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Isotropic minimal immersions of spheres into spheres

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1. Introduction.

It is an interesting and important problem to investigate the existence and the rigidity of a minimal immersion of a Riemannian manifold into a unit sphere. In the case of an n-dimensional sphere S^{n}, associated with each positive integer s, there exists an isometric minimal immersion $\psi_{n,s}: S_{k(s)}^{n}\rightarrow S_{1}^{m(s)}$, where S_{c}^{l} denotes an l-dimensional sphere with constant sectional curvature c and $k(s)$ and $m(s)$ are given as follows;

$$k(s)=\frac{n}{s(s+n-1)}\,,$$

$$m(s)=(2s+n-1)\frac{(s+n-2)\,!}{s\,!(n-1)\,!}-1\,.$$

$\psi_{n,s}$ is given by s-th eigenfunctions of the Laplacian Δ on S^{n} (T. Takahashi [9]). These immersions $\psi_{n,s}$ are called "standard minimal immersions" (cf. § 2).

It will be convenient to say that a minimal immersion $\varphi: S_{k}^{n}\rightarrow S_{1}^{l}\subset \boldsymbol{R}^{l+1}$ is *full* if $\varphi(S_{k}^{n})$ is not contained in a hyperplane of $\boldsymbol{R}^{l+1}$ and that two such immersions φ_{1}, φ_{2} are *equivalent* if there exists an isometry ρ of S_{1}^{l} such that $\varphi_{2}=\rho\circ\varphi_{1}$. For the rigidity of the immersion $\psi_{n,s}$, do Carmo and Wallach ([4]) showed the following result.

THEOREM ([4]). *In the case of* $s=1$, 2, *and* 3, *the immersion* $\psi_{n,s}$ *is rigid. Namely any isometric minimal immersion* φ *of* $S_{k(s)}^{n}$ *into* S_{1}^{l} *is equivalent to* $\psi_{n,s}$. *However when* $n\geqq 3$ *and* $s\geqq 4$ *the immersion* $\psi_{n,s}$ *is not rigid. That is, the set of equivalence classes of isometric minimal immersions of* $S_{k(s)}^{n}$ *into* S_{1}^{l} *can be smoothly parametrized by a compact convex body* $L\subset W$ *in a vector space* W, *with* $\dim W=N(n, s)\geqq 18$.

In this note we consider characterizations of the standard minimal immersion in such a broad class of minimal immersions. First we characterize it by making use of the notion of isotropic immersions introduced by B. O'Neill ([5]). We say that an $\boldsymbol{R}^{k}$-valued symmetric multi-linear form B on $\boldsymbol{R}^{n}$ is *isotropic* if $\|B(u, u, \cdots, u)\|=$constant for any unit vectors u in $\boldsymbol{R}^{n}$ (cf. § 2). Then we have the following result.

THEOREM A. *Let $\varphi: S^n_{k(s)} \to S^l_1$ be an isometric minimal immersion. Assume that φ is full and that $n \geqq 3$ and $s \geqq 4$. If the degree of $\varphi \geqq \left[\frac{s}{2}\right]$, where $\left[\frac{s}{2}\right]$ is the largest integer less than or equal to $s/2$, and the j-th fundamental form B_j is isotropic for $2 \leqq j \leqq \left[\frac{s}{2}\right]$, then we have $l=m(s)$ and φ is equivalent to the standard minimal immersion $\psi_{n,s}$.*

We shall refer to the notions of higher fundamental forms and the degree of the immersion in § 2.

Next we characterize the standard minimal immersion using the concept of a helical geodesic immersion. Let $\varphi: M \to \overline{M}$ be an isometric immersion of a connnected complete Riemannian manifold M into a Riemannian manifold $\overline{M}$. If for each geodesic γ of M the curve $\varphi \circ \gamma$ in $\overline{M}$ has constant curvatures of osculating order d which are independent of γ, then φ is called a *helical geodesic immersion* of order d (K. Sakamoto [8]). It is known that a strongly harmonic manifold admits a helical geodesic minimal immersion into a sphere (A. Besse [2]). Sakamoto ([8]) stated that the study of helical geodesic immersions will be useful for the study of the conjecture that the harmonic manifolds are locally symmetric. In this paper we show the following result.

