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Semi-simple degree of symmetry and maps of degree one into a product of 2-spheres

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Introduction.

Recently many authors have shown that if a smooth closed manifold M admits a continuous map of degree one into a product of 1-spheres, then the compact connected Lie group which acts on M smoothly and almost effectively is a torus ([**5**], [**9**], [**11**]). The present note is motivated by this result. In this note, we shall study the semi-simple degree of symmetry of a manifold which admits a continuous map of degree one into a product of 2-spheres. Here the *semi-simple degree of symmetry* of a manifold is, by definition, the maximum dimension of the compact connected semi-simple Lie group which acts on the manifold smoothly and almost effectively.

We shall prove the following

THEOREM A. *Let M be a simply connected closed $2m$-dimensional topological manifold which admits a continuous map of degree one into a product of 2-spheres. Then $SO(3)$ or $SU(2)$ is only the compact connected simple Lie group which acts on M continuously and almost effectively. Therefore if a compact connected Lie group G acts on M continuously and almost effectively, then G is locally isomorphic to $T^{s}\times SU(2)\times\cdots\times SU(2)$.*

A typical example of M of Theorem A is a connected sum of $S^{2}\times\cdots\times S^{2}$ (m-times) and a $2m$-dimensional manifold. As for the connected sum, we shall obtain the following

THEOREM B. *Let M be as in Theorem* A *and N a simply connected closed $2m$-dimensional topological manifold which is not a rational homology sphere. Then the connected sum $X=M\# N$ does not admit any action of $SU(2)$.*

REMARK 1. As a corollary to Theorem B, we have the following

PROPOSITION. *Let N be as in Theorem* B. *Then the semi-simple degree of symmetry of $(S^{2}\times\cdots\times S^{2})\# N$ is zero.*

REMARK 2. Since the connected sum $M=(S^{2}\times\cdots\times S^{2})\#\Sigma^{2m}$ (Σ^{2m} : $2m$-dimensional homotopy sphere) is homeomorphic to $S^{2}\times\cdots\times S^{2}$, it admits a continuous action of $SU(2)$. But it does not necessarily admit a smooth action of $SU(2)$.

In fact, if M admits a smooth action of $SU(2)$, then M admits a Riemannian metric of strictly positive scalar curvature ([7]). On the other hand, Hitchen has proved that if a Spin manifold admits a Riemannian metric of strictly positive scalar curvature, then the invariant α defined in [8] is zero. It is known that there is a homotopy sphere Σ_0^n ($n=1, 2 \bmod 8$) with $\alpha(\Sigma_0^n)\neq 0$ (see [6], [8]). Consider the manifold $M=(S^2\times \cdots \times S^2)\,\#\,\Sigma_0^{2m}$ ($m=4k+1$). Since $\alpha(M)\neq 0$, M does not admit any smooth action of $SU(2)$.

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In this note, we shall restrict ourselves to continuous and almost effective actions and use the following notation.

$\boldsymbol{Q}$: the field of rational numbers

$H^*(X)$: the cohomology ring of X with coefficient $\boldsymbol{Q}$

T^s and T: s-dimensional torus and 1-dimensional torus, respectively

$SU(n)$ ($SO(n)$, $Sp(n)$): the group of all $n\times n$ special unitary (special orthogonal, symplectic, respectively) matrices

X^*: the orbit space of X under the action of a compact connected Lie group on X.

1. Preliminaries.

In this section, we recall some basic facts about the Leray spectral sequence of the orbit map.

Let G be a compact connected Lie group and act on a compact connected space X. Let $\pi: X\to X^*$ be the orbit map and $\{E_r^{p,q}, d_r\}$ be the Leray spectral sequence of the map π. Then we have $E_2^{p,q}=H^p(X^*: H^q(\pi))$, where $H^q(\pi)$ is the sheaf generated by the presheaf $U^*\to H^q(\pi^{-1}(U^*))$ for open set U^* in X^* (see [2]). Recall the stalk of $H^q(\pi)$ at $x^*\in X^*$ is $H^q(G(x))$, where $\pi(x)=x^*$, and the edge homomorphism $e: H^q(X)\to E_2^{0,q}$ is given by $e(a)(x^*)=$the image of a by the homomorphism $H^q(X)\to H^q(G(x))$ induced by the inclusion $G(x)\to X$ (see [2] for details).

