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Some nonlinear degenerate diffusion equations related to population dynamics

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1. Introduction.

For the study of the spatial distribution of organisms, there are a large number of spatially spreading population models in which biological interactions and diffusion are taken into account. Among them, several models include nonlinear diffusion processes called "density-dependent dispersal". From ecological aspects, the works by Gurney and Nisbet [8], Gurtin and MacCamy [9] are relevent here.

In the category of such models, we propose a population model which provides a nonlocal interaction

$$u_t=(D(u)u_x)_x+\left[\left(\int_{-\infty}^{\infty}K(x-\xi)u(\xi,t)d\xi\right)u\right]_x, \tag{1.1}$$

where $u(x, t)$ denotes the population density at position $x\in \boldsymbol{R}^1$ and at time t, $D(u)$ is the diffusion rate satisfying $D(0)=0$ and $D'(u)>0$, and $K(x)$ is an odd function such that $K(x)>0$ for $x>0$. For one example, we have

$$K(x)=\begin{cases} ke^{-sx}, & x>0, \\ -ke^{-sx}, & x<0, \end{cases}$$

for a non-negative constant s. The second term of (1.1) ecologically implies a kind of aggregative mechanism of the individuals, which is motivated by the notion of "the selfish avoidance of a predator can lead to aggregation" (see, Hamilton [10]). If we restrict D and K to the specific forms $D(u)=mu^{m-1}$ and

$$K(x)=\begin{cases} k, & x>0, \\ -k, & x<0, \end{cases}$$

where $m>1$ and $k>0$ are constants, then (1.1) is rewritten as

$$u_t=(u^m)_{xx}+k\left[\left(\int_{-\infty}^{x}u(\xi,t)d\xi-\int_{x}^{\infty}u(\xi,t)d\xi\right)u\right]_x.$$

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When $K=0$, the equation (1.1) occurs in the theory of flow through porous medium. Because of the degeneracy of diffusion at $u=0$ this equation possesses the property that an initial smooth distribution with compact support spreads out a finite speed and loses the smoothness (see, for instance, Aronson [2], Oleinik, Kalashnikov and Yui-Lin [13]). Concerning the regularity of solutions for the porous medium equation it was known that the best possible Hölder exponent of solutions is $\min[1, 1/(m-1)]$ (see, Aronson [1] and Gilding [4]).

Let the initial function $u_0(x)$ satisfy

$$\int_{-\infty}^{\infty} u_0(x)dx = c < +\infty .$$

Then, by the $L^1(\boldsymbol{R}^1)$-conservation, (1.1) is reduced to

$$u_t = (u^m)_{xx} + k\left[\left(2\int_{-\infty}^{x} u(\xi, t)d\xi - c\right)u\right]_x . \tag{1.2}$$

In this paper, we consider the following slightly more general equation than (1.2):

$$u_t = (u^m)_{xx} + \left[\phi'\left(\int_{-\infty}^{x} u(\xi, t)d\xi\right)u\right]_x \quad \text{in} \quad \boldsymbol{R}^1 \times (0, \infty) \tag{1.3}$$

subject to the initial condition

$$u(x, 0) = u_0(x) \quad \text{on} \quad \boldsymbol{R}^1, \tag{1.4}$$

where $\phi' = d\phi/ds$, and discuss the global existence and uniqueness of solution of the Cauchy problem (1.3), (1.4). The assumptions to be imposed on this problem are essentially $m>1$, ϕ is a smooth function on $\boldsymbol{R}^1$ and $u_0(x)$ is non-negative, bounded and integrable on $\boldsymbol{R}^1$.

In Section 2, we state our main results which consist of global existence and the regularity of non-negative solutions of (1.3), (1.4). In Section 3, we transform the problem (1.3), (1.4) through

$$v(x, t) = \int_{-\infty}^{x} u(\xi, t)d\xi$$

into the more convenient Cauchy problem described by

$$v_t = [(v_x)^m + \phi(v)]_x \quad \text{in} \quad \boldsymbol{R}^1 \times (0, \infty), \tag{1.5}$$

$$v(-\infty, t) = 0 \quad \text{and} \quad v(+\infty, t) = c \quad \text{for each} \quad t>0, \tag{1.6}$$

$$v(x, 0) = v_0(x) = \int_{-\infty}^{x} u_0(\xi)d\xi \quad \text{on} \quad \boldsymbol{R}^1. \tag{1.7}$$

In Section 4, we show the uniqueness of non-negative monotone increasing solution of the problem (1.5)-(1.7), so that the uniqueness of a non-negative solution of the original problem (1.3), (1.4) can be obtained. In Section 5, as an approximation to (1.5)-(1.7), we consider the first boundary value problem for certain non-degenerate parabolic equations in an expanding sequence of cylinders and give some estimates on the derivatives by using a method similar to the ones used by Aronson [**1**], Gilding [**4, 6**] and Gilding and Peletier [**7**] to obtain a sharp Hölder exponent of solutions. In Section 6, using the results obtained from the problem of non-degenerate case, we prove the global existence theorem for (1.5)-(1.7) and then assert the same theorem for the original problem (1.3), (1.4).

Finally, in Sections 7 and 8, we state the regularity of solutions of the problem (1.3), (1.4) and the well known comparison theorem of the problem (1.5)-(1.7).

The asymptotic behavior of a solution of (1.3), (1.4) and the finite speed of propagation of disturbances are investigated in [**12**].

2. Main results.

Throughout this paper, we make the following assumptions:

(A.1) $\phi\in C^4$;

(A.2) The initial function u_0 is non-negative, bounded and integrable on $\boldsymbol{R}^1$.

In the case of the porous medium equation, that is, the equation (1.1) when $\phi\equiv 0$, it is known that classical solutions of the Cauchy problem for this equation do not always exist. For this reason, we have to define solutions of our problem (1.3), (1.4) in some generalized sense.

DEFINITION 2.1. A solution $u(x, t)$ of the Cauchy problem (1.3), (1.4) is defined by a non-negative and bounded function on $\boldsymbol{R}^1\times[0, \infty)$ which satisfies the following conditions:

(i) $u\in C(\boldsymbol{R}^1\times(0, \infty))\cap L^\infty_{\mathrm{loc}}([0, \infty); L^1(\boldsymbol{R}^1))$;

(ii) u^m has a weak derivative $(u^m)_x\in L^\infty(\boldsymbol{R}^1\times[\tau, T])$ for any $0<\tau<T<\infty$;

(iii) $\int_{-\infty}^x u(\xi, t)d\xi\in C(\boldsymbol{R}^1\times[0, \infty))$, $\int_{-\infty}^\infty u(\xi, t)d\xi\in C([0, \infty))$ and

$$\lim_{t\to 0+}\int_{-\infty}^x u(\xi, t)d\xi=\int_{-\infty}^x u_0(\xi)d\xi \quad \text{for any} \quad -\infty<x\leqq+\infty;$$

(iv) u satisfies the integral identity

$$\int_0^\infty\int_{-\infty}^\infty\left\{uf_t-\left[(u^m)_x+\phi'\left(\int_{-\infty}^x u(\xi, t)d\xi\right)u\right]f_x\right\}dxdt=0$$

for all $f\in C^1(\boldsymbol{R}^1\times(0, \infty))$ with compact support in $\boldsymbol{R}^1\times(0, \infty)$.

At first we shall state the existence and uniqueness results.

THEOREM 2.1. *The problem* (1.3), (1.4) *has a unique solution* $u(x, t)$ *which has the following properties:*

(i) $\int_{-\infty}^\infty u(x, t)dx=\int_{-\infty}^\infty u_0(x)dx$ *for any* $t\in(0, \infty)$;

(ii) *For any* $\tau\in(0, \infty)$ *there exists a positive constant* C_1 *depending on* m,

ϕ, $\|u_0\|_{L^1}$, $\|u_0\|_{L^\infty}$ *and* τ *such that for* $x, y \in \boldsymbol{R}^1$ *and* $\tau \leqq s, t < \infty$

$$|u^m(x, s) - u^m(y, t)| \leqq C_1[|x-y| + |s-t|^{1/2}]. \tag{2.1}$$

If u_0^m *is Lipschitz continuous, then* (2.1) *holds in* $\boldsymbol{R}^1 \times [0, \infty)$, *where* C_1 *depends on the Lipschitz constant of* u_0^m *instead of* τ;

(iii) *In a neighbourhood of a point in* $\boldsymbol{R}^1 \times (0, \infty)$ *where* u *is positive,* u_x, $(u^m)_{xx}$ *and* u_t *exist and are continuous, that is,* u *is a classical solution of* (1.3).

The regularity result is mentioned as follows.

THEOREM 2.2. *Let* u *be a solution of the problem* (1.3), (1.4). *Then* u *has the following properties:*

(i) *For any* $\tau \in (0, \infty)$ *there exists a positive constant* C_2 *depending on* m, ϕ, $\|u_0\|_{L^1}$, $\|u_0\|_{L^\infty}$ *and* τ *such that for* $x, y \in \boldsymbol{R}^1$ *and* $\tau \leqq s, t < \infty$

$$|u^{m-1}(x, s) - u^{m-1}(y, t)| \leqq C_2[|x-y| + |s-t|^{1/2}]. \tag{2.2}$$

If u_0^{m-1} *is Lipschitz continuous, then* (2.2) *holds in* $\boldsymbol{R}^1 \times [0, \infty)$, *where* C_2 *depends on the Lipschitz constant of* u_0^{m-1} *instead of* τ;

(ii) *The derivative* $(u^m)_x$ *exists and is continuous on* $\boldsymbol{R}^1 \times (0, \infty)$. *Moreover, if* $1 < m < 2$ *then* u_x *exists and is continuous on* $\boldsymbol{R}^1 \times (0, \infty)$.

We remark that the regularity result just mentioned above is the best possible for the porous medium equation (see [**1**]).

