorname{codim}_{A'}R>2$ then there is an analytic set of codimension one in X that satisfies the condition for it to be blown down if the map $\tilde{p}: X\rightarrow X'$ existed. Fujita in [**3**] showed that the problem could be solved in the case $\operatorname{codim}_{A'}R>2$ where R is a submanifold of A' along which we blow up.

We need the non-negativity of the Kodaira dimension for the theorem to be true. In fact given any projective threefold A there is a $\boldsymbol{P}^1$-bundle X over A

with a threefold B as a hyperplane section of X but yet the main theorem is false for (X, B).

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§0. Background material.

The notation will be as in [**13**]. Those that are used more frequently will be given below.

(0.1) Given a sheaf $\mathfrak{S}$ of abelian groups on a topological space X we denote the global sections of $\mathfrak{S}$ over X by $\Gamma(\mathfrak{S})$ or by $H^0(\mathfrak{S})$.

(0.2) All spaces and manifolds are complex analytic, all dimensions are over $\boldsymbol{C}$. Given an analytic space X we denote the structure sheaf by $\mathcal{O}_X$.

Given a coherent analytic sheaf $\mathfrak{S}$ on an analytic space X, we let $h^i(\mathfrak{S})$ or $h^i(X, \mathfrak{S})$ denote $\dim H^i(X, \mathfrak{S})$. Assuming that X is smooth we let $h^{p,q}(X)=h^q(\wedge^p T_X^*)$ where T_X^* is the holomorphic cotangent bundle of X.

(0.3) Let X be a connected projective manifold. Let D be an effective Cartier divisor on X. Denote by $[D]$ the holomorphic line bundle associated to D. If L is a holomorphic line bundle, we denote by $|L|$ the linear system of all Cartier divisors associated to L.

If $D\in|L|$ and C is a curve in X, $L\cdot C=D\cdot C=c_1(L)[C]$ where $c_1(L)$ is the first Chern class of L. We denote by K_X the canonical bundle of X if X is a pure dimensional complex manifold. If X is a complex manifold and A is a submanifold of X we denote by $N_{A/X}$ or N_A the normal bundle of A in X, and if $f\subset A$ is a subspace then we denote by $N_{A,f}$ the normal bundle of A in X restricted to f.

(0.4) If $p: X\to Y$ is a morphism and $\mathfrak{S}$ is any locally free sheaf on Y of finite rank we denote by $p^*\mathfrak{S}$ the pullback of $\mathfrak{S}$. If $\mathfrak{S}$ is a locally free sheaf on X of finite rank we denote by $p_{(i)}\mathfrak{S}$ the i-th direct image of $\mathfrak{S}$ and sometimes we denote $p_{(0)}\mathfrak{S}$, the direct image sheaf, by $p_*\mathfrak{S}$.

(0.5) By F_r with $r\geqq 0$ we denote the Hirzebruch surfaces which are the unique $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ with a section E satisfying $E\cdot E=-r$. If $r\geqq 1$ we denote by $\tilde{F}_r$ the normal surface obtained from F_r by blowing down E. In case $r=1$, $\tilde{F}_1=\boldsymbol{P}^2$. If L is a line bundle in F_r then L is given by $[E]^a\otimes[f]^b$ where f is a fibre in F_r and $[E]^a\otimes[f]^b$ is ample if and only if $a>0$ and $b\geqq ar+1$. $[E]^a\otimes[f]^b$ is spanned by global sections if and only if $a\geqq 0$ and $b\geqq ar$. Given a line bundle L on $\tilde{F}_r$, the pullback of L to F_r is of the form $([E]\otimes[f]^r)^a$ for some integer a.

(0.6) THEOREM. *Let X be a reduced compact complex space, all of whose irreducible components have the same dimension. Assume that X is a local complete intersection and that $\pi: \tilde{X} \to X$ is a desingularization of X with $\tilde{X}$ Kähler. Let L be a holomorphic line bundle on X whose pullback to $\tilde{X}$ has a metric with a semipositive curvature form that has at least k positive eigenvalues at at least one point of each component of X. Then:*

$$H^j(X, L^{-1})=0 \qquad \textit{for} \quad j<\min\{k, \dim X-\sigma\}$$

where σ is the dimension of the singular set of X.

The proof is done using the dualizing sheaf, the Grauert-Riemenschneider canonical sheaf and Serre duality, see [**12**].

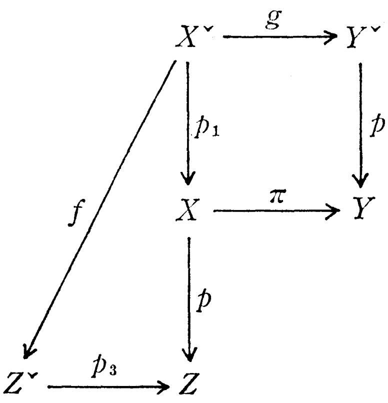
(0.7) LEMMA. *Let X, Y and Z be topological spaces. Let $\pi: X \to Y$ and $p: X \to Z$ be continuous maps with connected fibres. Assume that the induced maps $\pi_*: \pi_1 X \to \pi_1 Y$ and $p_*: \pi_1 X \to \pi_1 Z$ are surjective. Assume that $H_1(X, \boldsymbol{Z})/Torsion \cong H_1(Y, \boldsymbol{Z})/Torsion \cong H_1(Z, \boldsymbol{Z})$. Then there exist covering spaces $(X^\vee, p_1)$, $(Y^\vee, p_2)$, $(Z^\vee, p_3)$ of X, Y and Z respectively such that $(Z^\vee, p_3)$ corresponds to the commutator subgroup of $\pi_1 Z$ and the maps $p \circ p_1$ and $\pi \circ p_1$ lift to $Z^\vee$ and $Y^\vee$ respectively, with connected fibres.*