We have the following

PROPOSITION 1 (see [4]). *Let k be the dimension of a principal orbit. If the action has a singular orbit, then the edge homomorphism $e: H^k(X)\to E_2^{0,k}$ is trivial. In particular, we have $E_\infty^{0,k}=0$.*

PROOF. The first part follows from the existence of a slice and the connectedness of X. Note that the edge homomorphism is factored as follows; $H^k(X)\xrightarrow{\alpha} E_\infty^{0,k}\xrightarrow{\beta} E_2^{0,k}$. Since α is surjective and β is injective, we have $E_\infty^{0,k}=0$. This completes the proof of Proposition 1.

We have the following Propositions which are useful for the proof of Theorems A and B.

PROPOSITION 2. *Let M be a closed $2m$-dimensional topological manifold such that there are m elements $w_1, w_2, \cdots, w_m$ in $H^2(M)$ with the cup product $w_1 \cup \cdots \cup w_m \neq 0$. Assume the group $SU(2)$ acts on M with a torus T as a principal isotropy subgroup. Then there is no singular orbit.*

PROOF. Assume the contrary. Then it follows from Proposition 1 that $E_\infty^{0,2} = 0$, where $\{E_r^{p,q}, d_r\}$ is the Leray spectral sequence of the orbit map $\pi : M \to M^*$. Since $H^1(SU(2)(x)) = 0$ for every point x in M, we have $H^2(M^*) = E_\infty^{2,0} = H^2(M)$. Note that this isomorphism is induced by the orbit map. Hence $w_i = \pi^*(w_i')$ for $i = 1, \cdots, m$, where w_i' is an element of $H^2(M^*)$. Thus we have $w_1 \cup \cdots \cup w_m = \pi^*(w_1' \cup \cdots \cup w_m') = 0$, which is a contradiction. This completes the proof of Proposition 2.

PROPOSITION 3. *Let M be as in Proposition 2. Assume the group $SU(2)$ acts on M with a finite principal isotropy subgroup and a singular orbit. Then there is a point in M whose isotropy subgroup is a torus.*

PROOF. Assume the contrary. Since $H^i(SU(2)(x)) = 0$ for $i = 1, 2$ for every point x in M, it is easy to see that $H^2(M) = H^2(M^*)$ via the orbit map. The same argument as in Proposition 2 shows that this is impossible. This completes the proof of Proposition 3.

PROPOSITION 4. *Let M be a closed $3m$-dimensional topological manifold such that there are m elements $w_1, w_2, \cdots, w_m$ in $H^3(M)$ with $w_1 \cup \cdots \cup w_m \neq 0$. Assume the group $SU(2)$ acts on M with a finite principal isotropy subgroup and a singular orbit. Then there is a point x in M whose isotropy subgroup is a torus.*

Since the proof is similar to that of Proposition 3, we shall omit it.

PROPOSITION 5. *Let M be as in Proposition 4. Assume M is simply connected and the group $SU(3)$ acts on M with a finite principal isotropy subgroup. Then there is a singular orbit.*

PROOF. Assume the contrary. Since M is simply connected, it follows from a result in [3] (Theorem 1 in [3]) that the Leray sheaf of the orbit map is trivial, which means that $H^0(M^* : H^3(\pi)) = \boldsymbol{Q}$ and hence $\dim E_\infty^{0,3} \leqq 1$. It follows from the fact $H^i(SU(3)(x)) = 0$ for $i = 1, 2$ that we have the following exact sequence;

$$0 \longrightarrow E_\infty^{3,0} \longrightarrow H^3(M) \longrightarrow E_\infty^{0,3} \longrightarrow 0 .$$

Note that $E_\infty^{3,0} = H^3(M^*)$. Since $\dim E_\infty^{0,3} \leqq 1$, there are elements $w_1', \cdots, w_m'$ in $H^3(M)$ such that $w_1' \cup \cdots \cup w_m' \neq 0$ and $w_1', \cdots, w_{m-1}'$ are in $E_\infty^{3,0}$. Since $\dim M^* = \dim M - 8$, we have $w_1' \cup \cdots \cup w_{m-1}' = 0$, which is a contradiction. This completes the proof of Proposition 5.