3. Reduction of the problem (1.3), (1.4) to (1.5)-(1.7).

To obtain the existence, uniqueness and regularity results, we shall transform the problem (1.3), (1.4) into a certain Cauchy problem. Let $u(x, t)$ be a solution of (1.3), (1.4) with the initial function u_0 and define the function $v(x, t)$ on $\boldsymbol{R}^1 \times [0, \infty)$ by

$$v(x, t) = \int_{-\infty}^{x} u(\xi, t) d\xi. \tag{3.1}$$

Integrating formally the equation (1.3) from $-\infty$ to x, we find that the function v is a solution of the following problem:

$$v_t = [(v_x)^m + \phi(v)]_x \quad \text{in} \quad \boldsymbol{R}^1 \times (0, \infty), \tag{3.2}$$

$$v(-\infty, t) = 0 \quad \text{and} \quad v(+\infty, t) = c \quad \text{for each} \quad t \geqq 0, \tag{3.3}$$

$$v_x(x, t) \geqq 0 \quad \text{on} \quad \boldsymbol{R}^1 \times (0, \infty), \tag{3.4}$$

$$v(x, 0) = v_0(x) \quad \text{on} \quad \boldsymbol{R}^1, \tag{3.5}$$

where $v_0(x) = \int_{-\infty}^{x} u_0(\xi) d\xi$ and $c = \int_{-\infty}^{\infty} u_0(\xi) d\xi$. Conversely, if $v(x, t)$ is a solution of the problem (3.2)-(3.5) and the function $u(x, t)$ is defined by

$$u(x, t) = v_x(x, t), \tag{3.6}$$

then, differentiating formally the equation (3.2) with respect to x, we see that u is a solution of (1.3), (1.4). The details will be discussed below.

First we define a solution of the problem (3.2)-(3.5) with the initial function v_0, where v_0 satisfies the following condition:

(A.3) v_0 is a non-decreasing and Lipschitz continuous function on $\boldsymbol{R}^1$ such that for a constant c, $0 \leqq v_0 \leqq c$, $v_0(-\infty)=0$ and $v_0(+\infty)=c$.

DEFINITION 3.1. A solution of the problem (3.2)-(3.5) is defined by a continuous function $v(x, t)$ on $\boldsymbol{R}^1 \times [0, \infty)$ which satisfies

(i) $0 \leqq v(x, t) \leqq c$ on $\boldsymbol{R}^1 \times [0, \infty)$, and for each $t \in [0, \infty)$, $v(-\infty, t)=0$ and $v(+\infty, t)=c$;

(ii) $v(x, 0)=v_0(x)$ on $\boldsymbol{R}^1$;

(iii) v_x is non-negative and bounded on $\boldsymbol{R}^1 \times [0, \infty)$ and continuous on $\boldsymbol{R}^1 \times (0, \infty)$;

(iv) $((v_x)^m)_x \in L^\infty(\boldsymbol{R}^1 \times [\tau, T])$ for any $0<\tau<T<\infty$;

(v) $\int_0^\infty \int_{-\infty}^\infty \left\{v f_t - [(v_x)^m + \phi(v)] f_x\right\} dx dt = 0$

for all $f \in C^1(\boldsymbol{R}^1 \times (0, \infty))$ with compact support in $\boldsymbol{R}^1 \times (0, \infty)$.

Let u be a solution of the problem (1.3), (1.4) with the initial function u_0. By a similar calculation to that in the proof of Theorem 1 in [**5**], we have

PROPOSITION 3.1. *For each* $t \in [0, \infty)$

$$\int_{-\infty}^\infty u(x, t) dx = \int_{-\infty}^\infty u_0(x) dx .$$

By virtue of Proposition 3.1, we can obtain the following relation between solutions of two problems (1.3), (1.4) and (3.2)-(3.5).

PROPOSITION 3.2. *Let u be a solution of the problem* (1.3), (1.4). *Then the function v defined by* (3.1) *is a solution of the problem* (3.2)-(3.5). *Conversely, if v is a solution of the problem* (3.2)-(3.5), *then the function u defined by* (3.6) *is a solution of the problem* (1.3), (1.4).

PROOF. It is easy to prove the second part of the assertion. Hence, we only prove the first part.

Let u be a solution of (1.3), (1.4) with the initial function u_0. Define the function v by (3.1) and put

$$v_0(x) = \int_{-\infty}^x u_0(\xi) d\xi \quad \text{and} \quad c = \int_{-\infty}^\infty u_0(\xi) d\xi .$$

It can easily be seen that v satisfies the condition (i) of Definition 3.1 by using Proposition 3.1 and that the conditions (ii)-(iv) are fulfilled. Let us prove the condition (v). For any function $g \in C^1(\boldsymbol{R}^1 \times (0, \infty))$ of which support is included in a rectangle $(a, b) \times (0, T)$, we define the function $G(x, t)$ on $\boldsymbol{R}^1 \times (0, \infty)$ by

$$G(x, t)=\int_x^\infty g(\xi, t)d\xi .$$

For any positive integer N with $N>|a|, |b|$, we define the function $\chi_N(x)\in C^\infty(\boldsymbol{R}^1)$ such that $0\leqq\chi\leqq 1$, $\chi_N(x)=1$ for $|x|\leqq N$, $\chi_N(x)=0$ for $|x|\geqq N+1$, $\|\chi'_N\|_{L^\infty}\leqq M_1$ and $\|\chi''_N\|_{L^\infty}\leqq M_1$, where M_1 is a positive constant independent of N. Substituting the function $f(x, t)=\chi_N(x)G(x, t)$ into the condition (iv) of Definition 2.1, and integrating by parts, we obtain

$$\int_0^\infty\int_{-\infty}^\infty \left\{vg_t-[(v_x)^m+\phi(v)]g_x\right\}\chi_N dxdt$$

$$=\int_0^\infty\int_{-\infty}^\infty vG_t\chi'_N dxdt+\int_0^\infty\int_{-\infty}^\infty (v_x)^m G\chi''_N dxdt$$

$$+\int_0^\infty\int_{-\infty}^\infty \phi(v)G\chi''_N dxdt= \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

Noting that I is written as

$$\mathrm{I}=\int_0^T\int_{-N-1}^{-N} vG_t\chi'_N dxdt ,$$

we have

$$|\,\mathrm{I}\,|\leqq M_1\|G_t\|_{L^\infty}\int_0^T v(-N, t)dt\to 0 \quad \text{as} \quad N\to\infty.$$

Next, by using the facts that u is bounded on $\boldsymbol{R}^1\times[0, \infty)$ and $x\to u(x, t)$ is integrable on $\boldsymbol{R}^1$, we get

$$|\,\mathrm{II}\,|\leqq M_1\|G\|_{L^\infty}\int_0^T\int_{-N-1}^{-N} u^m dxdt\to 0 \quad \text{as} \quad N\to\infty.$$

Lastly, integrating by parts and estimating the resulting equality, we obtain

$$|\,\mathrm{III}\,|\leqq M_1\|G\|_{L^\infty}\Big(\sup_{0\leq v\leq c}|\phi'(v)|\Big)\int_0^T\int_{-N-1}^{-N} udxdt\to 0 \quad \text{as} \quad N\to\infty.$$

Hence, it holds that for all $g\in C^1(\boldsymbol{R}^1\times(0, \infty))$ with compact support in $\boldsymbol{R}^1\times(0, \infty)$,

$$\int_0^\infty\int_{-\infty}^\infty \{vg_t-[(v_x)^m+\phi(v)]g_x\}\,dxdt=0 .$$

Thus the proof is completed.

Proposition 3.2 has established a one-to-one correspondence between two solutions of (1.3), (1.4) and (3.2)-(3.5), which are mutually combined by (3.1) and (3.6). Hereafter, in order to show the existence, uniqueness and regularity results of the problem (1.3), (1.4) we may consider the problem (3.2)-(3.5).

4. Uniqueness.

THEOREM 4.1. *There exists at most one solution of the problem* (3.2)-(3.5). *Hence, the problem* (1.3), (1.4) *has at most one solution.*

PROOF. We first remark that the solution v of the problem (3.2)-(3.5) satisfies that

$$((v_x)^m)_x,\ v_t \in L^\infty(\boldsymbol{R}^1\times[\tau, T]) \quad \text{for any} \quad 0<\tau<T<\infty$$

and

$$v_t=[(v_x)^m+\phi(v)]_x \quad \text{a.e. in} \quad \boldsymbol{R}^1\times(0, \infty).$$

Let v and w be solutions of the problem (3.2)-(3.5) with the same initial function v_0. By the remark mentioned above, we have

$$(v-w)_t=[(v_x)^m-(w_x)^m]_x+[\phi(v)-\phi(w)]_x \quad \text{a.e. in} \quad \boldsymbol{R}^1\times(0, \infty). \tag{4.1}$$

Multiply (4.1) by $[v(x, t)-w(x, t)]\chi_N(x)$ by using the cut-off function $\chi_N(x)$ used in Proposition 3.2 and integrate over $\boldsymbol{R}^1\times[\tau, T]$, where positive constants τ and T are arbitrarily fixed. Then in the resulting equation the integration by parts yields

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty}&[v(x, T)-w(x, T)]^2\chi_N(x)dx\\ &\leqq\frac{1}{2}\int_{-\infty}^{\infty}[v(x, \tau)-w(x, \tau)]^2\chi_N(x)dx\\ &\quad-\int_{\tau}^{T}\int_{-\infty}^{\infty}[(v_x)^m-(w_x)^m](v-w)\chi_N' dxdt\\ &\quad+\int_{\tau}^{T}\int_{-\infty}^{\infty}(v-w)\chi_N[\phi(v)-\phi(w)]_x dxdt.\end{aligned}$$

Letting $\tau\to 0$ in this inequality, we find from $v(x, 0)=w(x, 0)$ that

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty}&[v(x, T)-w(x, T)]^2\chi_N(x)dx\\ &\leqq-\int_{0}^{T}\int_{-\infty}^{\infty}[(v_x)^m-(w_x)^m](v-w)\chi_N' dxdt\\ &\quad+\int_{0}^{T}\int_{-\infty}^{\infty}(v-w)\chi_N[\phi(v)-\phi(w)]_x dxdt=\mathrm{I}+\mathrm{II}.\end{aligned} \tag{4.2}$$