PROOF. Using the following two diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\phi & \longrightarrow & \pi_1 X & \xrightarrow{\phi} & H_1(X, \boldsymbol{Z})/\text{Torsion} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\ 0 & \longrightarrow & \ker\eta & \longrightarrow & \pi_1 Y & \xrightarrow{\eta} & H_1(Y, \boldsymbol{Z})/\text{Torsion} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\phi & \longrightarrow & \pi_1 X & \xrightarrow{\phi} & H_1(X, \boldsymbol{Z})/\text{Torsion} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\ 0 & \longrightarrow & [\pi_1 Z, \pi_1 Z] & \longrightarrow & \pi_1 Z & \longrightarrow & H_1(Z, \boldsymbol{Z}) & \longrightarrow & 0 \end{array}$$

and the hypothesis we have $\ker\eta = \pi_* \ker\phi$ and $[\pi_1 Z, \pi_1 Z] = p_* \ker\phi$. Let $K = [\pi_1 Z, \pi_1 Z]$, $K' = \ker\phi$, $K'' = \ker\eta$. Let $(X^\vee, p_1)$, $(Y^\vee, p_2)$ and $(Z^\vee, p_3)$ be the covering spaces of X, Y, Z associated to the subgroups K, K', K'', of $\pi_1 Z$, $\pi_1 X$ and $\pi_1 Y$ respectively. Noting that $p \circ p_1$ lifts to a map $X^\vee$ to $Z^\vee$ since $(p \circ p_1)_*(\pi_1 X^\vee) = K = p_{3*} \pi_1 Z^\vee$ we get a continuous map $f: X^\vee \to Z^\vee$. Moreover $\pi \circ p_1$ lifts to a map $X^\vee$ to $Y^\vee$ since $(\pi \circ p_1)_*(\pi_1 X^\vee) = K'' = p_{2*}(\pi_1 Y^\vee)$ hence we get a continuous map $g: X^\vee \to Y^\vee$. Thus we have the following diagram



We note that the maps p and π have connected fibres thus f and g have connected fibres. □

(0.8) LEMMA. *We have the same conditions as in* (0.7) *except that* X, Y *and* Z *are now normal analytic spaces,* π *is a proper modification and* p *is a holomorphic map. Moreover assume that* $\mathrm{Sing}(Y)$ *is a finite set of points and that* $Z^{\vee}$ *is Stein. Then the map* $F^{\vee}=f\circ g^{-1}: Y^{\vee}\to Z^{\vee}$ *is holomorphic. Moreover we get a holomorphic map* $F: Y\to Z$.

PROOF. By (0.7) there exist covering spaces $(X^{\vee}, p_1)$, $(Y^{\vee}, p_2)$ and $(Z^{\vee}, p_3)$ of X, Y and Z respectively, moreover the maps $p\circ p_1$ and $\pi\circ p_1$ lift to $Z^{\vee}$ and $Y^{\vee}$ respectively. Let f and g denote these lifted maps. Note that they are holomorphic. Moreover, g^{-1} exists as a meromorphic map since the map g has connected fibres and $g^{-1}: Y^{\vee}-p_2^{-1}(\mathrm{Sing}\,Y)\to X^{\vee}-(\pi\circ p_1^{-1})(\mathrm{Sing}\,Y)$ is biholomorphic. Consider $f\circ g^{-1}: Y^{\vee}\to Z^{\vee}$. $(f\circ g^{-1})|_{Y^{\vee}-p_2^{-1}(\mathrm{Sing}\,Y)}$ is holomorphic, $p_2^{-1}(\mathrm{Sing}\,Y)$ is an analytic set in $Y^{\vee}$ and $\mathrm{codim}(p_2^{-1}\mathrm{Sing}(Y))\geqq 2$. Hence by Riemann's extension theorem $f\circ g^{-1}$ extends to a holomorphic map from $Y^{\vee}$ to $Z^{\vee}$. Let $F^{\vee}=f\circ g^{-1}$. Note that $X^{\vee}$, $Y^{\vee}$ and $Z^{\vee}$ are regular covering spaces thus:

$$A(X^{\vee}, p_1)\cong\pi_1X/K'\cong H_1(X, \boldsymbol{Z})/\mathrm{Torsion}$$

$$A(Y^{\vee}, p_2)\cong\pi_1Y/K''\cong H_1^*(Y, \boldsymbol{Z})/\mathrm{Torsion}$$

$$A(Z^{\vee}, p_3)\cong\pi_1Z/K\cong H_1(Z, \boldsymbol{Z})$$

where $A(-, p_i)$ denotes the group of the deck transformations. The above three groups are isomorphic to one another since each one of them is isomorphic to $H_1(Z, \boldsymbol{Z})$. Denote such group by G. G acts transitively on $X^{\vee}$, $Y^{\vee}$, $Z^{\vee}$ thus $X^{\vee}/G\cong X$, $Y^{\vee}/G\cong Y$, $Z^{\vee}/G\cong Z$. Denote by F the map obtained from $F^{\vee}: Y^{\vee}\to Z^{\vee}$ after we have considered the action of G on $Y^{\vee}$ and $Z^{\vee}$. Thus $F: Y\to Z$. The map F is a holomorphic map since $F^{\vee}$ and the maps $\pi_1: Y^{\vee}\to Y^{\vee}/G$ and $\pi_2: Z^{\vee}\to Z^{\vee}/G$ are holomorphic. Moreover we note that $F=p\circ\pi^{-1}$. In fact it is straightforward to see that $(f\circ g^{-1})_G=f_G\circ g_G^{-1}$ and $f_G\circ g_G^{-1}=p\circ\pi^{-1}$. Thus $F=$