2. Proof of Theorem A.

Let M be a closed $2m$-dimensional topological manifold with a map of degree one into a product of 2-spheres $S^2\times\cdots\times S^2$ (m-times). We shall construct a principal T^m-bundle $\tilde{M}$ over M as follows. Put

$$N_i=\underbrace{S^3\times\cdots\times S^3}_{i\text{-times}}\times\underbrace{S^2\times\cdots\times S^2}_{(m-i)\text{-times}}\quad(i=0,1,\cdots,m).$$

Consider N_{i+1} as a principal T-bundle over N_i ($i=0,\cdots,m-1$). Let M_1 be the pull-back of the bundle $N_1\to N_0$ by the given map $f:M\to N_0$ of degree one and $f_1:M_1\to N_1$ the bundle map covering f. It is easy to see that f_1 is a map of degree one. Inductively we can construct a sequence of manifolds $M_0=M$, M_1, $\cdots$, $M_m=\tilde{M}$ and a sequence of maps $f_0=f$, $f_1,\cdots,f_m=\tilde{f}$ such that $f_i:M_i\to N_i$ is a map of degree one and $p_i:M_i\to M_{i-1}$ is a principal T-bundle which is the pull-back of $N_i\to N_{i-1}$ by the map f_{i-1} for $i=1,\cdots,m$.

Let $\{a_{i1},\cdots,a_{ii}\}$ and $\{b_{i1},\cdots,b_{i\,m-i}\}$ be the natural basis of $H^3(N_i)$ and $H^2(N_i)$, respectively and put $\bar{a}_{ij}=f_i^*(a_{ij})$, $\bar{b}_{ij}=f_i^*(b_{ij})$.

It follows from a result in [**10**] (Theorem 4.1 in [**10**]) that the action of a simply connected compact semi-simple Lie group on M_i can be lifted over M_{i+1} ($i=0,1,\cdots,m-1$).

Now we shall prove the following Propositions which are basic for the proof of Theorems A and B.

PROPOSITION 6. *Let M be a simply connected closed $2m$-dimensional topological manifold with a map of degree one into a product of 2-spheres. Assume M admits an action of $SU(2)$. Then the lifting of the action over $\tilde{M}$ is almost free; in other words, all isotropy subgroups are finite.*

PROOF. Put $G=SU(2)$. Let $\phi:G\times M\to M$ be the given action and ϕ_i the lifting of ϕ over M_i. Put $\phi_m=\tilde{\phi}$. Let H_ϕ or H_{ϕ_i} be a principal isotropy subgroup of ϕ or ϕ_i, respectively.

We shall first prove that $H_{\tilde{\phi}}$ is finite. Assume the contrary. If $\tilde{\phi}$ has a singular orbit, i.e. a fixed point, then ϕ has also a fixed point. This contradicts Proposition 2. If $\tilde{\phi}$ has no singular orbit, it can be proved that $\tilde{\phi}$ has a unique orbit S^2 and $\tilde{M}$ is equivariantly homeomorphic to $S^2\times\tilde{M}^*$, which is easily seen to be a contradiction. In fact, assume that there is a point $\tilde{x}$ in $\tilde{M}$ such that $G_{\tilde{x}}=N_T$ (N_T=the normalizer of T). It follows from the arguments in [**1**] (Lemma 2.4 and Theorem 2.6 in [**1**]) that there is a map $\alpha:\tilde{M}\to G/N_T$ such that $\alpha^*:H^*(G/N_T:A)\to H^*(\tilde{M}:A)$ is injective for any coefficient group A. Since $H^1(G/N_T:\boldsymbol{Z}_2)=\boldsymbol{Z}_2$ and $H^1(\tilde{M}:\boldsymbol{Z}_2)=0$, this is impossible. Hence the orbit map $\tilde{\pi}:\tilde{M}\to\tilde{M}^*$ is a fibre bundle with S^2 as fibre and N_T/T as the structure group. Since $\tilde{M}^*$ is simply connected we have $\tilde{M}=S^2\times\tilde{M}^*$. Thus we have proved that $H_{\tilde{\phi}}$ is finite.