We here note that $0\leqq v,\ w\leqq c$ on $\boldsymbol{R}^1\times[0, \infty)$, and the functions v_x and w_x are non-negative and bounded on $\boldsymbol{R}^1\times[0, \infty)$ and belong to $L^1(\boldsymbol{R}^1\times[0, T])$. Hence, letting $N\to\infty$, we have

$$\mathrm{I}=-\int_0^T\int_{N\leq|x|\leq N+1}[(v_x)^m-(w_x)^m](v-w)\chi_N' dxdt\to 0$$

and

$$\mathrm{II}\to\int_0^T\int_{-\infty}^{\infty}(v-w)[\phi(v)-\phi(w)]_x dxdt.$$

Therefore, the function $x\to[v(x, T)-w(x, T)]$ belongs to $L^2(\boldsymbol{R}^1)$ and the inequality (4.2) yields

$$\frac{1}{2}\int_{-\infty}^{\infty}[v(x, T)-w(x, T)]^2dx \leqq \int_0^T\int_{-\infty}^{\infty}(v-w)[\phi(v)-\phi(w)]_x dxdt. \tag{4.3}$$

By noting that $v(x, T)-w(x, T)\to 0$ as $x\to\pm\infty$ the integration by parts on the right hand side of (4.3) permits us to rewrite (4.3) as

$$\frac{1}{2}\int_{-\infty}^{\infty}[v(x, T)-w(x, T)]^2dx \leqq -\int_0^T\int_{-\infty}^{\infty}[\phi(v)-\phi(w)](v-w)_x dxdt. \tag{4.4}$$

We now estimate the right hand side of (4.4). Let us put

$$\phi(v(x, t))-\phi(w(x, t))=A(x, t)[v(x, t)-w(x, t)],$$

where

$$A(x, t)=\int_0^1\phi'(\theta v(x, t)+(1-\theta)w(x, t))d\theta.$$

Then, it follows from integration by parts that

$$-\int_0^T\int_{-\infty}^{\infty}[\phi(v)-\phi(w)](v-w)_x dxdt=\frac{1}{2}\int_0^T\int_{-\infty}^{\infty}A_x(v-w)^2dxdt. \tag{4.5}$$

Here we note from the definition of A that

$$|A_x| \leqq \sup_{0\leqq\sigma\leqq c}|\phi''(\sigma)|\cdot\sup_{\substack{0<t<\infty\\ -\infty<x<\infty}}|(v+w)_x| \equiv K<+\infty. \tag{4.6}$$

Combining (4.4) with (4.5) and (4.6) yields that for any $0<T<\infty$

$$\int_{-\infty}^{\infty}[v(x, T)-w(x, T)]^2dx \leqq K\int_0^T\int_{-\infty}^{\infty}[v(x, t)-w(x, t)]^2dxdt,$$

which implies

$$[v(x, T)-w(x, T)]^2=0 \quad \text{for} \quad x\in \boldsymbol{R}^1 \quad \text{and} \quad T\in(0, \infty).$$

This completes the proof.

5. Auxiliary lemmas for the existence and regularity.

As will be shown later, we shall construct a solution of the problem (3.2)-(3.5) as a limit of a sequence of solutions of the first boundary value problems for certain non-degenerate parabolic equation in an expanding sequence of cylinders. For this purpose, we prepare some lemmas.

We first introduce some notations which will be used later. Let Q be a domain in $\boldsymbol{R}^1\times(0, \infty)$. We denote by $C^{2,1}(Q)$ the set of functions $u(x, t)$ defined on Q which are continuous with their derivatives u_t, u_x and u_{xx}. Analogously, for the closure of Q, say $\bar{Q}$, we introduce the notation $C^{2,1}(\bar{Q})$. For a function $u(x, t)$ on Q we introduce the notation

$$|u|_{\alpha,Q}=\sup_Q|u(x, t)|+\sup_{(x,s),(y,t)\in Q}\frac{|u(x, s)-u(y, t)|}{[|x-y|^2+|s-t|]^{\alpha/2}}$$

where $0<\alpha\leqq 1$. If u_x, u_{xx} and u_t exist in Q we introduce

$$|u|_{2+\alpha,Q}=|u|_{\alpha,Q}+|u_x|_{\alpha,Q}+|u_{xx}|_{\alpha,Q}+|u_t|_{\alpha,Q}.$$

The set of all functions for which $|u|_{\alpha,Q}<\infty$ is denoted by $C^{\alpha,\alpha/2}(\bar{Q})$. By $C^{2+\alpha,1+\alpha/2}(\bar{Q})$ we mean the set of all functions such that $|u|_{2+\alpha,Q}<\infty$. For a positive integer n and a positive number T we put

$$Q_n(T)=(-n, n)\times(0, T] \quad\text{and}\quad Q_n=(-n, n)\times(0, \infty).$$

Let $v_0(x)$ be an infinitely differentiable function on $[-n, n]$ such that $v_0'(x)\geqq 0$ on $[-n, n]$, $0\leqq v_0(x)\leqq c$ on $[-n, n]$, $v_0(x)=0$ for $-n\leqq x\leqq -n+1$ and $v_0(x)=c$ for $n-1\leqq x\leqq n$. For any fixed sufficiently small $\varepsilon>0$, we consider the following problem:

$$v_t=[(v_x+\varepsilon)^m+\phi(v)]_x \quad\text{in}\quad Q_n, \tag{5.1}$$

$$v(-n, t)=0 \quad\text{and}\quad v(n, t)=c \quad\text{for}\quad t\in[0, \infty), \tag{5.2}$$

$$v(x, 0)=v_0(x) \quad\text{for}\quad x\in[-n, n]. \tag{5.3}$$

LEMMA 5.1. *The problem* (5.1)-(5.3) *has a unique* (*classical*) *solution* v *in* Q_n *satisfying the following properties:*

(i) $0\leqq v\leqq c$ *in* $\bar{Q}_n$;

(ii) $v_x\geqq 0$ *in* $\bar{Q}_n$ *and* $v_x>0$ *in* Q_n;

(iii) *There exists an* α *with* $0<\alpha\leqq 1$ *such that* $v\in C^{2+\alpha,1+\alpha/2}(\overline{Q_n(T)})$ *for any fixed* T *with* $0<T<\infty$;

(iv) $v_{xx}\in C^{2,1}(Q_n)$.

PROOF. Let $f(p)$ be the smooth function on $\boldsymbol{R}^1$ such that $f(p)=m(p+\varepsilon)^{m-1}$ for $p\geqq 0$, $f(p)\geqq m(\varepsilon/2)^{m-1}$ on $\boldsymbol{R}^1$ and there are positive constants ν and μ satisfying

$$\nu(|p|+\varepsilon)^{m-1}\leqq f(p)\leqq \mu(|p|+\varepsilon)^{m-1} \quad\text{on}\quad \boldsymbol{R}^1.$$

Then, for the equation

$$v_t=f(v_x)v_{xx}+\phi'(v)v_x \quad\text{in}\quad Q_n \tag{5.4}$$

and the initial-boundary conditions (5.2) and (5.3), Theorem 4.1 in [**11**; p. 558] shows that there exists uniquely a function v having the property (iii). The standard maximum principle yields that $0\leqq v\leqq c$ in $\bar{Q}_n$. Since $\phi\in C^4$, the property (iv) can be shown by virtue of a standard argument in [**3**].

Finally, we verify the property (ii) which yields that v is a solution of the problem (5.1)-(5.3) in Q_n. Differentiate the equation (5.1) with respect to x and write $w=v_x$. We then have

$$w_t=f(v_x)w_{xx}+[f'(v_x)v_{xx}+\phi'(v)]w_x+[\phi''(v)v_x]w \quad\text{in}\quad Q_n.$$

We note that $w(t, 0)=v_0'(x)\geqq 0$ in $[-n, n]$ and that $w(\pm n, t)\geqq 0$ for $t\in[0, \infty)$ by using the property (i) and the boundary condition (5.2). Hence, applying the maximum principle, we know that

$$w(x, t) \geqq 0 \quad \text{in} \quad \bar{Q}_n \quad \text{and} \quad w(x, t)>0 \quad \text{in} \quad Q_n,$$

which implies the property (ii). This completes the proof.

Next we shall give the boundedness of v_x by using Bernstein's method. A technique similar to ours was used by Aronson [1] and Gilding and Peletier [7] to obtain a sharp Hölder exponent for solutions of the porous medium equation.

LEMMA 5.2. *Let v be a solution of the problem* (5.1)-(5.3). *Then we have*

$$0 \leqq v_x \leqq C_1 \quad on \quad \overline{Q_{n-1}}$$

where C_1 is a constant depending only on m, ϕ, c and $\|v_0'\|_{L^\infty}$.