$p\circ\pi^{-1}$ (by $(f\circ g^{-1})_G: Y\to Z$ we denote the map obtained from $f\circ g^{-1}: Y^{\vee}\to Z^{\vee}$ after we have considered the action of G on $Y^{\vee}$ and $Z^{\vee}$). □

§1. The main theorem.

(1.0) Throughout this section we assume:

a) X is a four dimensional connected projective manifold,

b) L is an ample line bundle with at least one smooth $A\in|L|$,

c) the Kodaira dimension of X is non-negative i.e. $\Gamma(K_X^n)\neq 0$ for some integer $n>0$.

(1.1) LEMMA. *Let X, A and L be as in* (1.0). *Assume that A is the blow-up of a smooth projective threefold A' with center a smooth curve R_g of genus $g\geqq 1$. Let Y be the exceptional divisor of this blow-up and let f' be a fibre of Y. Then the closure D of the union of all deformations of f' in X is a normal, irreducible, reduced divisor on X such that:*

a) *D intersects A transversely in Y, and*

b) *$Y\subset D_{\mathrm{reg}}$.*

PROOF. From $f'\subset Y\subset A$ and the fact that $N_{Y/A,f'}=\mathcal{O}_{f'}(-1)$ we have the exact sequence

$$0\longrightarrow\mathcal{O}_{f'}\longrightarrow N_{f'/A}\longrightarrow\mathcal{O}_{f'}(-1)\longrightarrow 0 \tag{1.1.1}$$

where $\mathcal{O}_{f'}$ is the trivial bundle and $N_{f'/A}$ is the normal bundle of f' in A. By the long exact cohomology sequence associated to (1.1.1) we have

$$h^0(N_{f'/A})=1 \quad\text{and}\quad H^1(f', N_{f'/A})=0\,.$$

From $f'\subset A\subset X$ we have the short exact sequence

$$0\longrightarrow N_{f'/A}\longrightarrow N_{f'}\longrightarrow\mathcal{O}_{f'}(a)\longrightarrow 0 \tag{1.1.2}$$

where $N_{f'}$ is the normal bundle of f' in X and $\mathcal{O}_{f'}(a)$ is the a-th power of the hyperplane section bundle on $f'\cong\boldsymbol{P}^1$, and where $a=L\cdot f'>0$. By the long exact cohomology sequence associated to (1.1.2) it follows that

$$h^0(N_{f'})=a+2\geqq 3 \quad\text{and}\quad H^1(f', N_{f'})=0\,. \tag{1.1.3}$$

From (1.1.3) it follows that there exist deformations of f' in X. Let D be the closure of the union of all the deformations of f' in X.

Claim. D is a divisor in X.

Proof of Claim. Since $\Gamma(N_{f'})$ is naturally identified with $T_{\mathcal{H},\alpha}$, where $\mathcal{H}$ is the irreducible component of the Hilbert scheme of X parametrizing flat deformations of f' with $\alpha\in\mathcal{H}$ corresponding to f' and containing the deformations of f' on Y, we have $\dim T_{\mathcal{H},\alpha}=\dim\Gamma(N_{f'})\geqq 3$ thus $\dim\mathcal{H}\geqq 3$. From (1.1.3) using Kodaira-Spencer theory, it follows that $\dim D \geqq 2$. But $\dim D=2$ does not occur since this would imply that Y was a component of D and that deformations of most fibres of Y remain in Y. This implies that $\dim \mathscr{H}<2$ for a generic f' on Y. Finally $\dim D \neq 4$ ([**13**], (0.7.2)). In fact if the deformations of f' filled out an open set of X, then since nK_X is effective it follows that

(1.1.4) $$K_X \cdot f' \geqq 0 .$$

By (1.1.1) and (1.1.2) we have $\det N_{f'}=\mathcal{O}_{f'}(a-1)$. By the adjunction formula

$$K_{f'}=K_{X|f'} \otimes \det N_{f'}$$

thus $-2=K_X \cdot f'+a-1$, i.e., $K_X \cdot f'=-2-a+1 \leqq -2$ which contradicts (1.1.4).