Next we shall prove that $\tilde{\phi}$ has no singular orbit. We consider the following two cases separately.

1. H_ϕ is positive dimensional.
2. H_ϕ is a finite group.

Case 1: It follows from Proposition 2 that there is no fixed point of ϕ. Since M is simply connected, the same arguments as before show that ϕ has no orbit of type N_T and M is equivariantly homeomorphic to $S^2\times M^*$.

It is clear that there is an index j, say $j=1$, such that $w=f^*(b_{01})$ is not in $\operatorname{Im}\pi^*$. We may assume that w corresponds to a generator of $H^2(S^2)$. Then the homomorphism $i_x^*: H^2(M)\to H^2(G(x))$ induced by the inclusion sends w to a generator of $H^2(G(x))$ for every point x in M. Hence we have $i_x^*(\bar{b}_{01})\neq 0$ for every point x in M and $p_1: p_1^{-1}(G(x))\to G(x)$ is a non-trivial T-bundle for every point x in M. This means that $p_1^{-1}(G(x))=G(x_1)$ for every point x_1 in $p_1^{-1}(x)$ and hence ϕ_1 has no singular orbit. Thus $\tilde{\phi}$ has no singular orbit.

Case 2: Assume that $\tilde{\phi}$ has a singular orbit. Then ϕ has also a singular orbit. It follows from Proposition 3 that ϕ has an orbit of type T.

LEMMA 1. *There is a point x in M such that the homomorphism $i_x^*f^*: H^2(N_0)\to H^2(G(x))$ is not zero.*

PROOF. Assume that $i_x^*f^*$ is trivial for every point x in M. Then we have $e(a)(x^*)=i^*(a)=0$ for every element a in $\operatorname{Im}f^*$, where $e: H^2(M)\to E_2^{0,2}$ is the edge homomorphism of the Leray spectral sequence of the orbit map for ϕ. This implies that $\operatorname{Im}f^*$ is contained in $\operatorname{Ker}\{H^2(M)\to E_\infty^{0,2}\}$ and hence $\operatorname{Im}f^*$ is contained in $\operatorname{Im}\{E_\infty^{2,0}\to H^2(M)\}=\operatorname{Im}\pi^*$, where $\pi: M\to M^*$ is the orbit map. This is easily seen to be a contradiction. This completes the proof of Lemma 1.

Fix a point x in M such that $i_x^*f^*$ is not zero. We may assume $i_x^*f^*(b_{01})\neq 0$. Consider the lifting ϕ_1. Choose a point x_1 of M_1 such that $p_1(x_1)=x$. Then we have the following

LEMMA 2. *The inclusion $i_{x_1}: G(x_1)\to M_1$ induces non-trivial homomorphism $i_{x_1}^*: H^3(M_1)\to H^3(G(x_1))$.*

PROOF. It follows from the assumption that $p^{-1}(G(x))=G(x_1)$. Then Lemma follows from the following commutative diagram;

$$
\begin{array}{ccccccccc}
\longrightarrow & H^3(N_0) & \longrightarrow & H^3(N_1) & \xrightarrow{\approx} & H^2(N_0) & \longrightarrow \\
 & \downarrow f^* & & \downarrow f_1^* & & \downarrow f^* & \\
\longrightarrow & H^3(M) & \longrightarrow & H^3(M_1) & \longrightarrow & H^2(M) & \longrightarrow \\
 & \downarrow i_x^* & & \downarrow i_{x_1}^* & & \downarrow i_x^* & \\
\longrightarrow & H^3(G(x)) & \longrightarrow & H^3(G(x_1)) & \longrightarrow & H^2(G(x)) & \longrightarrow
\end{array}
$$

where the horizontal sequences are Gysin sequences. This completes the proof of Lemma 2.