THEOREM B. Let $\varphi: S^n_{k(s)} \to S^l_1$ be a helical geodesic minimal immersion. Assume that φ is full. Then φ is equivalent to the standard minimal immersion $\psi_{n,s}$; in particular the order of the helical geodesic immersion φ is equal to s and $l=m(s)$.

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2. The standard minimal immersions and their properties.

2.1. The standard minimal immersions. Let $M=G/K$ be an n-dimensional compact homogeneous Riemannian manifold with an irreducible linear isotropy group and V^s the s-th eigenspace of the Laplacian Δ_M corresponding to the s-th eigenvalue λ_s. We define an inner product $\langle\ ,\ \rangle$ in V^s by

$$\langle f, h\rangle = \int_M f \cdot h \, d\mu \qquad f, h \in V^s .$$

For simplicity, we normalize the canonical measure $d\mu$ of (M, g) in such a way that $\int_M d\mu = \dim V^s = m(s)+1$. Let $\{f_0, f_1, \cdots, f_{m(s)}\}$ be an orthonormal basis for V^s and define a map $\psi: M \to \boldsymbol{R}^{m(s)+1}$ by $\psi(p)=(f_0(p), \cdots, f_{m(s)}(p))$, $p \in M$. The transitive action G on M induces a natural action on V^s by $(g \cdot f)(p)=f(g^{-1}p)$, $g \in G$, $p \in M$. It is easily seen that $\sum_{i=0}^{m(s)} f_i^2(p)=1$ for all $p \in M$, i.e., $\psi(M) \subset S_1^{m(s)}$.

The irreducibility of the linear isotropy action of K and the G-invariance of the metric g imply that ψ is an isometric immersion of (M, cg) into $S_1^{m(s)}$. By a theorem of Takahashi ([**9**]) ψ is then a minimal immersion of (M, cg) into $S_1^{m(s)}$ and $c=\lambda_s/n$. We shall call this isometric minimal immersion ψ of $(M, (\lambda_s/n)g)$ into $S_1^{m(s)}$ *the s-th standard minimal immersion* of M.

The standard minimal immersion can be described in other words as follows. Take an orthonormal basis $\{f_0, f_1, \cdots, f_{m(s)}\}$ of V^s such that $e_0=\psi(eK)=(f_0(eK), \cdots, f_{m(s)}(eK))$, where e is the identity element of G. Let A be an isometry of V^s into $\boldsymbol{R}^{m(s)+1}$ such that $A(f_j)=e_j$, $j=0, 1, \cdots, m(s)$. Let G act on $\boldsymbol{R}^{m(s)+1}$ so that A is a G-isomorphism. Then by a simple computation we get $\psi(gK)=A(g\cdot f_0)$, $g\in G$. Since A is an isometry, we can consider ψ as an isometric minimal immersion of $(M, (\lambda_s/n)g)$ into a unit hypersphere in V^s defined by $\psi(gK)=g\cdot f_0$, $g\in G$.

Let $\varphi: M\to\bar{M}$ be an isometric immersion of a Riemannian homogeneous space $M=G/K$ into a Riemannian manifold of constant sectional curvature $\bar{M}$. We say that φ is *equivariant* if there exists a continuous homomorphism ρ of G into the isometry group $I(\bar{M})$ of $\bar{M}$ such that

$$\varphi(g\cdot p)=\rho(g)\varphi(p) \qquad p\in M,\ g\in G\,.$$

It is easily seen that the standard minimal immersion is naturally equivariant.