The above argument also shows that $N_{f'}$ is not spanned since otherwise by Kodaira-Spencer theory the deformations of f' are dense in X. It is straightforward to see that $N_{f'}=\mathcal{O}_{f'}(-1) \oplus \mathcal{L}_{f'}$ where $\mathcal{L}_{f'}$ is a rank two vector bundle on f'. This shows that a union U of small deformations of f' in X gives a complex manifold that meets A transversely in Y, which implies that D meets A transversely in Y, by the same argument as in [**13**] p. 23. Since the intersection is transverse and Y is smooth, the singularities of D are in $D-A$, but A is ample. Therefore Sing(D) is a finite set of closed points. Hence D is normal being a divisor with isolated singularities in a manifold of dimension $\geqq 3$. For a proof, see [**13**] p. 67. □

(1.2) LEMMA. *Let X, A, L, Y and D be as in* (1.1). *Let $p: Y \rightarrow R_g$ be the restriction of the blow-up $p: A \rightarrow A'$. Then p extends to a holomorphic map from D to R_g.*

PROOF. Let $\tilde{D}$ be a desingularization of D.

Claim 1. $\dim \mathrm{Alb}(Y)=\dim \mathrm{Alb}(\tilde{D})$.

Proof of Claim 1. Let $\bar{L}=[Y]$ be the ample line bundle on D determined by Y. Since $\pi_* \mathcal{O}_{\tilde{D}} \cong \mathcal{O}_D$ we have

$$H^0(\tilde{D}, \pi^* \bar{L}^n) \cong H^0(D, \pi_*(\pi^* \bar{L}^n)) \cong H^0(D, \bar{L}^n) \quad \text{for} \quad n \gg 0 .$$

Note that $\pi^* \bar{L}^n$ is spanned by global sections and the map $\Phi_{\pi^* \bar{L}^n}: \tilde{D} \rightarrow \boldsymbol{P}_C$ is given by the following composition

$$\tilde{D} \xrightarrow{\pi} D \xrightarrow{\Phi_{\bar{L}^n}} \boldsymbol{P}_C .$$

Moreover note that $\dim \Phi_{\pi^* \bar{L}^n}(\tilde{D})=3$ since $\pi(\tilde{D})=D$ and $\Phi_{\bar{L}^n}$ is an embedding where by $\Phi_{\bar{L}^n}$ we denote the map associated to the linear system given by $\bar{L}^n$. Since $\pi^* \bar{L}^n$ is spanned by global sections and $\Phi_{\pi^* \bar{L}^n}$ has three dimensional image this implies

$$h^i(K_{\tilde{D}} \otimes \pi^* \bar{L})=0 \quad \text{for} \quad i>\dim \tilde{D}-3$$

therefore

$$h^1(K_{\tilde{D}}\otimes\pi^*\bar{L})=h^2(K_{\tilde{D}}\otimes\pi^*\bar{L})=0$$

and by Serre duality

$$(*)\qquad h^1(\tilde{D},\ (\pi^*\bar{L})^{-1})=h^2(\tilde{D},\ (\pi^*\bar{L})^{-1})=0\,.$$

Using $(*)$, the fact that $(\pi^*\bar{L})^{-1}\approx\mathcal{O}_{\tilde{D}}(-Y)$ and the long exact cohomology sequence associated to

$$0\longrightarrow\mathcal{O}_{\tilde{D}}(-Y)\longrightarrow\mathcal{O}_{\tilde{D}}\longrightarrow\mathcal{O}_Y\longrightarrow 0$$

we have

$$(1.2.1)\qquad H^1(\mathcal{O}_{\tilde{D}})\cong H^1(\mathcal{O}_Y)\,.$$

From this last fact and Hodge theory it follows that $H^0(\tilde{D},\ T^*_{\tilde{D}})\cong H^0(Y,\ T^*_Y)$. Thus

$$\dim\mathrm{Alb}(Y)=\dim\mathrm{Alb}(\tilde{D})\,.$$

We have the following diagram

$$\begin{array}{ccccc} Y & \longrightarrow & R_g & \longrightarrow & \mathrm{Alb}(Y) \\ \downarrow i & & \downarrow j & & \downarrow \mathrm{Alb}(i) \\ \tilde{D} & \xrightarrow{\ \alpha\ } & \alpha(\tilde{D}) & \longrightarrow & \mathrm{Alb}(\tilde{D}) \end{array}$$

where α is the Albanese map. In the above diagram we use the fact that $\mathrm{Alb}(Y)\cong J(R_g)$ where $J(R_g)$ denotes the Jacobian variety of R_g.

Claim 2. $\dim\alpha(\tilde{D})=1$.

Proof of Claim 2. If $\dim\alpha(\tilde{D})\neq 1$ then we would have 2 cases:

1) $\dim\alpha(\tilde{D})=0$ which does not occur since $\alpha(\tilde{D})$ generates $\mathrm{Alb}(\tilde{D})$ and $\dim\mathrm{Alb}(\tilde{D})=\dim\mathrm{Alb}(Y)=g>0$.

2) $\dim\alpha(\tilde{D})\geqq 2$.

If $\dim\alpha(\tilde{D})\geqq 2$, one can conclude that $H^0(\tilde{D},\ \Omega^2_{\tilde{D}})>0$ by Ueno's theory. But this is impossible because $H^2(Y,\ \mathcal{O}_Y)=0$ and $H^2(\tilde{D},\ (\pi^*\bar{L})^{-1})=0$.

Claim 3. $\alpha(\tilde{D})$ is isomorphic to R_g via j.