PROOF. Define the function $\varphi(w)$ by

$$\varphi(w)=-2c+6ce\int_0^w e^{-\xi^q} d\xi,$$

where q is a constant satisfying

$$q[1-(m-1)2^{-q}]=2.$$

Here we note $q>2$. Let us determine the range of variation $[w_1, w_2]$ of w when $\varphi(w)$ varies from 0 to c. w_1 and w_2 are given by

$$\int_0^{w_1} e^{-\xi^q} d\xi=\frac{1}{3e} \quad \text{and} \quad \int_0^{w_2} e^{-\xi^q} d\xi=\frac{1}{2e}.$$

It is obvious that $\varphi(w_1)=0$, $\varphi(w_2)=c$ and

$$\frac{1}{3e}<w_1<w_2<\frac{1}{2}.$$

For $w \in [w_1, w_2]$ we have

$$\varphi'=6cee^{-w^q}>0, \quad \varphi''=-6ceqw^{q-1}e^{-w^q}<0,$$
$$\frac{\varphi''}{\varphi'}=-qw^{q-1}, \quad \left(\frac{\varphi''}{\varphi'}\right)'=-q(q-1)w^{q-2}<0. \tag{5.5}$$

We now define $w(x, t)$ by

$$v(x, t)=\varphi(w(x, t)) \quad \text{for} \quad x \in \boldsymbol{R}^1 \quad \text{and} \quad t \in [0, \infty). \tag{5.6}$$

Substituting (5.6) into (5.1), we have

$$w_t=m(\varphi' w_x+\varepsilon)^{m-1} w_{xx}+m\frac{\varphi''}{\varphi'}(\varphi' w_x+\varepsilon)^{m-1}(w_x)^2+\phi'(\varphi) w_x. \tag{5.7}$$

Differentiate (5.7) with respect to x and then put $p=w_x$, which is non-negative. Then, we have in Q_n

$$\begin{aligned} p_t-m(\varphi' p+\varepsilon)^{m-1} p_{xx} &= m(m-1)(\varphi' p+\varepsilon)^{m-2}(\varphi' p_x+\varphi'' p^2) p_x \\ &+m(m-1)(\varphi' p+\varepsilon)^{m-2}(\varphi' p_x+\varphi'' p^2)\frac{\varphi''}{\varphi'} p^2 \\ &+m(\varphi' p+\varepsilon)^{m-1}\left(\frac{\varphi''}{\varphi'}\right)' p^3+2m(\varphi' p+\varepsilon)^{m-1}\frac{\varphi''}{\varphi'} p p_x+\phi' p_x+\phi''\varphi' p^2. \end{aligned} \tag{5.8}$$

We put $z(x, t)=\chi(x)p(x, t)$, where $\chi(x)$ is a smooth function on $\boldsymbol{R}^1$ such that $0\leqq\chi(x)\leqq 1$ on $\boldsymbol{R}^1$, $\chi(x)=1$ for $|x|\leqq n-1$ and $\chi(x)=0$ for $|x|\geqq n-1/2$. For any fixed T with $0<T<\infty$, let us estimate the value of $z(x, t)$ in $\overline{Q_n(T)}=[-n, n]\times[0, T]$. Suppose that the point where the function z takes the maximum in $\overline{Q_n(T)}$ lies on the lower base of $\overline{Q_n(T)}$. We then have

$$0\leqq z(x, t)\leqq\|p(\cdot, 0)\|_{L^\infty} \quad \text{on} \quad \overline{Q_n(T)}$$

and hence

$$0\leqq v_x(x, t)\leqq e\|v_0'\|_{L^\infty} \quad \text{on} \quad \overline{Q_{n-1}(T)}. \tag{5.9}$$

Let the maximum of z be attained either inside $Q_n(T)$ or on the upper base of $Q_n(T)$. At this point, say (x_0, t_0), we have

$$z_x=\chi p_x+\chi_x p=0 \tag{5.10}$$

and

$$m(\varphi'p+\varepsilon)^{m-1}z_{xx}-z_t\leqq 0,$$

which can be rewritten in the form

$$\chi\{p_t-m(\varphi'p+\varepsilon)^{m-1}p_{xx}\}\geqq 2m(\varphi'p+\varepsilon)^{m-1}\chi_x p_x+m(\varphi'p+\varepsilon)^{m-1}\chi_{xx}p. \tag{5.11}$$

Substituting (5.8) into (5.11) and then multiplying it by χ, we obtain

$$\begin{aligned}
&-m\Big(\frac{\varphi''}{\varphi'}\Big)'(\varphi'p+\varepsilon)^{m-1}\chi^2p^3-m(m-1)\frac{\varphi''}{\varphi'}\varphi''(\varphi'p+\varepsilon)^{m-2}\chi^2p^4\\
&\leqq -2m\Big\{(m-1)\varphi''(\varphi'p+\varepsilon)^{m-2}\chi_x\chi p^3+\frac{\varphi''}{\varphi'}(\varphi'p+\varepsilon)^{m-1}\chi_x\chi p^2\Big\}\\
&\quad +m\{(m-1)\varphi'(\varphi'p+\varepsilon)^{m-2}(\chi_x p)^2-(\varphi'p+\varepsilon)^{m-1}\chi_{xx}\chi p+2(\varphi'p+\varepsilon)^{m-1}(\chi_x)^2p\}\\
&\quad +\{\phi''\varphi'\chi^2p^2-\phi'\chi_x\chi p\} = \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned} \tag{5.12}$$

Here we used the relation (5.10) at $(x, t)=(x_0, t_0)$. It follows from (5.5) and the choice of q that

$$\begin{aligned}
&-m\Big(\frac{\varphi''}{\varphi'}\Big)'(\varphi'p+\varepsilon)^{m-1}\chi^2p^3-m(m-1)\frac{\varphi''}{\varphi'}\varphi''(\varphi'p+\varepsilon)^{m-2}\chi^2p^4\\
&\geqq m(\varphi'p+\varepsilon)^{m-2}\chi^2p^4\varphi'\Big\{-\Big(\frac{\varphi''}{\varphi'}\Big)'-(m-1)\Big(\frac{\varphi''}{\varphi'}\Big)^2\Big\}\\
&\geqq 6mcw_1^{q-2}(\varphi'p+\varepsilon)^{m-2}\chi^2p^4.
\end{aligned} \tag{5.13}$$

Next, by using (5.5) we get

$$\begin{aligned}
&\mathrm{I}\leqq 2m^2q\|\chi_x\|_{L^\infty}(\varphi'p+\varepsilon)^{m-1}\chi p^2,\\
&\mathrm{II}\leqq m(m+1)\max[(\|\chi_x\|_{L^\infty})^2, \|\chi_{xx}\|_{L^\infty}](\varphi'p+\varepsilon)^{m-1}p,\\
&\mathrm{III}\leqq 6ce[\max_{0\leq v\leq c}|\phi''(v)|]\chi^2p^2+\{[\max_{0\leq v\leq c}|\phi'(v)|]\|\chi_x\|_{L^\infty}\}\chi p.
\end{aligned} \tag{5.14}$$

Combining (5.12) with (5.13) and (5.14), we find that there exists a positive constant M_1, depending only on m, c, $\max_{0\le v\le c}|\phi'(v)|$, $\max_{0\le v\le c}|\phi''(v)|$, $\|\chi_x\|_{L^\infty}$ and $\|\chi_{xx}\|_{L^\infty}$, such that

$$(\varphi'p+\varepsilon)^{m-2}\chi^2p^4\leqq M_1\{(\varphi'p+\varepsilon)^{m-1}\chi p^2+(\varphi'p+\varepsilon)^{m-1}p+\chi^2p^2+\chi p\}.$$

Hence,

$$\chi^2p^4\leqq M_1\{(\varphi'p+1)\chi p^2+(\varphi'p+1)p+(\varphi'p+\varepsilon)^{2-m}(\chi^2p^2+\chi p)\}. \tag{5.15}$$

It is enough to assume $p\geqq 1$. The relation $\varphi'(w)=6cee^{-w^q}$ yields that

$$(\varphi'p+\varepsilon)^{2-m}\leqq\begin{cases}(6cep+1)^{2-m} & \text{if } 1<m<2,\\ (6c)^{2-m} & \text{if } m\geqq 2.\end{cases} \tag{5.16}$$

By (5.15) and (5.16), we obtain

$$\chi p\leqq M_2 \tag{5.17}$$

for a positive constant M_2 depending only on m, c, $\max_{0\le v\le c}|\phi'(v)|$, $\max_{0\le v\le c}|\phi''(v)|$, $\|\chi_x\|_{L^\infty}$ and $\|\chi_{xx}\|_{L^\infty}$. The inequality (5.17) implies

$$0\leqq v_x(x,t)\leqq 6ceM_2 \quad \text{on} \quad \overline{Q_{n-1}(T)}. \tag{5.18}$$

Since T is arbitrary and M_2 is independent of T, we obtain Lemma 5.2 by the inequalities (5.9) and (5.18).

LEMMA 5.3. *For any $\tau\in(0,\infty)$, (resp. $\tau=0$), there holds*

$$|((v_x+\varepsilon)^m)_x|\leqq C_2 \quad on \quad [-n+2, n-2]\times[\tau,\infty), \tag{5.19}$$

where C_2 is a constant depending only on m, ϕ, c, $\|v_0'\|_{L^\infty}$ and τ (resp. $\|((v_0')^m)'\|_{L^\infty}$).

PROOF. Differentiate (5.1) with respect to x and put $u=v_x$. We then have

$$u_t=[m(u+\varepsilon)^{m-1}u_x]_x+\phi'(v)u_x+\phi''(v)u^2 \quad \text{in} \quad Q_n. \tag{5.20}$$

If follows from (ii) of Lemma 5.1 that $u\geqq 0$ in $\overline{Q}_n$ and $u>0$ in Q_n.