2.2. Higher fundamental forms and degrees of isometric immersions. In this part, we define the higher fundamental forms and the degree of an isometric immersion (Wallach [**10**]). Let $\bar{M}$ be a Riemannian manifold of constant curvature. Let $\varphi: M\to\bar{M}$ be an isometric immersion of a Riemannian manifold M into $\bar{M}$. Let B_2 be the second fundamental form of φ at $p\in M$ and O_p^2 be the linear span of the image of B_2 in the normal space $N_p(M)$ of the immersion φ at $p\in M$. We call $\varphi_*T_pM+O_p^2$ the *second osculating space* at $p\in M$. We say that $p\in M$ is degree 2 regular if O_p^2 is of maximal dimension. Let $R_2\subset M$ be the set of all degree 2 regular points of M. Then R_2 is open in M. Let $p\in R_2$. Let N_2 be the normal projection in $N_p(M)$ relative to $N_p(M)=O_p^2+(O_p^2)^\perp$ (we write $v\to v^{N_2}\in(O_p^2)^\perp$). We define $B_3(u_1, u_2, u_3)=(\tilde{\nabla}_{u_1}(B_2(u_2, u_3)))^{N_2}$ for $u_1, u_2, u_3\in T_pM$ arbitrarily extended to the vector fields on M, where $\tilde{\nabla}$ denotes the Riemannian connection on $\bar{M}$. B_3 is well-defined and defines a symmetric tensor field on R_2. Let O_p^3 be the linear span of the image of B_3. We call B_3 *the third fundamental form* of φ at p and $\varphi_*T_pM+O_p^2+O_p^3$ *the third osculating space*. We call a point $p\in R_2$ degree 3 regular if $\dim O_p^3$ is maximal. We define B_j, O^j for $j=2, 3, \cdots$ by recursion as above on the space R_{j-1} of all degree $j-1$ regular points of M. We call B_j *the j-th fundamental form* of φ and $\varphi_*T_pM+O_p^2+\cdots+O_p^j$ *the j-th osculating space*. Clearly the above process must eventually stop since $\dim(\varphi_*T_pM+O_p^2+O_p^3+\cdots+O_p^d)\leqq\dim T_p\bar{M}$.

Let d be the first integer $\geqq 2$ such that $B_d \not\equiv 0$ but $B_{d+1} \equiv 0$. Then we call d the *degree* of φ and the set of all d-regular points will be called the set of all completely regular points of M, denoted $M' = R_d$. In particular, when φ is totally geodesic, i. e., $B_2 \equiv 0$, we say that φ has degree 1.

LEMMA 2.1 (Wallach [**10**]).

(1) *$B_j: \underbrace{T_pM \times T_pM \times \cdots \times T_pM}_{j\text{-times}} \to O_p^j$ is an O_p^j-valued symmetric j-linear form on T_pM for $p \in R_{j-1}$. Then B_j induces a linear map $S^j(T_pM) \to O_p^j$, where $S^j(T_pM)$ denotes the j-fold symmetric power of T_pM.*

(2) *Let $e_1, \cdots, e_n$ be an orthonormal basis of T_pM. Set $r_p = \sum_{i=1}^{n} e_i^2 \in S^2(T_pM)$. If $\varphi: M \to \bar{M}$ is minimal, then*

$$\ker B_j \supset r_p \cdot S^{j-2}(T_pM), \qquad j \geqq 2.$$

2.3. Higher fundamental forms of the standard minimal immersions. Let $\psi: M \to S_1^l$ be the standard minimal immersion of a compact homogeneous space $M = G/K$ defined in 2.1. Since ψ is equivariant, the set of all completely regular points of M coincides with M. Moreover the following properties hold.

LEMMA 2.2. (1) *B_j is G-invariant and commutes with $\rho(g)$,*

$$B_{j|g\cdot p}(g\cdot u_1, \cdots, g\cdot u_j) = \rho(g) B_{j|p}(u_1, \cdots, u_j)$$

$$\rho(g) O_p^j = O_{g\cdot p}^j$$

$$N_j \circ \rho(g) = \rho(g) N_j, \qquad g \in G.$$

In particular, $B_j: S^j(T_{eK}M) \to O_{eK}^j(M)$ is a K-homomorphism.

(2) *V^s admits an orthogonal direct sum decomposition*

$$V^s = \boldsymbol{R} \cdot \psi(eK) + \psi_* T_{eK}M + O_{eK}^2 + \cdots + O_{eK}^d,$$

where d is the degree of ψ.

REMARK 2.3. When $M = G/K$ is a compact rank 1 symmetric space, K acts transitively on the unit sphere of $T_{eK}M$. Then by Lemma 2.2 (1),

$$\|B_j(k\cdot u, \cdots, k\cdot u)\| = \|\rho(k) B_j(u, \cdots, u)\| = \|B_j(u, \cdots, u)\| \qquad k \in K.$$

Thus B_j is isotropic at eK and again by Lemma 2.2 (1) B_j is constant isotropic on M.

REMARK 2.4. The degrees of the standard minimal immersions of a compact rank 1 symmetric space M into spheres are computed.

(1) When $M = S^n$, the degree of the s-th standard minimal immersion ψ_s is s.

(2) When M is a complex projective space $P_n(\boldsymbol{C})$, a quaternion projective space $P_n(\boldsymbol{H})$, or a Cayley projective plane $P_2(\mathrm{Cay})$, the degree of ψ_s is $2s$.