Proof of Claim 3. Assume $g(\alpha(\tilde{D}))=g'>1$. If j is not an isomorphism then $\deg j\geqq 2$ and by Riemann-Hurwitz' theorem we have

$$(1.2.2)\qquad 2g-2=n(2g'-2)+\rho$$

where $g=g(R_g)$, $n=\deg j$ and ρ is the total ramification. Since $g=g'$ and $n\geqq 2$, from (1.2.2) we get a contradiction. Thus j is an isomorphism for $g>1$. Now assume that $g=1$. In this last case the map $j:\ R_g\rightarrow\alpha(\tilde{D})$ is a covering map by Riemann-Hurwitz' theorem. From the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ i\ } & \tilde{D} \\ {\scriptstyle p}\big\downarrow & \overset{h}{\searrow} & \big\downarrow{\scriptstyle \alpha} \\ R_g & \xrightarrow[\ j\ ]{} & \alpha(\tilde{D}) \end{array}$$

where $h=j\circ p$, if j is not an isomorphism, then the generic fibre of h is disconnected. Let $e'\in\alpha(\tilde{D})$ such that $h^{-1}(e')$ is disconnected. Let $S=\alpha^{-1}(e')$ be a smooth surface in $\tilde{D}$. Note that $\alpha^{-1}(e')$ is connected. For a proof, see [**18**]. Moreover note that $i(Y)\cong Y$ and $C=Y\cap S$ is disconnected. Denote by $\bar{L}_S$ the restriction of the line bundle $\bar{L}$ to S. $\bar{L}_S^m$ gives a birational map of S for $m\gg 0$. Looking at the long exact cohomology sequence associated to

$$0\longrightarrow \bar{L}_S^{-1}\longrightarrow \mathcal{O}_S\longrightarrow \mathcal{O}_C\longrightarrow 0$$

since $h^0(\bar{L}_S^{-1})=h^1(\bar{L}_S^{-1})=0$ we get $H^0(\mathcal{O}_S)\cong H^0(\mathcal{O}_C)$ but $H^0(\mathcal{O}_S)\cong \boldsymbol{C}$ which gives a contradiction since C is not connected. Hence j has to be an isomorphism and therefore we can identify R_g with $\alpha(\tilde{D})$ via j. Thus we get a holomorphic map $\tilde{p}:\tilde{D}\to R_g$ such that $\tilde{p}|_Y=p$.

Claim 4. $H^1(D, \boldsymbol{Z})\cong H^1(\tilde{D}, \boldsymbol{Z})$.

Proof of Claim 4. From (0.6) we get

$$H^i(D, \bar{L}^{-1})=0 \qquad \text{for} \quad i<\min\{k, \dim D-\sigma\}=3$$

since $k=3$ and $\sigma=\dim\mathrm{Sing}(D)=0$, thus $H^1(D, \bar{L}^{-1})=H^2(D, \bar{L}^{-1})=0$. Using this and the long exact cohomology sequence associated to

$$0\longrightarrow \mathcal{J}_Y\longrightarrow \mathcal{O}_D\longrightarrow \mathcal{O}_Y\longrightarrow 0$$

we get that $H^1(\mathcal{O}_D)\cong H^1(\mathcal{O}_Y)$. Since Y is ruled it follows that $H^2(\mathcal{O}_Y)=0$ which together with $H^2(\mathcal{J}_Y)=H^2(\mathcal{O}_D(-Y))=0$ gives $H^2(\mathcal{O}_D)=0$. Now using the Leray spectral sequence, $H^1(D, \mathcal{O}_D)\cong H^1(\tilde{D}, \mathcal{O}_{\tilde{D}})$ and $H^2(D, \mathcal{O}_D)=0$ it follows that $H^0(D, \pi_{(1)}\mathcal{O}_{\tilde{D}})=0$ which implies $\pi_{(1)}\mathcal{O}_{\tilde{D}}=0$ since $\pi_{(1)}\mathcal{O}_{\tilde{D}}$ is supported at a finite number of points. From

$$0\longrightarrow \boldsymbol{Z}\longrightarrow \mathcal{O}_{\tilde{D}}\longrightarrow \mathcal{O}_{\tilde{D}}^*\longrightarrow 0$$

we get

$$0\longrightarrow \pi_*\boldsymbol{Z}\longrightarrow \pi_*\mathcal{O}_{\tilde{D}}\xrightarrow{\ \beta\ }\pi_*\mathcal{O}_{\tilde{D}}^*\longrightarrow \pi_{(1)}\boldsymbol{Z}\longrightarrow \pi_{(1)}\mathcal{O}_{\tilde{D}}\longrightarrow\cdots.$$

Since the map β is onto and $\pi_{(1)}\mathcal{O}_{\tilde{D}}=0$ we get that $\pi_{(1)}\boldsymbol{Z}=0$ and by the Leray spectral sequence we get $H^1(D, \boldsymbol{Z})\cong H^1(\tilde{D}, \boldsymbol{Z})$.