Define the function $\varphi(u)$ by

$$\varphi(u)=\int_0^u\frac{a(s)}{\theta(s)}ds \quad \text{for} \quad 0\leqq u\leqq C_1,$$

where C_1 is the constant used in Lemma 5.2. The form of $\varphi(u)$ is the one introduced by Gilding [**6**]. Here a and θ are respectively specified as

$$a(s)=m(s+\varepsilon)^{m-1}$$

and

$$\theta(s)=\left[\int_0^s ra'(r)dr+2sa(C_1)-sa(s)+s+1\right]^{1/2}$$

for $0\leqq s\leqq C_1$. We then have the following relations:

$$\theta'(s)=\frac{1}{2}[2a(C_1)-a(s)+1]\frac{1}{\theta(s)}>0,$$

$$\theta''(s)=-\frac{1}{2}[a'(s)+2(\theta'(s))^2]\frac{1}{\theta(s)}<0. \tag{5.21}$$

$$0 \leqq a'(s)\theta(s) \leqq -2\theta^2(s)\theta''(s), \quad 0 \leqq a(s)\theta'(s) \leqq -\theta^2(s)\theta''(s),$$
$$0 \leqq -\frac{a(s)}{\theta(s)\theta''(s)} \leqq \frac{2}{m-1}(s+\varepsilon).$$

Since $0 \leqq u(x, t) \leqq C_1$ in $\overline{Q_{n-1}}$ by Lemma 5.2, we can define the function $w(x, t)$ by

$$w(x, t)=\varphi(u(x, t)) \quad \text{for} \quad (x, t) \in \overline{Q_{n-1}}. \tag{5.22}$$

Substituting (5.22) into (5.20), we get

$$w_t=a(u)w_{xx}+\theta'(u)(w_x)^2+\phi' w_x+\frac{a(u)}{\theta(u)}\phi'' u^2.$$

Differentiate this equation with respect to x and then multiply it by w_x. Then, writing $p=w_x$, we obtain

$$\begin{aligned} &\frac{1}{2}(p^2)_t-a(u)pp_{xx} \\ &=\left[\frac{a'(u)}{a(u)}\theta(u)+2\theta'(u)\right]p^2p_x+\frac{1}{a(u)}\theta(u)\theta''(u)p^4 \\ &\quad+\left\{3\phi'' u+\left[\frac{a'(u)}{a(u)}-\frac{\theta'(u)}{\theta(u)}\right]\phi'' u^2\right\}p^2+\phi' pp_x+\frac{a(u)}{\theta(u)}\phi''' u^3 p. \end{aligned} \tag{5.23}$$

We put $z(x, t)=\chi^2(x, t)p^2(x, t)$. Here χ is a smooth function on $\boldsymbol{R}^1 \times [0, \infty)$ such that $0 \leqq \chi(x, t) \leqq 1$ on $\boldsymbol{R}^1 \times [0, \infty)$, $\chi(x, t)=1$ on $[-n+2, n-2] \times [\tau, \infty)$ and $\chi(x, t)=0$ on the outside of $[-n+3/2, n-3/2] \times [\tau/2, \infty)$, where τ is any fixed constant with $0<\tau<\infty$. For an arbitrary fixed T with $0<T<\infty$, let us consider a point (x_0, t_0) where z attains a positive maximum over $\overline{Q_{n-1}(T)}$. At the point (x_0, t_0) we have

$$z_x=0 \quad \text{and} \quad a(u)z_{xx}-z_t \leqq 0,$$

which yield

$$\chi p_x=-\chi_x p \tag{5.24}$$

and

$$\begin{aligned} &\chi^2\left\{\frac{1}{2}(p^2)_t-a(u)pp_{xx}\right\} \\ &\geqq a(u)\{\chi^2(p_x)^2+4\chi\chi_x pp_x+\chi p_{xx}p^2+(\chi_x)^2p^2\}+\chi\chi_t p^2. \end{aligned} \tag{5.25}$$

Substituting (5.23) into (5.25) and using (5.24), we obtain

$$\begin{aligned} -\frac{1}{a(u)}\theta(u)\theta''(u)\chi^2p^4 \leqq &-\left[\frac{a'(u)}{a(u)}\theta(u)+2\theta'(u)\right]\chi_x\chi p^3 \\ &+\left\{-\phi'\chi\chi_x+3\phi'' u\chi^2+\left[\frac{a'(u)}{a(u)}-\frac{\theta'(u)}{\theta(u)}\right]\phi'' u^2\chi^2\right. \\ &\left.+a(u)[-2(\chi_x)^2+\chi\chi_{xx}]-\chi\chi_t\right\}p^2+\frac{a(u)}{\theta(u)}\phi''' u^3\chi^2 p. \end{aligned}$$

Noting $\theta''<0$, at this point we have

$$(5.26)\qquad \chi^2 p^4 \leqq \left[\frac{a'(u)}{\theta''(u)}+\frac{2a(u)\theta'(u)}{\theta(u)\theta''(u)}\right]\chi_x \chi p^3$$
$$-\frac{a(u)}{\theta(u)\theta''(u)}\left[\frac{a'(u)}{a(u)}-\frac{\theta'(u)}{\theta(u)}\right]\phi'' u^2\chi^2 p^2$$
$$-\frac{a(u)}{\theta(u)\theta''(u)}\{-\phi'\chi\chi_x+3\phi'' u\chi^2+a(u)[-2(\chi_x)^2+\chi\chi_{xx}]-\chi\chi_t\}p^2$$
$$+\frac{-a(u)}{\theta(u)\theta''(u)}\cdot\frac{a(u)}{\theta(u)}\phi''' u^3\chi^2 p.$$

It is enough to assume $|p|\geqq 1$. It follows from (5.21) that

$$\left|\frac{a'(u)}{\theta''(u)}+\frac{2a(u)\theta'(u)}{\theta(u)\theta''(u)}\right|\leqq 4\theta(C_1),$$
$$\left|\frac{a(u)}{\theta(u)\theta''(u)}\left[\frac{a'(u)}{a(u)}-\frac{\theta'(u)}{\theta(u)}\right]\right|\leqq\frac{-1}{\theta^2(u)\theta''(u)}|a'(u)\theta(u)-a(u)\theta'(u)|\leqq 3,$$
$$\left|\frac{a(u)}{\theta(u)\theta''(u)}\cdot\frac{a(u)}{\theta(u)}\right|\leqq\frac{2}{m-1}(C_1+1)a(C_1).$$

Combining (5.26) with the inequalities mentioned just above and noting $|p|\geqq 1$, we have

$$(5.27)\qquad (\chi p)^2\leqq 4\theta(C_1)\|\chi_x\|_{L^\infty}\chi|p|+3C_1^2\max_{0\leqq v\leqq c}|\phi(v)|+M_1,$$

where M_1 is a positive constant depending only on m, c, C_1, $\max_{0\leqq v\leqq c}|\phi'(v)|$, $\max_{0\leqq v\leqq c}|\phi''(v)|$, $\max_{0\leqq v\leqq c}|\phi'''(v)|$, $\|\chi_x\|_{L^\infty}$, $\|\chi_{xx}\|_{L^\infty}$ and $\|\chi_t\|_{L^\infty}$. The inequality (5.27) implies that

$$(\chi p)^2\leqq M_2$$

for a positive constant M_2 depending on C_1 and M_1. We note that

$$((u+\varepsilon)^m)_x=a(u)u_x=\theta(u)w_x.$$

Hence, at the point (x_0, t_0) we obtain

$$|\chi((u+\varepsilon)^m)_x|\leqq\theta(C_1)M_2^{1/2}\leqq\{[2mC_1(C_1+1)^{m-1}+C_1+1]M_2\}^{1/2}.$$

Putting $C_2=\{[2mC_1(C_1+1)^{m-1}+C_1+1]M_2\}^{1/2}$, we get the proof of the first part.

To prove the second part of the assertion we observe that $((v_x+\varepsilon)^m)_x$ is bounded at $t=0$. Hence we may take a function χ which depends only on x and allow z to attain its maximum at a point on the lower base of $Q_{n-1}(T)$. Except these consideration, the proof is the same. Thus the proof is completed.

To show the regularity result we shall need the following

LEMMA 5.4. *For any* $\tau\in(0,\infty)$ (*resp.* $\tau=0$) *it holds*

$$(5.28)\qquad |((v_x+\varepsilon)^{m-1})_x|\leqq C_3 \quad on \quad [-n+2, n-2]\times[\tau,\infty),$$

where C_3 *is a constant depending only on* m, ϕ, c, $\|v_0'\|_{L^\infty}$ *and* τ *(resp.* $\|((v_0')^{m-1})'\|_{L^\infty}$*).*

PROOF. Differentiate (5.1) with respect to x and put $u=v_x$. We then have

$$u_t=((u+\varepsilon)^m)_{xx}+\phi'(v)u_x+\phi''(v)u^2.$$

By putting $w=(u+\varepsilon)^{m-1}$, this equation is rewritten as

$$w_t=mww_{xx}+\frac{m}{m-1}(w_x)^2+\phi'w_x+(m-1)\phi'w^{1-1/(m-1)}(w^{1/(m-1)}-\varepsilon)^2. \tag{5.29}$$

Consider the function $\varphi(z)$ defined by

$$\varphi(z)=-2M_1+6eM_1\int_0^z e^{-\xi q}d\xi\,.$$

Here $M_1=(C_1+1)^{m-1}$, where C_1 is the constant used in Lemma 5.2, and q satisfies

$$6q(6e)^{1-q}\geqq(m-1)m^{-2}.$$

We define the function $z(x, t)$ by

$$w(x, t)=\varphi(z(x, t)) \quad \text{on} \quad \overline{Q_{n-1}}.$$

It follows from (5.29) that

$$z_t-m\varphi z_{xx}=m\Big[\varphi\frac{\varphi''}{\varphi'}+\frac{1}{m-1}\varphi'\Big](z_x)^2+\phi'z_x$$
$$+(m-1)\phi''\frac{1}{\varphi'}\varphi^{1-1/(m-1)}(\varphi^{1/(m-1)}-\varepsilon)^2\,.$$

By using the method almost analogous to the one used to prove Lemmas 5.2 and 5.3, we can prove that for any fixed $\tau>0$

$$|z_x|\leqq M_2 \quad \text{on} \quad [-n+2, n-2]\times[\tau, \infty)\,, \tag{5.30}$$

where a positive constant M_2 depends only on m, ϕ, c, $\|v_0'\|_{L^\infty}$ and τ.

The inequality (5.30) yields

$$|w_x|\leqq M_3 \quad \text{on} \quad [-n+2, n-2]\times[\tau, \infty)\,,$$

for a constant M_3 depending on M_2, which proves the first part of Lemma 5.4. In the same way as Lemma 5.3, we can prove the second part of Lemma 5.4. Thus we have established the lemma.

In order to show that the Hölder continuity of v with respect to t holds independently of n and ε, we use the following result due to Gilding [**4**].