Do Carmo and Wallach ([**4**]) showed the above result in the case of a sphere and K. Mashimo ([**6**], [**7**]) calculated the degree for the other cases.

3. Higher fundamental forms of isotropic minimal immersions.

Let $\psi_{n,s}: M^n_{k(s)} \to S_1^{m(s)}$ be the standard minimal immersion of an n-dimensional compact rank 1 symmetric space M into a unit sphere corresponding to the s-th eigenvalue, where $M^n_{k(s)}$ has the induced Riemannian metric by $\psi_{n,s}$. Let $\varphi: M^n_{k(s)} \to S_1^l$ be an another minimal immersion corresponding to the same eigenvalue. We compare the higher fundamental forms of φ with those of $\psi_{n,s}$ when the higher fundamental forms of φ are isotropic. Namely we show the following.

PROPOSITION 3.1. *We denote by B_j and $\mathring{B}_j$ the j-th fundamental forms of φ and $\psi_{n,s}$ respectively. Let i be an integer such that $2 \leqq i \leqq$ the minimum of the degree of φ and the degree of $\psi_{n,s}$. If B_k is isotropic for $2 \leqq k \leqq i$ at every degree $k-1$ regular point $p \in R_{k-1}$ with respect to φ, then we have*

$$\langle B_k(u_1, \cdots, u_k), B_k(v_1, \cdots, v_k)\rangle = \langle \mathring{B}_k(u_1, \cdots, u_k), \mathring{B}_k(v_1, \cdots, v_k)\rangle,$$

$2 \leqq k \leqq i$, $u_1, \cdots, u_k, v_1, \cdots, v_k \in T_pM$ at every point $p \in R_{k-1}$. In particular, the set of all degree k regular points with respect to φ coincides with M for $2 \leqq k \leqq i$.

As preliminaries we state two well-known lemmas.

LEMMA 3.2. *Let B be an $\boldsymbol{R}^k$-valued symmetric j-linear form on $\boldsymbol{R}^n$. B is λ-isotropic, i.e., $\|B(x, \cdots, x)\| = \lambda$ for any unit vector $x \in \boldsymbol{R}^n$, if and only if*

$$\mathcal{S}_{2j}\{\langle B(u_1, \cdots, u_j), B(u_{j+1}, \cdots u_{2j})\rangle\} = \lambda^2 \mathcal{S}_{2j}\{\langle u_1, u_2\rangle \cdots \langle u_{2j-1}, u_{2j}\rangle\}$$
$$\text{for } u_1, \cdots, u_{2j} \in \boldsymbol{R}^n,$$

where $\mathcal{S}_{2j}$ denotes the symmetrizer of order 2j.

Next we recall the equations of Gauss and Ricci. We prepare notations. Let $\varphi: M \to S_1^l$ be an isometric immersion. We denote by $\tilde{\nabla}$ and ∇ the covariant differentiations on S_1^l and M respectively. $\nabla^\perp$ denotes the covariant differentiation with respect to the induced connection in the normal bundle. We define the covariant differentiation $\overline{\nabla}$ on $T(M) \oplus N(M)$ as follows: For any $N(M)$-valued tensor field S of type $(0, k)$, we define

$$(\overline{\nabla}_X S)(Y_1, \cdots, Y_k) = \nabla^\perp_X(S(Y_1, \cdots, Y_k)) - \sum_{i=1}^k S(Y_1, \cdots, \nabla_X Y_i, \cdots, Y_k)$$

and $\overline{\nabla} S$ is also defined by $(\overline{\nabla} S)(X, Y_1, \cdots, Y_k) = (\overline{\nabla}_X S)(Y_1, \cdots, Y_k)$.

LEMMA 3.3.

(1) *Gauss equation:*

$$\langle R(X, Y)Z, W\rangle = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle$$
$$+\langle B_2(X, W), B_2(Y, Z)\rangle - \langle B_2(X, Z), B_2(Y, W)\rangle,$$

where R denotes the curvature tensor with respect to ∇.

(2) *Ricci equation:*

$$\langle R^{\perp}(X, Y)\xi, \eta\rangle=\langle[H_{\xi}, H_{\eta}](X), Y\rangle,$$

where $R^{\perp}$ denotes the curvature tensor with respect to $\nabla^{\perp}$ and H_{ξ} denotes the second fundamental tensor corresponding to the normal vector field ξ. H_{ξ} is related to B_2 as $\langle H_{\xi}X, Y\rangle=\langle B_2(X, Y), \xi\rangle$.