It follows from the assumption that ϕ_1 has also a singular orbit. Then it follows from Proposition 1 that the edge homomorphism $e_1: H^3(M_1)\to E_2^{0,3}$ of the Leray spectral sequence of the orbit map $M_1\to M_1^*$ is trivial. This means that the homomorphism $i_y^*: H^3(M_1)\to H^3(G(y))$ induced by the inclusion must be trivial for every point y in M_1. This contradicts Lemma 2. This contradiction shows that $\tilde{\phi}$ has no singular orbit. This completes the proof of Proposition 6.

Now we have the following

PROPOSITION 7. *Let M be as in Proposition 6. Then the Leray spectral sequence of the orbit map $\tilde{M}\to\tilde{M}^*$ collapses and $H^*(\tilde{M})$ is isomorphic to $H^*(\tilde{M}^*)\otimes H^*(S^3)$ as algebras.*

PROOF. Since the action $\tilde{\phi}$ is almost free, it follows from a result in [**3**] (Theorem 1 in [**3**]) that the second term of the spectral sequence is given by $E_2^{p,q}=H^p(\tilde{M}^*)\otimes H^q(S^3)$. The edge homomorphism $e: H^3(\tilde{M})\to E_2^{0,3}$ is proved to be surjective. In fact, assume the contrary. Then we have $E_\infty^{0,3}=0$, because $\dim E_2^{0,3}=1$ and hence $H^3(\tilde{M}^*)=H^3(\tilde{M})$ via the orbit map, which is easily proved to be a contradiction. Thus the spectral sequence collapses. It follows from the arguments of the Leray-Hirsch Theorem that $H^*(\tilde{M})$ is isomorphic to $H^*(\tilde{M}^*)\otimes H^*(S^3)$ as algebras, which completes the proof of Proposition 7.

Now we shall prove Theorem A. It is sufficient to show that $SU(3)$ and $Sp(2)$ can not act on M non-trivially. Since the arguments for $SU(3)$ and $Sp(2)$ are completely parallel, we shall consider only the case of $SU(3)$.

Assume $G=SU(3)$ acts on M non-trivially. Denote this action by ψ. Let ϕ be an action of a subgroup K which is locally isomorphic to $SU(2)$ obtained from the restriction of ψ and ψ_i, ϕ_i the lifting of ψ, ϕ over M_i, respectively. Put $\tilde{\psi}=\psi_m$ and $\tilde{\phi}=\phi_m$.

It follows from Proposition 6 that $\tilde{\phi}$ is almost free for any subgroup K, and hence the identity component of any isotropy subgroup is the identity or a torus which is not contained in a subgroup locally isomorphic to $SU(2)$.

We have the following several observations.

(1) Consider the action $\tilde{\phi}$. It follows from Proposition 7 that $H^*(\tilde{M})$ is isomorphic to $H^*(\tilde{M}^*)\otimes H^*(S^3)$. It is easy to see that there is an index h, say $h=1$, such that $\tilde{f}^*(a_{m1})$ is not contained in $H^*(\tilde{M}^*)$. We may assume that $\tilde{w}=\tilde{f}^*(a_{m1})$ corresponds to a generator of $H^3(S^3)$. Then the homomorphism $i_{\tilde{x}}^*: H^3(\tilde{M})\to H^3(K(\tilde{x}))$ induced by the inclusion $i_{\tilde{x}}$ sends $\tilde{w}$ to a generator of $H^3(K(\tilde{x}))$ for every point $\tilde{x}$ in $\tilde{M}$.

(2) The homomorphism $j_{\tilde{x}}^*: H^3(\tilde{M})\to H^3(G(\tilde{x}))$ induced by the inclusion $j_{\tilde{x}}$ sends $\tilde{w}$ to a non-zero element of $H^3(G(\tilde{x}))$ for every point $\tilde{x}$ in $\tilde{M}$.

This follows from (1) and the following commutative diagram;

$$\begin{array}{ccc} H^3(\tilde{M}) & \xrightarrow{i_{\tilde{x}}^*} & H^3(K(\tilde{x})) \\ & {\scriptstyle j_{\tilde{x}}^*}\searrow \quad \nearrow{\scriptstyle k^*} & \\ & H^3(G(\tilde{x})) & \end{array}$$

where $k: K(\tilde{x}) \to G(\tilde{x})$ is the natural map.