LEMMA 5.5. *Let* $z\in C^{2,1}((a, b)\times(\tau, T))\cap C^0([a, b]\times[\tau, T])$ *be a solution of the equation*

$$z_t=A(x, t)z_{xx}+B(x, t)z_x+f(x, t) \quad \text{in} \quad (a, b)\times(\tau, T)\,,$$

where $-\infty<a<b<\infty$, $0\leqq\tau<T<\infty$, *and let* A, B *and* f *be continuous on* $[a, b]\times[\tau, T]$ *such that*

$$0<A(x, t)\leqq\mu, \quad |B(x, t)|\leqq\mu \quad \text{and} \quad |f(x, t)|\leqq\mu \quad \text{in} \quad [a, b]\times[\tau, T]$$

for some positive constant μ. *If* z *is Hölder continuous with respect to* x *in* $[a, b]\times[\tau, T]$ *with an exponent* $\alpha\in(0, 1]$ *and a Hölder constant* M_1, *then for any* $0<d<(b-a)/2$ *it holds that for* $\tau\leqq s<t\leqq s+\delta\leqq T$ *and* $x\in[a+d, b-d]$

$$|z(x, s)-z(x, t)|\leqq M_2|s-t|^{\alpha/2},$$

where

$$\delta=\frac{d^2}{4\mu(1+d)} \quad \text{and} \quad M_2=2\{M_1[2\mu(1+d)^{1/2}]^{\alpha}+\mu\delta^{1-\alpha/2}\}.$$

Combining Lemmas 5.2 and 5.3 with Lemma 5.5, we have

LEMMA 5.6. *Let* v *be a solution of the problem* (5.1)-(5.3). *Then* v *satisfies the following*:

(i) $|v(x, s)-v(x, t)|\leqq C_4|s-t|^{1/2}$ *on* $\overline{Q_{n-3}}$ *for a positive constant* C_4 *depending only on* m, ϕ, c *and* $\|v_0'\|_{L^\infty}$;

(ii) *For any* $\tau\in(0, \infty)$ (*resp.* $\tau=0$) *there exists a constant* C_5 *which depends only on* m, ϕ, c, $\|v_0'\|_{L^\infty}$ *and* τ (*resp.* $\|((v_0')^m)'\|_{L^\infty}$) *such that for* $|x|\leqq n-3$ *and* $\tau\leqq s$, $t<\infty$

$$|(v_x+\varepsilon)^m(x, s)-(v_x+\varepsilon)^m(x, t)|\leqq C_5|s-t|^{1/2}.$$

PROOF. At first we shall prove the assertion (i). The equation (5.1) is rewritten as

$$v_t=m(v_x+\varepsilon)^{m-1}v_{xx}+\phi'(v)v_x \quad \text{in} \quad Q_{n-2}.$$

It follows from Lemma 5.2 that

$$0<m(v_x+\varepsilon)^{m-1}\leqq m(C_1+1)^{m-1} \quad \text{on} \quad \overline{Q_{n-2}}$$

and

$$|v(x, t)-v(y, t)|\leqq C_1|x-y| \quad \text{on} \quad \overline{Q_{n-2}}.$$

Hence, Lemma 5.5 leads to the assertion (i).

Next we put $w=(v_x+\varepsilon)^m$. The function w satisfies the equation

$$w_t=mw^{1-1/m}w_{xx}+\phi'(v)w_x+m\phi''(v)w^{1-1/m}(w^{1/m}-\varepsilon)^2 \quad \text{in} \quad Q_{n-2}.$$

By Lemmas 5.2 and 5.3, Lemma 5.5 can be applied to the equation mentioned above in Q_{n-2}, which states the assertion (ii).

Using Lemmas 5.2, 5.5 and 5.6 and then employing the same argument as Lemma 5.6, we have

LEMMA 5.7. *Let* v *be a solution of the problem* (5.1)-(5.3). *For any* $\tau\in(0, \infty)$ (*resp.* $\tau=0$) *there exists a constant* C_6 *which depends only on* m, ϕ, c, $\|v_0'\|_{L^\infty}$ *and* τ (*resp.* $\|((v_0')^{m-1})'\|_{L^\infty}$) *such that for* $|x|\leqq n-3$ *and* $\tau\leqq s$, $t<\infty$

$$|(v_x+\varepsilon)^{m-1}(x, s)-(v_x+\varepsilon)^{m-1}(x, t)|\leqq C_6|s-t|^{1/2}.$$

Let us consider the following Cauchy problem in place of the problem (5.1)-(5.3):

$$v_t=[(v_x+\varepsilon)^m+\phi(v)]_x \quad \text{in} \quad \boldsymbol{R}^1\times(0, \infty), \tag{5.31}$$

(5.32) $$v(x, 0)=v_{0}(x) \quad \text{on} \quad \boldsymbol{R}^{1},$$

where ε is a positive constant and v_{0} is a smooth function on $\boldsymbol{R}^{1}$ having bounded derivatives up to the third order such that $0\leqq v_{0}(x)\leqq c$ on $\boldsymbol{R}^{1}$, $v_{0}(-\infty)=0$, $v_{0}(+\infty)=c$ and $v_{0}'(x)\geqq 0$ on $\boldsymbol{R}^{1}$.

LEMMA 5.8. *The problem* (5.31), (5.32) *has a unique classical solution* v *such that:*

(i) $0\leqq v(x, t)\leqq c$ *on* $\boldsymbol{R}^{1}\times[0, \infty)$;

(ii) $v_{x}(x, t)\geqq 0$ *on* $\boldsymbol{R}^{1}\times[0, \infty)$;

(iii) *There exists* $\alpha'\in(0,1]$ *such that* $v\in C^{2+\alpha', 1+\alpha'/2}(\boldsymbol{R}^{1}\times[0, T])$ *for any* $T\in(0, \infty)$;

(iv) $v_{xx}\in C^{2,1}(\boldsymbol{R}^{1}\times(0, \infty))$;

(v) *There exists a constant* C_{7} *which depends only on* m, ϕ, c *and* $\|v_{0}'\|_{L^{\infty}}$ *such that for* $x, y\in\boldsymbol{R}^{1}$ *and* $0\leqq s, t<\infty$

$$|v(x, s)-v(y, t)|\leqq C_{7}[|x-y|+|s-t|^{1/2}];$$

(vi) *For any* $\tau\in(0, \infty)$ (*resp.* $\tau=0$) *there exists a constant* C_{8} *which depends only on* m, ϕ, c, $\|v_{0}'\|_{L^{\infty}}$ *ane* τ (*resp.* $\|((v_{0}')^{m})'\|_{L^{\infty}}$) *such that for* $x, y\in\boldsymbol{R}^{1}$ *and* $\tau\leqq s, t<\infty$

$$|(v_{x}+\varepsilon)^{m}(x, s)-(v_{x}+\varepsilon)^{m}(y, t)|\leqq C_{8}[|x-y|+|s-t|^{1/2}];$$

(vii) *For any* $\tau\in(0, \infty)$ (*resp.* $\tau=0$) *there exists a constant* C_{9} *which depends only on* m, ϕ, c, $\|v_{0}'\|_{L^{\infty}}$ *and* τ (*resp.* $\|((v_{0}')^{m-1})'\|_{L^{\infty}}$) *such that for* $x, y\in\boldsymbol{R}^{1}$ *and* $\tau\leqq s, t<\infty$

$$|(v_{x}+\varepsilon)^{m-1}(x, s)-(v_{x}+\varepsilon)^{m-1}(y, t)|\leqq C_{9}[|x-y|+|s-t|^{1/2}].$$

PROOF. Take the sequence of functions $\{v_{0,n}(x)\}$ with $v_{0,n}\in C^{\infty}(\boldsymbol{R}^{1})$ satisfying:

(i) $v_{0,n}$ is non-decreasing on $\boldsymbol{R}^{1}$;

(ii) $0\leqq v_{0,n}(x)\leqq c$ on $\boldsymbol{R}^{1}$, $v_{0,n}(x)=0$ for $x\leqq -n+1$ and $v_{0,n}(x)=c$ for $x\geqq n-1$;

(iii) $v_{0,n}(x)\rightarrow v_{0}(x)$ as $n\rightarrow\infty$ uniformly on $\boldsymbol{R}^{1}$;

(iv) $\|v_{0,n}'\|_{L^{\infty}}\leqq M_{1}\|v_{0}'\|_{L^{\infty}}$, $\|((v_{0,n}')^{m})'\|_{L^{\infty}}\leqq M_{1}\|((v_{0}')^{m})'\|_{L^{\infty}}$ and $\|((v_{0,n}')^{m-1})'\|_{L^{\infty}}\leqq M_{1}\|((v_{0}')^{m-1})'\|_{L^{\infty}}$, where M_{1} is a constant independent of n.