(3) *Ricci formula:*

$$\begin{aligned}&\bar{\nabla}^2 S(U, V, X_1, \cdots, X_k)-\bar{\nabla}^2 S(V, U, X_1, \cdots, X_k)\\ &\quad=\bar{R}(U, V)S(X_1, \cdots, X_k)\\ &\quad=R^{\perp}(U, V)(S(X_1, \cdots, X_k))-\sum_{i=1}^{k} S(X_1, \cdots, R(U, V)X_i, \cdots, X_k).\end{aligned}$$

We prove Proposition 3.1 inductively. First we start under the assumption that the second fundamental form B_2 of φ is λ-isotropic. We recall $\mathring{B}_2$ is constant isotropic (Remark 2.3). Using Lemma 3.2 and Gauss equation we obtain

$$\begin{aligned}(3.1)\qquad &3\langle B_2(u, v), B_2(x, y)\rangle\\ &\quad=\lambda^2\{\langle u, v\rangle\langle x, y\rangle+\langle u, x\rangle\langle y, v\rangle+\langle u, y\rangle\langle v, x\rangle\}\\ &\quad\quad-\langle R(u, y)v, x\rangle-\langle R(u, x)v, y\rangle+\langle u, x\rangle\langle y, v\rangle\\ &\quad\quad-\langle u, v\rangle\langle x, y\rangle+\langle u, y\rangle\langle x, v\rangle-\langle u, v\rangle\langle x, y\rangle.\end{aligned}$$

Since φ is minimal,

$$0=3\left\langle \sum_{i=1}^{n} B_2(e_i, e_i), \sum_{j=1}^{n} B_2(e_j, e_j)\right\rangle=\lambda^2 n(n+2)+2(\tau-n(n-1)),$$

where τ is the scalar curvature of M. Then we have

$$\lambda^2=2(n(n-1)-\tau)/n(n+2).$$

Therefore the right-hand-side of (3.1) for φ coincides with that of $\psi_{n,s}$. Thus we have $\langle B_2(u, v), B_2(x, y)\rangle=\langle\mathring{B}_2(u, v), \mathring{B}_2(x, y)\rangle$, which implies that the dimension of O_p^2 of φ is equal to that of $\psi_{n,s}$ at every point $p\in M$. Therefore every point of M is degree 2 regular for the immersion φ.

Next step we shall show that $\langle\bar{\nabla}B_2(x, y, z), B_2(u, v)\rangle=\langle\bar{\nabla}\mathring{B}_2(x, y, z), \mathring{B}_2(u, v)\rangle=0$ on M. Since $\psi_{n,s}$ is equivariant, it is sufficient to prove this at the origin $o=eK$ of M. For an arbitrary vector $x\in T_oM$, we denote by γ the geodesic of M such that $\gamma(0)=o$ and $\gamma'(0)=x$. Let (G, K) be a symmetric pair corresponding to M and $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ the canonical decomposition. Then the geodesic γ is described in such a way that $\gamma(t)=\exp tX\cdot o$, where $\exp tX$ denotes a one-parameter subgroup of G and X is a vector in $\mathfrak{m}$ corresponding to x. Moreover $(\exp tX)\cdot y$ is a parallel vector field along γ, which we denote by Y. Similarly we set $Z=(\exp tX)\cdot z$, $U=(\exp tX)\cdot u$, etc. Then

$$\langle \mathring{B}_2(Y, Z), \mathring{B}_2(U, V)\rangle = \langle \mathring{B}_2((\exp tX)y, (\exp tX)z), \mathring{B}_2((\exp tX)u, (\exp tX)v)\rangle$$
$$= \langle \rho(\exp tX)\mathring{B}_2(y, z), \rho(\exp tX)\mathring{B}_2(u, v)\rangle$$
$$= \langle \mathring{B}_2(y, z), \mathring{B}_2(u, v)\rangle .$$