This claim implies $H_1(D, \boldsymbol{Z})/\mathrm{Torsion}\cong H_1(\tilde{D}, \boldsymbol{Z})/\mathrm{Torsion}$. By $H^1(\mathcal{O}_{R_g})\cong H^1(\mathcal{O}_{\tilde{D}})$ and by Hodge theory we get

$$(1.2.3) \qquad H^1(R_g, \boldsymbol{C}) \cong H^1(\tilde{D}, \boldsymbol{C}).$$

The map $\tilde{p}: \tilde{D} \to R_g$ is proper with connected fibres and R_g is a Riemann surface thus the fundamental group of $\tilde{D}$ maps onto the fundamental group of R_g. The above fact together with (1.2.3) implies that

$$(1.2.4) \qquad H_1(\tilde{D}, \boldsymbol{C}) \cong H_1(R_g, \boldsymbol{C}).$$

Moreover since $\pi_1 \tilde{D}$ maps onto $\pi_1 R_g$ it follows that $\tilde{p}_*: H_1(\tilde{D}, \boldsymbol{Z}) \to H_1(R_g, \boldsymbol{Z})$ is onto and using (1.2.4) we have $\ker(\tilde{p}_*) = \mathrm{Torsion}(H_1(\tilde{D}, \boldsymbol{Z}))$. Thus $H_1(R_g, \boldsymbol{Z}) \cong H_1(\tilde{D}, \boldsymbol{Z})/\mathrm{Torsion}$. Note that the fundamental group of $\tilde{D}$ maps onto the fundamental group of D. Since D has isolated singularities, loops in D can be moved away from the singular points, but $D - \mathrm{Sing}(D)$ is isomorphic to $\tilde{D} - \pi^{-1}(\mathrm{Sing}(D))$. Thus loops in D come from loops in $\tilde{D}$. We can apply (0.7) and (0.8) with $X = \tilde{D}$, $Y = D$ and $Z = R_g$. Thus we have a holomorphic map $F^\vee = f \circ g^{-1}: D^\vee \to R_g^\vee$ as in (0.8) and $F: D^\vee/G \to R_g^\vee/G$ is holomorphic. Note that $D^\vee/G \cong D$ and $R_g^\vee/G \cong R_g$ thus $F: D \to R_g$. Moreover F extends $p: Y \to R_g$ since $f_G = \tilde{p}$ where $f_G: \tilde{D} \to R_g$ is obtained from $f: \tilde{D}^\vee \to R_g^\vee$ after we consider the action of G on $\tilde{D}^\vee$ and $R_g^\vee$, and $\tilde{p}$ is as in (1.2). Denote the map F by $\tilde{p}$. □

(1.3) LEMMA. *Let X, A, L, Y and D be as in* (1.2). *Then all the fibres of the map $\tilde{p}: D \to R_g$ are smooth.*

PROOF. The map $\tilde{p}: D \to R_g$ is flat, see [**5**] Prop. 9.7 p. 257. This implies that the Hilbert polynomial of the fibres D_x is independent of x (see [**5**] theorem (9.9) p. 261) thus the Hilbert polynomial of the singular fibres F is equal to the Hilbert polynomial of the smooth fibres F'; in particular $\chi(\mathcal{O}_{F'}) = \chi(\mathcal{O}_F)$. Note that F' intersects Y transversely in f', where f' is a fibre of Y. Moreover f' is ample in F' and $f' \cong \boldsymbol{P}^1$ thus by Scorza's lemma, see [**11**], F' is either F_r with $r \geqq 0$ or $\boldsymbol{P}^2$. Thus $\chi(\mathcal{O}_{F'}) = 1$. Note that the singular fibre F intersects Y transversely in f' and f' is a smooth Cartier divisor on F which implies that $\mathrm{Sing}(F)$ is in the complement of Y which is ample in D thus $\mathrm{Sing}(F)$ is a finite set of closed points. Since F is a local complete intersection and has only isolated singular points, F is normal by Serre's criterion. Thus F is either F_r with $r \geqq 0$ or $\tilde{F}_r$, $r \geqq 1$, where F_r is as in (0.5).

Assume $F = \tilde{F}_r$. Let $N_{F/D}$ be the normal bundle of F in D. $N_{F/D}$ is trivial since F is a fibre of $\tilde{p}$. Note that since $f' \cong \boldsymbol{P}^1$ then $f' = (E + rf)^a$ for some integer a, where E and f are as in (0.5). We know that

$$(1.3.1) \qquad F \cap Y = f'$$

and such intersection is transverse in D therefore $N_{f'/F} = N_{Y/D, f'}$. From (1.3.1) we see that

$$(1.3.2) \qquad N_{f'/D} = N_{F/D, f'} \oplus N_{Y/D, f'} = \mathcal{O}_{f'} \oplus \mathcal{O}_{f'}(ra^2)$$

since

(1.3.3) $$N_{Y/D,f'}=N_{f'/F}=\mathcal{O}_{f'}(ra^2).$$

We know that $D\cap A=Y$ and such intersection is transverse in X thus $N_{A/X,Y}=N_{Y/D}$ which implies that $N_{A/X,f'}=N_{Y/D,f'}$ thus by (1.3.3)

(1.3.4) $$N_{A/X,f'}=\mathcal{O}_{f'}(ra^2).$$

From (1.1.1) it follows that det $N_{f'/A}=\mathcal{O}_{f'}(-1)$ and from (1.1.2) and (1.3.4) we have

(1.3.5) $$\det N_{f'}=\det N_{f'/A}\otimes N_{A/X,f'}=\mathcal{O}_{f'}(ra^2-1).$$