Consider the following problem:

(5.33) $$v_{t}=[(v_{x}+\varepsilon)^{m}+\phi(v)]_{x} \quad \text{in} \quad Q_{n},$$

(5.34) $$v(-n, t)=0 \quad \text{and} \quad v(n, t)=c \quad \text{for} \quad t\in[0, \infty),$$

(5.35) $$v(x, 0)=v_{0,n}(x) \quad \text{for} \quad x\in[-n, n].$$

It is shown by Lemma 5.1 that there exists a unique classical solution v_{n} of (5.33)-(5.35) satisfying the properties (i)-(iv) of Lemma 5.1. Moreover, it is found that v_{n} has the properties of Lemmas 5.2-5.4 and Lemmas 5.6 and 5.7. We rewrite (5.33) as

$$(v_{n})_{t}=A(x, t)(v_{n})_{xx}+B(x, t)(v_{n})_{x} \quad \text{in} \quad Q_{n},$$

where $A=m((v_n)_x+\varepsilon)^{m-1}$ and $B=\phi'(v_n)$. By Lemmas 5.2-5.4 we find that

$$m\varepsilon^{m-1}\leqq A(x, t)\leqq m(C_1+1)^{m-1} \quad \text{on} \quad \overline{Q_{n-3}}$$

and that A, B and B_x are Hölder continuous with respect to the parabolic distance on $\overline{Q_{n-3}}$ with the Hölder constant depending only on C_i $(i=1, 2, \cdots, 6)$. Let T be an arbitrary fixed positive number and let x_1 be any point in $[-n+4, n-4]$, and then put $D=(x_1-1, x_1+1)\times(0, T]$, $R=(x_1-1, x_1+1)\times\{t=0\}$ and $R_0=(x_1-1/2, x_1+1/2)\times\{t=0\}$. By virtue of Theorem 4 [3, p. 121] we see that there exists a constant M_2, depending only on C_i $(i=1, 2, \cdots, 6)$, ε and T such that for some $\alpha\in(0, 1]$

$$|v_l|_{2+\alpha, D_1}\leqq M_2,$$

where $D_1=[x_1-1/2, x_1+1/2]\times[0, T]$ and $l\geqq n$. Since x_1 is an arbitrary point in $[-n+4, n-4]$, we obtain

$$|v_l|_{2+\alpha, \overline{Q_{n-4}(T)}}\leqq M_2 \quad \text{for} \quad l\geqq n.$$

By using Ascoli-Arzela's theorem and a diagonal process, from $\{v_n\}$ we can select a subsequence $\{v_{n_j}\}$ which converges with respect to the norm of the Hölder space $C^{2+\alpha', 1+\alpha'/2}$ $(\alpha'<\alpha)$ on any domain $[-n_0, n_0]\times[0, T]$, and then we know the limit function $v\in C^{2+\alpha', 1+\alpha'/2}(\boldsymbol{R}^1\times[0, T])$ for any $T>0$. Moreover, Lemmas 5.1-5.4 and Lemmas 5.6-5.7 conclude that the limit function v is a classical solution of the problem (5.31), (5.32) and satisfies the properties (i)-(vii). The uniqueness is derived from the usual maximum principle. By the uniqueness of solutions, we see that the original sequence $\{v_n\}$ converges to v as $n\to\infty$. Thus the proof is completed.

6. Existence.

We are now in a position to prove the existence theorem for the problem (3.2)-(3.5). The result is the following

THEOREM 6.1. *Let v_0 be a function on $\boldsymbol{R}^1$ satisfying the assumption* (A.3) *in Section* 3. *Then there exists a unique solution v of the problem* (3.2)-(3.5) *which has the following properties:*

(i) *For $x, y\in\boldsymbol{R}^1$ and $0\leqq s, t<\infty$*

$$|v(x, s)-v(y, t)|\leqq C_7[|x-y|+|s-t|^{1/2}],$$

where C_7 depends only on m, ϕ, $\|v_0\|_{L^\infty}$ and $\|v_0'\|_{L^\infty}$;

(ii) *For any $\tau\in(0, \infty)$ (resp. $\tau=0$) there exists a constant C_8 which depends only on m, ϕ, $\|v_0\|_{L^\infty}$, $\|v_0'\|_{L^\infty}$ and τ (resp. $\|((v_0')^m)'\|_{L^\infty}$) such that for $x, y\in\boldsymbol{R}^1$ and $\tau\leqq s, t<\infty$*

$$|(v_x)^m(x, s)-(v_x)^m(y, t)|\leqq C_8[|x-y|+|s-t|^{1/2}];$$

(iii) *In a neighbourhood of a point in $\boldsymbol{R}^1\times(0, \infty)$ where v_x is positive, the* *function* $u=v_x$ *is a classical solution for the equation*

$$u_t=\left[(u^m)_x+\phi'\left(\int_{-\infty}^{x} u(\xi, t)d\xi\right)u\right]_x.$$

Theorem 6.1 implies Theorem 2.1 which shows the existence of solutions for the original problem (1.3), (1.4). In fact, for a given function $u_0(x)$ on $\boldsymbol{R}^1$ satisfying the assumption (A.2) in Section 2 we put

$$v_0(x)=\int_{-\infty}^{x} u_0(\xi)d\xi \quad \text{and} \quad c=\int_{-\infty}^{\infty} u_0(\xi)d\xi .$$

Then the assumption (A.3) is fulfilled. Theorem 6.1 states that there exists a unique solution v of the problem (3.2)-(3.5) with the initial function v_0. Proposition 3.2 in Section 3 implies that the function $u=v_x$ is the solution of the problem (1.3), (1.4) with the initial function u_0. The assertion (i) of Theorem 2.1 is derived from Proposition 3.1. The assertions (ii) and (iii) of Theorem 2.1 follow from the assertions (ii) and (iii) of Theorem 6.1, respectively.

PROOF OF THEOREM 6.1. Let ε be a sufficiently small positive number. We can construct a sequence of functions $\{v_{0,\varepsilon}(t)\}$ such that:

(i) $v_{0,\varepsilon}$ is a smooth function on $\boldsymbol{R}^1$ having the bounded derivatives up to the third order;

(ii) $v_{0,\varepsilon}$ is non-decreasing on $\boldsymbol{R}^1$;

(iii) $v_{0,\varepsilon}(-\infty)=0$ and $v_{0,\varepsilon}(+\infty)=c$;

(iv) $v_{0,\varepsilon}(x)\to v_0(x)$ as $\varepsilon\to 0$ uniformly on $\boldsymbol{R}^1$;

(v) $\|v'_{0,\varepsilon}\|_{L^\infty}\leqq M_1\|v'_0\|_{L^\infty}$;

(vi) If $\|((v'_0)^m)'\|_{L^\infty}<\infty$, then $\|((v'_{0,\varepsilon})^m)'\|_{L^\infty}\leqq M_2\|((v'_0)^m)'\|_{L^\infty}$;

(vii) If $\|((v'_0)^{m-1})'\|_{L^\infty}<\infty$, then $\|((v'_{0,\varepsilon})^{m-1})'\|_{L^\infty}\leqq M_3\|((v'_0)^{m-1})'\|_{L^\infty}$, where M_i $(i=1, 2, 3)$ are constants independent of ε.

Consider the following problem:

(6.1) $$v_t=[(v_x+\varepsilon)^m+\phi(v)]_x \quad \text{in} \quad \boldsymbol{R}^1\times(0, \infty),$$

(6.2) $$v(x, 0)=v_{0,\varepsilon}(x) \quad \text{on} \quad \boldsymbol{R}^1.$$

By Lemma 5.8 we find that there exists a unique solution v_ε of the problem (6.1), (6.2) such that v_ε satisfies the properties (i)-(vi) of Lemma 5.8. Applying Ascoli-Arzela's theorem and a diagonal process, from $\{v_\varepsilon\}$ we can select a subsequence $\{v_{\varepsilon_j}\}$ which converges to a limit function v uniformly on any compact set in $\boldsymbol{R}^1\times[0, \infty)$. Moreover, we obtain that

$$(v_{\varepsilon_j})_x\to v_x \text{ uniformly on any compact set in } \boldsymbol{R}^1\times(0, \infty) \text{ as } \varepsilon_j\to 0.$$

It follows from the properties (i)-(vi) of Lemma 5.8 that the function v satisfies the following properties:

(i) v is continuous on $\boldsymbol{R}^1\times[0, \infty)$ and differentiable on $\boldsymbol{R}^1\times(0, \infty)$;

(ii) $0\leqq v(x, t)\leqq c$ on $\boldsymbol{R}^1\times[0, \infty)$ and $v_x(x, t)\geqq 0$ on $\boldsymbol{R}^1\times(0, \infty)$;

(iii) For $x, y \in \boldsymbol{R}^1$ and $0 \leqq s, t < \infty$

$$|v(x, s) - v(y, t)| \leqq C_7[|x-y| + |s-t|^{1/2}],$$

where C_7 depends only on m, ϕ, $\|v_0\|_{L^\infty}$ and $\|v_0'\|_{L^\infty}$;

(iv) For any $\tau \in (0, \infty)$ there exists a constant C_8 which depends only on m, ϕ, $\|v_0\|_{L^\infty}$, $\|v_0'\|_{L^\infty}$ and τ such that for $x, y \in \boldsymbol{R}^1$ and $\tau \leqq s, t < \infty$

$$|(v_x)^m(x, s) - (v_x)^m(y, t)| \leqq C_8[|x-y| + |s-t|^{1/2}].$$

If $\|((v_0')^m)'\|_{L^\infty} < \infty$, then the inequality mentioned above holds on $\boldsymbol{R}^1 \times [0, \infty)$ for the same C_8 as the above except that τ is replaced by $\|((v_0')^m)'\|_{L^\infty}$.

Multiply (6.1) by a function $f \in C^1(\boldsymbol{R}^1 \times (0, \infty))$ with compact support in $\boldsymbol{R}^1 \times (0, \infty)$ and integrate the resulting equation over $\boldsymbol{R}^1 \times (0, \infty)$. Using integration by parts and letting $\varepsilon_j \to 0$, we obtain that the limit function v satisfies the integral identity

$$\int_0^\infty \int_{-\infty}^\infty \{v f_t - [(v_x)^m + \phi(v)] f_x\} \, dx dt = 0.$$

To show that v is a solution of the problem (3.2)-(3.5), we must prove that for each $t \in [0, \infty)$, $v(-\infty, t) = 0$ and $v(+\infty, t) = c$. We shall show the following:

LEMMA 6.2. *For any $T \in (0, \infty)$ we have $v(x, t) \to 0$ as $x \to -\infty$ and $v(x, t) \to c$ as $x \to +\infty$ uniformly in $t \in [0, T]$.*

PROOF. Since $v_0(-\infty) = 0$, for an arbitrary fixed constant $\delta > 0$ there exists a positive constant M_1, which is independent of ε_j, such that

$$0 \leqq v_{0,\varepsilon_j}(x) \leqq \delta + M_1 e^x \quad \text{for} \quad x \in \boldsymbol{R}^1.$$