Therefore $\langle \mathring{B}_2(Y, Z), \mathring{B}_2(U, V)\rangle$=constant along γ. Since $\langle B_2(Y, Z), B_2(U, V)\rangle = \langle \mathring{B}_2(Y, Z), \mathring{B}_2(U, V)\rangle$=constant along γ, we have $\langle \overline{\nabla} B_2(x, y, z), B_2(u, v)\rangle + \langle B_2(y, z), \overline{\nabla} B_2(x, u, v)\rangle = 0$. Since the above equation holds for any vectors x, y, z, u and v, and $\overline{\nabla} B_2$ and B_2 are symmetric,

$$\langle \overline{\nabla} B_2(x, y, z), B_2(u, v)\rangle = -\langle B_2(y, z), \overline{\nabla} B_2(x, u, v)\rangle$$
$$= \langle \overline{\nabla} B_2(u, y, z), B_2(x, v)\rangle$$
$$= -\langle B_2(u, z), \overline{\nabla} B_2(y, x, v)\rangle$$
$$= \langle \overline{\nabla} B_2(v, u, z), B_2(y, x)\rangle$$
$$= -\langle B_2(u, v), \overline{\nabla} B_2(x, y, z)\rangle .$$

Therefore we get $\langle \overline{\nabla} B_2(x, y, z), B_2(u, v)\rangle = 0$. Similarly we have $\langle \overline{\nabla} \mathring{B}_2(x, y, z), \mathring{B}_2(u, v)\rangle = 0$. By the definition of the third fundamental form, we obtain $B_3 = \overline{\nabla} B_2$ and $\mathring{B}_3 = \overline{\nabla} \mathring{B}_2$.

Next we shall show that

$$\text{(3.2)} \qquad \langle B_3(X, Z, W), B_3(Y, U, V)\rangle - \langle B_3(Y, Z, W), B_3(X, U, V)\rangle$$
$$= \langle R^{\perp}(X, Y)B_2(Z, W) - B_2(R(X, Y)Z, W) - B_2(Z, R(X, Y)W), B_2(U, V)\rangle$$

and the same equation holds for the third fundamental form $\mathring{B}_3$ of $\psi_{n,s}$. Since, for any vector fields Y, Z, W, U and V, $\langle \overline{\nabla} B_2(Y, Z, W), B_2(U, V)\rangle = 0$, differentiating it with respect to X, we have

$$\langle \overline{\nabla}\overline{\nabla} B_2(X, Y, Z, W), B_2(U, V)\rangle = -\langle \overline{\nabla} B_2(Y, Z, W), \overline{\nabla} B_2(X, U, V)\rangle$$
$$= -\langle B_3(Y, Z, W), B_3(X, U, V)\rangle .$$

This, together with Lemma 3.3 (3), gives (3.2). For $\mathring{B}_3$, the situation is quite similar. By Lemma 3.3 (2),

$$\langle R^{\perp}(X, Y)B_2(Z, W), B_2(U, V)\rangle = \langle [H_{B_2(Z,W)}, H_{B_2(U,V)}](X), Y\rangle .$$

Moreover

$$\langle H_{B_2(Z,W)}X, Y\rangle = \langle B_2(X, Y), B_2(Z, W)\rangle = \langle \mathring{B}_2(X, Y), \mathring{B}_2(Z, W)\rangle$$
$$= \langle H_{\mathring{B}_2(Z,W)}X, Y\rangle .$$

Thus we have

$$\langle R^{\perp}(X, Y)B_2(Z, W)-B_2(R(X, Y)Z, W)-B_2(Z, R(X, Y)W), B_2(U, V)\rangle$$
$$=\langle R^{\perp}(X, Y)\mathring{B}_2(Z, W)-\mathring{B}_2(R(X, Y)Z, W)-\mathring{B}_2(Z, R(X, Y)W), \mathring{B}_2(U, V)\rangle .$$

Now we shall prove Proposition 3.1 for B_3. We recall the third fundamental form $\mathring{B}_3$ of $\psi_{n,s}$ is constant isotropic. Namely there exists a constant λ_0 such that for any unit tangent vector u of M, $\|\mathring{B}_3(u, u, u)\|=\lambda_0$. On the other hand we assume that B_3 of φ is λ-isotropic and λ is not necessarily constant on M. By Lemma 3.2,

$$\mathcal{S}_6\{\langle B_3(u_1, u_2, u_3), B_3(u_4, u_5, u_6)\rangle\}=\lambda^2\mathcal{S}_6\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\}$$

for $u_1, u_2, \cdots, u_6\in T_pM$ at an arbitrary point $p\in M$. This, together with (3.2), yields

$$(3.3)\qquad 6!\langle B_3(u_1, u_2, u_3), B_3(u_4, u_5, u_6)\rangle$$
$$=\lambda^2\mathcal{S}_6\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\}+\langle R^{\perp}(u_3, u_4)B_2(u_1, u_2)$$
$$-B_2(R(u_3, u_4)u_1, u_2)-B_2(u_1, R(u_3, u_4)u_2), B_2(u_5, u_6)\rangle+\cdots .$$