From $f'\subset D\subset X$ and (1.3.2) we have

(1.3.6) $$\det N_{f'}=\mathcal{O}_{f'}(ra^2)\otimes N_{D/X,f'}.$$

From

$$0\longrightarrow N_{F/D}\longrightarrow N_{F/X}\longrightarrow N_{D/X,F}\longrightarrow 0$$

and the fact that $N_{F/D}$ is trivial it follows $\det N_{F/X}=N_{D/X,F}$. Thus

(*) $$\det N_{F/X,f'}=N_{D/X,f'}$$

and (1.3.6) becomes

$$\det N_{f'}=\mathcal{O}_{f'}(ra^2)\otimes\det N_{F/X,f'}.$$

Combining the above with (1.3.5) we have

(1.3.7) $$\det N_{F/X,f'}=\mathcal{O}_{f'}(-1).$$

Note that $\det N_{F/X}=(E+rf)^b$ for some integer b since $F=\tilde{F}_r$. Thus $\det N_{F/X,f'}=\mathcal{O}_{f'}(abr)$ and by (1.3.7) we have $abr=-1$. Note that $r\geqq 1$, a and b are integers thus $r=1$. Therefore $F=\tilde{F}_1$. Thus we conclude that F is smooth since F can be either F_r with $r\geqq 0$ or $\tilde{F}_1$. □

(1.4) LEMMA. *Let X, A, L, Y and D be as in* (1.0) *and* (1.2). *Then the fibres of $\tilde{p}$ are biholomorphic to $\boldsymbol{P}^2$. Moreover $L_{\boldsymbol{P}^2}\cong\mathcal{O}_{\boldsymbol{P}^2}(1)$.*

PROOF. By (1.3) the fibres F of $\tilde{p}$ are smooth and F is either F_r with $r\geqq 0$ or $\boldsymbol{P}^2$. Assume $F=F_r$. Knowing that $f'(\cong\boldsymbol{P}^1)$ is ample in F_r, we have $f'=E+(r+k)f$ with $k>0$, and as in (1.3), using $F=F_r$, instead of $F=\tilde{F}_r$, we get

(1.4.1) $$\det N_{F_r/X,f'}=\mathcal{O}_{f'}(-1).$$

Denote by L_{F_r} the restriction of the line bundle L to F_r. Thus L_{F_r} is ample since L is ample. Moreover $f'\in|L_{F_r}|$ since $f'=F_r\cap A$. Let E and f be as in (0.5). From $f\subset F_r\subset D$ we get

$$0 \longrightarrow \mathcal{O}_f \longrightarrow N_{f/D} \longrightarrow \mathcal{O}_f \longrightarrow 0$$

which implies that $N_{f/D}$ is spanned by global sections and $H^1(N_{f/D})=0$. From $f\subset D\subset X$ we have

$$(1.4.2) \qquad 0 \longrightarrow N_{f/D} \longrightarrow N_f \longrightarrow N_{D,f} \longrightarrow 0\,.$$

Thus $\det N_f = N_{D,f}$ since $\det N_{f/D}$ is trivial. We will show that $N_{D,f}$ is not spanned by global sections. Assume it is, i.e. $N_D\cdot f\geqq 0$, this implies, from the long exact cohomology sequence associated to (1.4.2), that N_f is spanned by global sections and $H^1(N_f)=0$ which is impossible by an earlier argument used in (1.1). Hence

$$(1.4.3) \qquad N_D\cdot f<0\,.$$

The line bundle L_{F_r} is ample thus $L_{F_r}=[E]\otimes[(r+k)f]$ with $k>0$. Since L_{F_r} is spanned we can find a smooth rational curve $C\in|[E]\otimes[(r+k-1)f]|=|L_{F_r}-f|$. Let N_C denote the normal bundle of C in X.

Claim. $\Gamma(N_C)$ is spanned by global sections and $H^1(N_C)=0$.

Proof of Claim. From $C\subset F_r\subset D$ we have

$$0 \longrightarrow \mathcal{O}_C(a) \longrightarrow N_{C/D} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

where $a=C\cdot C=r+2k-2\geqq 0$, thus $N_{C/D}$ is spanned and $H^1(N_{C/D})=0$. From $C\subset D\subset X$ we have

$$(1.4.4) \qquad 0 \longrightarrow N_{C/D} \longrightarrow N_C \longrightarrow N_{D/X,C} \longrightarrow 0\,.$$

Note that $N_D\cdot C=N_D\cdot(L_{F_r}-f)=N_D\cdot L_{F_r}-N_D\cdot f=N_D\cdot f'-N_D\cdot f$, thus by (1.4.1) and (1.4.3) we have $N_D\cdot C=-1-N_D\cdot f\geqq 0$. Hence by (1.4.3) and (1.4.4) above N_C is spanned and $H^1(N_C)=0$.

Using a similar argument used with f we get $K_X\cdot C\geqq 0$. $K_X\cdot C=K_X\cdot(L_{F_r}-f)=K_X\cdot L_{F_r}-K_X\cdot f$ thus

$$(1.4.5) \qquad K_X\cdot L_{F_r}\geqq K_X\cdot f\,.$$

From the adjunction formula, and $f'=E+(r+k)f$ and $\deg(\det N_{F_r/X,f'})=-1$ it follows $K_X\cdot L_{F_r}=K_X\cdot f'=-1-(r+2k)$. Hence from (1.4.5) we get $K_X\cdot f\leqq -1-(r+2k)\leqq -3$. Again by the adjunction formula it follows $-2=(K_X+D)\cdot f=K_X\cdot f+D\cdot f\leqq -3-1$ which is a contradiction. Thus $F=\boldsymbol{P}^2$.