We consider the auxiliary function

$$w(x, t) = M_1 e^{x + \gamma t} + \delta - v_{\varepsilon_j}(x, t),$$

where γ is a constant satisfying

$$\gamma \geqq m(C_7 + 1)^{m-1} + \max_{0 \leq \sigma \leq c} |\phi'(\sigma)|.$$

Let T be an arbitrary fixed positive number. Then we see that

$$w(x, 0) \geqq 0 \quad \text{for} \quad x \in \boldsymbol{R}^1$$

and

$$|w(x, t)| \leqq M_2 e^{M_2 |x|^2} \quad \text{for} \quad x \in \boldsymbol{R}^1 \quad \text{and} \quad t \in [0, T],$$

where M_2 is a positive constant depending on T. It follows from the choice of γ that

$$\begin{aligned} Lw &\equiv w_t - m((v_{\varepsilon_j})_x + \varepsilon_j)^{m-1} w_{xx} + \phi'(v_{\varepsilon_j}) w_x \\ &= M_1 e^{x + \gamma t}[\gamma - m((v_{\varepsilon_j})_x + \varepsilon_j)^{m-1} - \phi'(v_{\varepsilon_j})] \\ &\geqq 0. \end{aligned}$$

Applying the maximum principle, we obtain

$$w(x, t) \geqq 0 \quad \text{for} \quad x \in \boldsymbol{R}^1 \quad \text{and} \quad t \in [0, T],$$

which yields that

$$0 \leqq v_{\varepsilon_j}(x, t) \leqq \delta + M_1 e^{x+\gamma t} \quad \text{for} \quad x \in \boldsymbol{R}^1 \quad \text{and} \quad t \in [0, T].$$

Letting $\varepsilon_j \to 0$, we have

$$0 \leqq v(x, t) \leqq \delta + M_1 e^{x+\gamma t} \quad \text{for} \quad x \in \boldsymbol{R}^1 \quad \text{and} \quad t \in [0, T].$$

Hence, for a sufficiently large positive number N_1 we have

$$0 \leqq v(x, t) \leqq 2\delta \quad \text{for} \quad t \in [0, T], \quad x \leqq -N_1$$

which implies that

$$v(x, t) \to 0 \quad \text{as} \quad x \to -\infty \quad \text{uniformly in} \quad t \in [0, T].$$

Next, using the auxiliary function

$$w(x, t) = M_1 e^{-x+\gamma t} + \delta - [c - v_{\varepsilon_j}(x, t)],$$

we obtain similarly

$$0 \leqq v(x, t) \leqq 2\delta \quad \text{for} \quad t \in [0, T] \quad \text{and} \quad x \geqq N_1,$$

which implies that

$$v(x, t) \to c \quad \text{as} \quad x \to +\infty \quad \text{uniformly in} \quad t \in [0, T].$$

Thus the proof of Lemma 6.2 is completed.

Let us return to the proof of Theorem 6.1. We only prove the assertion (iii) of Theorem 6.1. By using a method similar to that used to prove the assertion (ii) of Theorem 3 in [7], we see that v_{xx}, $((v_x)^m)_{xx}$ and v_{xt} exist and are continuous in a neighbourhood of a point in $\boldsymbol{R}^1 \times (0, \infty)$ where v_x is positive, and that in this neighbourhood v is a classical solution for the equation

$$v_t = [(v_x)^m + \phi(v)]_x.$$

Differentiating this equation with respect to x and then putting $u = v_x$, we obtain that in this neighbourhood

$$u_t = \left[(u^m)_x + \phi'\left(\int_{-\infty}^{x} u(\xi, t)d\xi\right)u\right]_x.$$

Here we note

$$v(x, t) = \int_{-\infty}^{x} u(\xi, t)d\xi.$$

Finally we remark that the original sequence $\{v_\varepsilon\}$ converges to v by using the uniqueness of solutions for the problem (3.2)-(3.5). Thus Theorem 6.1 is completed.

7. Regularity.

We have constructed a solution of the problem (3.2)-(3.5). We state here some regularity properties of this solution. The technique is similar to that used in the porous medium equation (see Aronson [1], Gilding and Peletier [7]).

THEOREM 7.1. *Let v be a solution of the problem* (3.2)-(3.5). *Then v has the following properties:*

(i) *For any positive number τ there exists a constant C_9 which depends only on m, ϕ, $\|v_0\|_{L^\infty}$, $\|v_0'\|_{L^\infty}$ and τ such that for $x, y \in \boldsymbol{R}^1$ and $\tau \leqq s, t < \infty$*

$$|(v_x)^{m-1}(x, s)-(v_x)^{m-1}(y, t)| \leqq C_9[|x-y|+|s-t|^{1/2}].$$

If $\|((v_0')^{m-1})'\|_{L^\infty}<\infty$, the inequality mentioned just above holds on $\boldsymbol{R}^1\times[0, \infty)$ for the same C_9 as the above except that τ is replaced by $\|((v_0')^{m-1})'\|_{L^\infty}$;

(ii) *The derivatives $((v_x)^m)_x$ and v_t exist and are continuous on $\boldsymbol{R}^1\times(0, \infty)$ and v is a classical solution for the equation*

$$v_t=[(v_x)^m+\phi(v)]_x \quad in \quad \boldsymbol{R}^1\times(0, \infty);$$

(iii) *If $1<m<2$, then v_{xx} exists and is continuous on $\boldsymbol{R}^1\times(0, \infty)$.*

Combining Theorem 7.1 with Proposition 3.2, we obtain Theorem 2.2 for solutions of the original problem (1.4), (1.5).

PROOF OF THEOREM 7.1. Let v_ε be a solution of the problem (6.1), (6.2) constructed in the proof of Theorem 6.1. It follows from Lemma 5.8 that for any $\tau\in(0, \infty)$ (resp. $\tau=0$) there exists a constant C_9 depending only on m, ϕ, $\|v_0\|_{L^\infty}$, $\|v_0'\|_{L^\infty}$ and τ (resp. $\|((v_0')^{m-1})'\|_{L^\infty}$) such that for $x, y\in\boldsymbol{R}^1$ and $\tau\leqq s, t<\infty$

$$|((v_\varepsilon)_x)^{m-1}(x, s)-((v_\varepsilon)_x)^{m-1}(y, t)| \leqq C_9[|x-y|+|s-t|^{1/2}].$$

Letting $\varepsilon\to 0$, we obtain the assertion (i) of Theorem 7.1.

Next, by a method similar to that used in the proof of the assertion (iii) of Theorem 3 in [7], we see that $((v_x)^m)_x$ exists and is continuous on $\boldsymbol{R}^1\times(0, \infty)$ and that v_{xx} exists and is continuous on $\boldsymbol{R}^1\times(0, \infty)$ if $1<m<2$. Since v is a solution of the problem (3.2)-(3.5) we have

$$\int_0^\infty\int_{-\infty}^\infty v f_t dx dt=-\int_0^\infty\int_{-\infty}^\infty [((v_x)^m)_x+\phi'(v)v_x] f dx dt$$

for all $f\in C(\boldsymbol{R}^1\times(0, \infty))$ with compact support in $\boldsymbol{R}^1\times(0, \infty)$. This relation implies that v_t exists and is continuous on $\boldsymbol{R}^1\times(0, \infty)$, and that in the classical sense v satisfies the equation

$$v_t=[(v_x)^m+\phi(v)]_x \quad \text{in} \quad \boldsymbol{R}^1\times(0, \infty).$$

This completes the proof.

8. Comparison theorem.

By using the construction and uniqueness of solutions for the problem (3.2)-(3.5), we shall show the comparison theorem for the problem (3.2)-(3.5).

THEOREM 8.1. *For each $i=1, 2$ let v_{0i} be non-decreasing and Lipschitz continuous function on $\boldsymbol{R}^1$ such that $0\leqq v_{0i}(x)\leqq c_i$ on $\boldsymbol{R}^1$, $v_{0i}(-\infty)=0$ and $v_{0i}(+\infty)=c_i$.*

Let v_1 *and* v_2 *be two solutions of the problem* (3.2)-(3.5) *with the corresponding initial functions* v_{01} *and* v_{02}. *Suppose that*

$$v_{01}(x) \leqq v_{02}(x) \quad on \quad \boldsymbol{R}^1.$$

Then

$$v_1(x, t) \leqq v_2(x, t) \quad on \quad \boldsymbol{R}^1 \times [0, \infty).$$

Proof. For each $i=1, 2$ the solution v_i is constructed as the limit function of a sequence of functions $\{v_{i\varepsilon}\}$, where $v_{i\varepsilon}$ is a classical solution of the problem (6.1), (6.2) in Section 6 with the initial function $v_{0i\varepsilon}(x)$. Here, for each $i=1, 2$ $v_{0i\varepsilon}(x)$ is a smooth function on $\boldsymbol{R}^1$ satisfying the following properties:

(i) $v_{0i\varepsilon}$ is non-decreasing;

(ii) $v_{0i\varepsilon}(-\infty)=0$ and $v_{0i\varepsilon}(+\infty)=c_i$;

(iii) $v_{0i\varepsilon}(x) \to v_{0i}(x)$ as $\varepsilon \to 0$ uniformly on $\boldsymbol{R}^1$;

(iv) $\|v'_{0i\varepsilon}\|_{L^\infty} \leqq M_1 \|v'_i\|_{L^\infty}$, where M_1 is independent of ε and i.

Moreover, since $v_{01}(x) \leqq v_{02}(x)$ on $\boldsymbol{R}^1$ we can suppose that for each ε

$$v_{01\varepsilon}(x) \leqq v_{02\varepsilon}(x) \quad \text{on} \quad \boldsymbol{R}^1.$$

By the standard maximum principle for parabolic equations, we obtain that for each ε

$$v_{1\varepsilon}(x, t) \leqq v_{2\varepsilon}(x, t) \quad \text{on} \quad \boldsymbol{R}^1 \times [0, \infty).$$

Letting $\varepsilon \to 0$, we have

$$v_1(x, t) \leqq v_2(x, t) \quad \text{on} \quad \boldsymbol{R}^1 \times [0, \infty).$$

This completes the proof.

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