Similarly

$$(3.4)\qquad 6!\langle \mathring{B}_3(u_1, u_2, u_3), \mathring{B}_3(u_4, u_5, u_6)\rangle$$
$$=\lambda_0^2\mathcal{S}_6\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\}+\langle R^{\perp}(u_3, u_4)\mathring{B}_2(u_1, u_2)$$
$$-\mathring{B}_2(R(u_3, u_4)u_1, u_2)-\mathring{B}_2(u_1, R(u_3, u_4)u_2), \mathring{B}_2(u_5, u_6)\rangle+\cdots .$$

Since the immersion φ is minimal, for an orthonormal basis $\{e_1, \cdots, e_n\}$ we get

$$0=(6!)\sum_{i,j,k}\langle B_3(e_i, e_j, e_j), B_3(e_i, e_k, e_k)\rangle$$
$$=\lambda^2 48(n^3+6n^2+8n)+(\text{term not containing } \lambda).$$

Similarly $0=\lambda_0^2 48(n^3+6n^2+8n)+(\text{the same term as above})$. We remark that terms besides the first term of the right-hand-side of (3.3) are equal to those of the right-hand-side of (3.4). Therefore we have $\lambda=\lambda_0$ and again by comparing (3.3) and (3.4) we get $\langle B_3(u_1, u_2, u_3), B_3(u_4, u_5, u_6)\rangle=\langle \mathring{B}_3(u_1, u_2, u_3), \mathring{B}_3(u_4, u_5, u_6)\rangle$. Also this implies the dimension of O_p^3 of φ is equal to that of $\psi_{n,s}$ at every point $p\in M$. Therefore every point of M is degree 3 regular for the immersion φ.

Now we will apply the mathematical induction. For this we set the assumptions of the induction as follows;

(1) We assume that $j\geqq 3$ and that every point of M is degree j regular for the immersion φ. Moreover we assume that

$$\langle B_k(u_1, \cdots, u_k), B_k(v_1, \cdots, v_k)\rangle=\langle \mathring{B}_k(u_1, \cdots, u_k), \mathring{B}_k(v_1, \cdots, v_k)\rangle$$

for $2\leqq k\leqq j$ at every point of M.

This implies that the vector bundle over M which is given by restricting the tangent bundle of S_1^l to M admits the following orthogonal decomposition;

$$TS_1^l|_M = TM + O^2 + O^3 + \cdots + O^j + Q_j ,$$

where O^k, $2 \leqq k \leqq j$, denotes the vector bundle whose fibre at p consists of O_p^k and Q_j denotes the vector bundle whose fibre at p consists of the orthogonal complement of $\sum_{k=2}^{j} O_p^k$ in $N_p(M)$.

(2) Next we assume that $\bar{\nabla} B_{k-1}(u_1, \cdots, u_k)$ has the components only in O^{k-2} and O^k for $3 \leqq k \leqq j$. By definition $B_k(u_1, \cdots, u_k)$=the component of $\bar{\nabla} B_{k-1}(u_1, \cdots, u_k)$ in O^k. On the other hand we define a tensor field D_k as follows; $D_k(u_1, \cdots, u_k)$=the component of $\bar{\nabla} B_{k-1}(u_1, \cdots, u_k)$ in O^{k-2}. Similarly we denote by $\mathring{D}_k$ the corresponding tensor field for $\psi_{n,s}$. Moreover we assume that $\langle D_k(u_1, \cdots, u_k), D_k(v_1, \cdots, v_k)\rangle = \langle \mathring{D}_k(u_1, \cdots, u_k), \mathring{D}_k(v_1, \cdots, v_k)\rangle$ for $3 \leqq k \leqq j$.

(3) Finally we assume that B_{j+1} is λ-isotropic.

Under these assumptions, we shall show that the same statements hold for B_{j+1}. We proceed on the same way as the process from B_2 to B_3.