Denote by $\mathcal{L}$ the $\det N_{\boldsymbol{P}^2/X}$ which is a line bundle in $\boldsymbol{P}^2$ thus $\mathcal{L}=\mathcal{O}_{\boldsymbol{P}^2}(\beta)$ with $\beta\in\boldsymbol{Z}$ moreover $\mathcal{L}_{f'}=\mathcal{O}_{f'}(-1)$ by (1.4.1) hence

$$(1.4.6) \qquad \mathcal{L}\cdot f'=-1\,.$$

Noting that $f'\in|L_{\boldsymbol{P}^2}|$ and that $L_{\boldsymbol{P}^2}$ is ample, i.e., $L_{\boldsymbol{P}^2}=\mathcal{O}_{\boldsymbol{P}^2}(\alpha)$, $\alpha>0$ and $\alpha\in\boldsymbol{Z}$ we have $\mathcal{L}\cdot f'=\mathcal{L}\cdot L_{\boldsymbol{P}^2}=\alpha\beta$. From (1.4.6) $\alpha\beta=-1$ giving $\alpha=\pm 1$ hence $\alpha=1$

since $\alpha>0$ therefore $L_{P^2}=\mathcal{O}_{P^2}(1)$. □

(1.5) LEMMA. *Let X, A, L, Y and D be as in* (1.2). *Then the map $\tilde{p}: D\to R_g$ is a $\boldsymbol{P}^2$-bundle.*

PROOF. By (1.3) and (1.4) the fibres of $\tilde{p}$ are smooth and biholomorphic to $\boldsymbol{P}^2$. Moreover there exists a line bundle in D, L_D such that $L_D|_F\cong\mathcal{O}_{P^2}(1)$. The map $\tilde{p}$ is flat by (1.4) hence by Hironaka's theorem $\tilde{p}: D\to R_g$ is a $\boldsymbol{P}^2$-bundle, see [7] Theorem 1.8, p. 10.

(1.6) MAIN THEOREM. *Let X be a connected four dimensional projective manifold. Let A be an ample divisor in X. Assume that the Kodaira dimension of X is non-negative. Assume that A is the blow-up of a smooth projective three-fold A' with center a curve R_g of genus $g\geqq 1$, where R_g is a submanifold of A'. Then there exists a smooth four dimensional manifold X' such that A' lies on X' as a divisor and such that X is the blow up of X' with center R_g. For the divisor A' to be ample it suffices to have $N^*_{R_g/X'}$ not ample.*

PROOF. By (1.1), (1.2), (1.3), (1.4) and (1.5) there exists a divisor D in X such that:

1) $D\cap A=Y$, where Y is the exceptional divisor on A over R_g

2) the natural projection $p: Y\to R_g$ extends to a surjective holomorphic map $\tilde{p}: D\to R_g$

3) $\tilde{p}$ makes D a $\boldsymbol{P}^2$-bundle over R_g, where $2=\mathrm{codim}_{A'}R_g$. Moreover each fibre Y_x of Y over $x\in R_g$ is a hyperplane on $D_x=\tilde{p}^{-1}(x)\cong\boldsymbol{P}^2$.

Now it is straightforward to see that 1), 2) and 3) imply that $N_{D,P^2}\cong\mathcal{O}_{P^2}(-1)$, see [3] (5.3). This is enough to ensure the existence of a manifold X' such that A' lies in X' as a divisor and X is the blow-up of X' with center R_g. Thus we have a map $\tilde{p}': X\to X'$ which blows down D, see [8]. In order to show that the divisor A' is ample on X', under the assumption that the dual bundle of $N_{R_g/X'}$ is not ample, it is enough to show that the restriction of $[A']$ to R_g is ample, see [3] Prop. 5.6. Thus, assume that $[A']|_{R_g}$ is not ample, i.e., $[A']\cdot R_g\leqq 0$. We have $p^*[A']=[A]+[D]$ since A is the proper transform of A' in X. Therefore

$$[A]|_D=p^*[A']|_D-[D]|_D=p^*[A']|_D+\zeta \tag{1.6.1}$$

where ζ is the tautological line bundle on $\boldsymbol{P}(N^*_{R_g/X'})$ and $N^*_{R_g/X'}$ is the dual bundle of $N_{R_g/X'}$. From (1.6.1) and using $p^*[A']|_D=p^*[A'|_{R_g}]$ we get

$$\zeta=[A]|_D+p^*([A'|_{R_g}]^{-1})\,. \tag{1.6.2}$$

Thus ζ is ample since $[A]|_D$ is ample and $[A'|_{R_g}]^{-1}$ is semipositive which implies $N^*_{R_g/X'}$ is ample contradicting our hypothesis. □

(1.7) REMARK. For A' to be ample, it suffices to have $N^*_{R_g/X'}$ not ample.

The referee of this paper gave the following example in which the divisor A' is not ample even though its proper transform A in X is ample.

Let C be an elliptic curve. Let $\mathcal{L}$ be a very ample line bundle on C. Let $X'=\boldsymbol{P}_C(\mathcal{L}\oplus\mathcal{L}\oplus\mathcal{L}\oplus\mathcal{O})$ and let ζ be the tautological line bundle on X'. Let S be the section of $X'\to C$ defined by the quotient $\mathcal{O}$. One can find a smooth $A'\in|2\zeta|$ which contains S. Let X be the blow-up of X' with center S and let A be the proper transform of A' on X. Then one can check that A is an ample divisor on X but A' is not ample on X'.

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