(3) The possible type of the rational cohomology ring of orbit of the action $\tilde{\phi}$ is that of $S^3 \times S^5$. In other words, the action $\tilde{\phi}$ has no singular orbit.

This follows from (2) and the following Proposition for which the author is indebted to the referee.

PROPOSITION 8. *Let U be a closed subgroup of $SU(3)$. If U is positive dimensional, then we have $H^3(SU(3)/U)=0$.*

PROOF. We may assume that U is connected. For the proof of the Proposition, it is sufficient to show the followings;

(i) $H^*(SU(3)/N(SU(2))) \cong H^*(CP^2)$, where $N(SU(2))$ is the normalizer of $SU(2)$ in $SU(3)$ and CP^2 is the 2-dimensional complex projective space.

(ii) $H^*(SU(3)/SU(2)) \cong H^*(S^5)$

(iii) $H^*(SU(3)/SO(3)) \cong H^*(S^5)$

(iv) $H^*(SU(3)/T^2) \cong Q[u_1, u_2]/(u_1^3, u_1^2+u_1u_2+u_2^2)$ $(\deg u_1 = \deg u_2 = 2)$

and

(v) $H^*(SU(3)/T) \cong H^*(S^2 \times S^5)$,

where the notation "$\cong$" means "isomorphic as rings".

(i) and (ii) are well known. (iii) follows from the fact $H^*(U(3)/SO(3)) \cong H^*(S^1 \times S^5)$. We shall prove (iv). Let S be the standard maximal torus of $SU(3)$. Then we can identify $SU(3)/S$ with the hypersurface $H'_{2,2}$ in $CP^2 \times CP^2$;

$$H'_{2,2} = \{[x_1, x_2, x_3] \times [y_1, y_2, y_3];\ x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 = 0\}.$$

Let $\pi_i: H'_{2,2} \to CP^2 \times CP^2 \to CP^2$ be the projection to the i-th component, γ the canonical complex line bundle over CP^2 and $c_1 = c_1(\gamma)$ the first Chern class of γ. Define $u_i = \pi_i^*(c_1)$ for $i=1, 2$. Then we have $H^*(SU(3)/S: \boldsymbol{Z}) = \boldsymbol{Z}[u_1, u_2]/(u_1^3, u_1^2+u_1u_2+u_2^2)$, which implies (iv). In fact, let ζ be the complex 2-plane bundle over CP^2 defined by

$$E(\zeta) = \{[x] \times y \in CP^2 \times \boldsymbol{C}^3;\ x \text{ and } y \text{ are orthogonal}\}.$$

Note that $H'_{2,2}$ is the associated projective bundle $CP(\zeta)$ of ζ. Then $\zeta \oplus \gamma$ is trivial and hence the total Chern class $c(\zeta) = 1 - c_1 + c_1^2$. Let $\hat{\zeta}$ be the canonical complex line bundle over $H'_{2,2}$. Then it is easy to see that $c_1(\hat{\zeta}) = \pi_2^*(c_1) = u_2$. Now we have an isomorphism;

$$H^*(H'_{2,2}: \boldsymbol{Z}) = H^*(CP^2: \boldsymbol{Z})[t]/(c_2(\zeta) - c_1(\zeta)t + t^2),$$

under which $c_1(\hat{\zeta})$ is mapped to t. This induces an isomorphism;

$$H^*(H'_{2,2}:\boldsymbol{Z})=\boldsymbol{Z}[u_1, u_2]/(u_1^3, u_1^2+u_1u_2+u_2^2),$$

as desired.

Now we shall prove (v). We use the same notations as above. It is clear that every 1-dimensional toral subgroup of $SU(3)$ is conjugate to the subgroup $D(a, b)$ defined as follows;

$$D(a, b)=\left\{\begin{pmatrix} z^a & & \\ & z^b & \\ & & \bar{z}^{a+b}\end{pmatrix};\ a, b\in \boldsymbol{Z},\ z\in \boldsymbol{C},\ |z|=1\right\}.$$

We may assume $a\geqq b\geqq 0$ and a, b are relatively prime. Consider the principal bundle $\pi: SU(3)/D(a, b)\to SU(3)/S$. First assume $b\neq 0$. Let i_1 and i_2 be monomorphisms $SU(2)\to SU(3)$ defined as follows;

$$i_1\begin{pmatrix} x & y \\ u & v\end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \\ 0 & u & v\end{pmatrix} \quad\text{and}\quad i_2\begin{pmatrix} x & y \\ u & v\end{pmatrix}=\begin{pmatrix} x & 0 & y \\ 0 & 1 & 0 \\ u & 0 & v\end{pmatrix}$$

respectively and put $T'=i_1^{-1}(S)$, $T''=i_2^{-1}(S)$. Then we have the following commutative diagram;

$$\begin{array}{ccccc} SU(3)/\boldsymbol{Z}_a & \xrightarrow{\bar{i}_1} & SU(3)/D(a, b) & \xleftarrow{\bar{i}_2} & SU(3)/\boldsymbol{Z}_b \\ \downarrow \pi' & & \downarrow \pi & & \downarrow \pi'' \\ SU(2)/T' & \xrightarrow{i_1} & SU(3)/S & \xleftarrow{i_2} & SU(2)/T'' \end{array}$$

where $\bar{i}_1$, $\bar{i}_2$ are bundle maps and π', π'' are projections. It follows from the above diagram and the definition of u_i that the Euler class e of π is given by $e=bu_1+au_2$. Hence the homomorphism $\theta: H^2(SU(3)/S:\boldsymbol{Z})\to H^4(SU(3)/S:\boldsymbol{Z})$ defined by $\theta(c)=c\cdot e$ is injective. It follows from the Gysin sequence of π with rational coefficient that $H^*(SU(3)/T)=H^*(S^2\times S^5)$. If $b=0$, then the bundle π may be assumed to be reduced to the fibering $S^2\to SU(3)/T\to S^5$, which means the conclusion. This completes the proof of Proposition 8.

It is clear that the observation (3) contradicts Proposition 5. This completes the proof of Theorem A.

3. Proof of Theorem B.

Let $g: M\to S^2\times S^2\times \cdots \times S^2$ (m-times) be a map of degree one and $c: X=M\# N\to M$ the collapsing map. Then the composition $g\circ c$ has degree one. As before, we can construct a T^m-bundle $\tilde{X}$ over X and a map $\tilde{f}: \tilde{X}\to S^3\times S^3\times \cdots \times S^3$ of degree one. We have the following diagram of fibre bundles and bundle maps;

$$(\sharp)\qquad \begin{array}{ccccc} \tilde{X} & \xrightarrow{\ \tilde{c}\ } & \tilde{M} & \xrightarrow{\ \tilde{g}\ } & S^3\times S^3\times \cdots \times S^3 \\ {\scriptstyle \tilde{p}}\big\downarrow & & \big\downarrow & & \big\downarrow \\ X & \xrightarrow[\ c\ ]{} & M & \xrightarrow[\ g\ ]{} & S^2\times S^2\times \cdots \times S^2 \end{array}$$

where $\tilde{M}$ is the T^m-bundle over M constructed from g and $\tilde{f}=\tilde{g}\circ\tilde{c}$.

We have the following observations.

(1) $\tilde{X}$ is homeomorphic to the space

$$(\tilde{M}-\operatorname{int} D^{2m}\times T^m)\bigcup_{S^{2m-1}\times T^m}(N-\operatorname{int} D^{2m})\times T^m .$$

(2) Consider the following commutative diagram;

$$\begin{array}{ccccc} & & H^k(\tilde{X},(N-\operatorname{int} D^{2m})\times T^m)\cong H^k(\tilde{M}-\operatorname{int} D^{2m}\times T^m, S^{2m-1}\times T^m) & & \\ & & {\scriptstyle j_0^*}\big\downarrow \quad \swarrow{\scriptstyle r^*} \qquad\qquad {\scriptstyle j_2^*}\big\downarrow & & \\ H^k(\tilde{X}, \tilde{M}-\operatorname{int} D^{2m}\times T^m) & \xrightarrow{\ j_1^*\ } & H^k(\tilde{X}) & \xrightarrow[\ i_1^*\ ]{} & H^k(\tilde{M}-\operatorname{int} D^{2m}\times T^m) \\ \cong\big\downarrow & \nearrow{\scriptstyle q^*} & {\scriptstyle i_0^*}\big\downarrow & & {\scriptstyle i_2^*}\big\downarrow \\ H^k((N-\operatorname{int} D^{2m})\times T^m, S^{2m-1}\times T^m) & \xrightarrow[\ j_3^*\ ]{} & H^k((N-\operatorname{int} D^{2m})\times T^m) & \xrightarrow[\ i_3^*\ ]{} & H^k(S^{2m-1}\times T^m) \end{array}$$

Here the vertical and horizontal sequences are exact, and q and r are the collapsing maps: $\tilde{X}\to\tilde{X}/\tilde{M}-\operatorname{int} D^{2m}\times T^m$ and $\tilde{X}\to\tilde{X}/(N-\operatorname{int} D^{2m})\times T^m$, respectively and the other maps are the inclusions. Then it follows from the diagram $(\sharp)$ that $\operatorname{Im} f^*$ is contained in $\operatorname{Im} r^*=\operatorname{Ker} i_0^*$.

The observations (1) and (2) are direct consequences of the definition of $\tilde{X}$.

(3) Let $r=\min\{r' : H^{r'}(N)\neq 0\}$. Since N is not a rational homology sphere and simply connected, we have $1\leqq r\leqq m$. Choose elements $a'\in H^r(N)$ and $b'\in H^{2m-r}(N)$ such that $a'\cup b'\neq 0$. Since $a'\times[T^m]\in H^{m+r}((N-\operatorname{int} D^{2m})\times T^m)$ and $b'\times 1\in H^{2m-r}((N-\operatorname{int} D^{2m})\times T^m)$ are in $\operatorname{Ker} i_3^*$, there exist a and b in $H^*(\tilde{X})$ such that $i_0^*(a)=a'\times[T^m]$ and $i_0^*(b)=b'\times 1$. Then we have $a\cup b\neq 0$.

In fact, consider the space $Y=\tilde{X}/\tilde{M}-\operatorname{int} D^{2m}\times T^m$ obtained from collapsing $\tilde{M}-\operatorname{int} D^{2m}\times T^m$ to a point. It is clear that Y is homeomorphic to the space $(N-\operatorname{int} D^{2m})\times T^m/S^{2m-1}\times T^m$. Let c and d be elements of $H^*(Y)$ corresponding to $a'\times[T^m]$ and $b'\times 1$ via the isomorphism $H^*(Y)=H^*((N-\operatorname{int} D^{2m})\times T^m, S^{2m-1}\times T^m)$, respectively. It is clear that $c\cup d\neq 0$ and $q^*(c)=a$ and $q^*(d)=b$, which implies $a\cup b=q^*(c\cup d)\neq 0$, because q is a map of degree one. This completes the proof of the observation (3).

Now assume $G=SU(2)$ acts on X. Then it follows from Propositions 6 and 7 that $H^*(\tilde{X})$ is isomorphic to $H^*(\tilde{X}^*)\otimes H^*(S^3)$. It is easy to see that there is an element $\tilde{w}$ in $H^3(\tilde{X})$ such that $\tilde{w}$ is contained in $\operatorname{Im}\tilde{f}^*$, but not in $\operatorname{Im}\tilde{\pi}^*$, where $\tilde{\pi}:\tilde{X}\to\tilde{X}^*$ is the orbit map. It follows from (2) that $i_0^*(\tilde{w})=0$. Since $H^*(\tilde{X})=$

$H^*(\tilde{X}^*)+\tilde{w}H^*(\tilde{X}^*)$ and $i_0^*(\tilde{w})=0$, a and b can be chosen in $\mathrm{Im}\,\tilde{\pi}^*$; in other words, $a=\tilde{\pi}^*(a'')$ and $b=\tilde{\pi}^*(b'')$ where a'' and b'' are in $H^*(\tilde{X}^*)$. This implies that $a\cup b=\tilde{\pi}^*(a''\cup b'')=0$, which is a contradiction. Thus we have completed the proof of Theorem B.

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