Step 1.

$$\langle \bar{\nabla} B_j(u_0, u_1, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle + \langle B_j(u_1, \cdots, u_j), \bar{\nabla} B_j(u_0, v_1, \cdots, v_j)\rangle = 0 .$$

This can be proved in the same way as in the case $j=2$ by using the assumption (1), i.e.,

$$\langle B_j(u_1, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle = \langle \mathring{B}_j(u_1, \cdots, u_j), \mathring{B}_j(v_1, \cdots, v_j)\rangle .$$

Step 2.

$$\langle \bar{\nabla} B_j(x, y, u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle = \langle \bar{\nabla} B_j(y, x, u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle .$$

By the assumption (2), $\bar{\nabla} B_{j-1} = B_j + D_j$. Since

$$\bar{\nabla} D_j(x, y, u_2, \cdots, u_j) = \nabla_x^{\perp}(D_j(Y, U_2, \cdots, U_j)) - D_j(\nabla_x Y, U_2, \cdots, U_j) - \cdots - D_j(Y, U_2, \cdots, \nabla_x U_j)$$

and

$$D_j(Y, U_2, \cdots, U_j) \in O^{j-2}, \quad \bar{\nabla} D_j(x, y, u_2, \cdots, u_j) \in \sum_{k=2}^{j-1} O^k .$$

Thus we have

$$\begin{aligned} &\langle \bar{\nabla} B_j(x, y, u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle \\ &= \langle \bar{\nabla}^2 B_{j-1}(x, y, u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle \\ &= \langle \bar{\nabla}^2 B_{j-1}(y, x, u_2, \cdots, u_j) + R^{\perp}(x, y) B_{j-1}(u_2, \cdots, u_j) \end{aligned}$$

$$-\sum_{i=2}^{j}B_{j-1}(u_2, \cdots, R(x, y)u_i, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle$$

$$=\langle\overline{\nabla}^2 B_{j-1}(y, x, u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle$$

$$+\langle R^{\perp}(x, y)B_{j-1}(u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle .$$

For $j\geqq 3$, $H_{B_j(v_1,\cdots,v_j)}=0$. Therefore by Ricci equation,

$$\langle R^{\perp}(x, y)B_{j-1}(u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle=0 .$$

Thus the statement of Step 2 holds.

Step 3. $\langle\overline{\nabla}B_j(u_0, u_1, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle=0$, that is, the component of $\overline{\nabla}B_j$ in O^j vanishes. By Step 2, $\langle\overline{\nabla}B_j(x, y, u_2, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle$ is symmetric with respect to $x, y, u_2, \cdots, u_j$, which, combined with Step 1, implies the assertion.

The following two facts are easily seen.

Step 4.

$$\langle\overline{\nabla}B_j(u_0, u_1, \cdots, u_j), B_{j-1}(v_1, \cdots, v_{j-1})\rangle$$
$$=-\langle B_j(u_1, \cdots, u_j), B_j(u_0, v_1, \cdots, v_{j-1})\rangle .$$

Step 5. For $2\leqq k\leqq j-2$, $\langle\overline{\nabla}B_j(u_0, u_1, \cdots, u_j), B_k(v_1, \cdots, v_k)\rangle=0$. By Step 3, Step 4, and Step 5, we see that $\overline{\nabla}B_j$ has the components only in O^{j-1} and Q_j. By the definition we set $B_{j+1}(u_0, u_1, \cdots, u_j)=$the Q_j-component of $\overline{\nabla}B_j(u_0, u_1, \cdots, u_j)$ and $D_{j+1}(u_0, u_1, \cdots, u_j)=$the O^{j-1}-component of $\overline{\nabla}B_j(u_0, u_1, \cdots, u_j)$. Similarly we can define $\mathring{B}_{j+1}$ and $\mathring{D}_{j+1}$ for the standard minimal immersion $\psi_{n,s}$. By Step 4, we have

$$\langle D_{j+1}(u_0, u_1, \cdots, u_j), B_{j-1}(v_1, \cdots, v_{j-1})\rangle$$
$$=-\langle B_j(u_1, \cdots, u_j), B_j(u_0, v_1, \cdots, v_{j-1})\rangle .$$

Similarly

$$\langle\mathring{D}_{j+1}(u_0, u_1, \cdots, u_j), \mathring{B}_{j-1}(v_1, \cdots, v_{j-1})\rangle$$
$$=-\langle\mathring{B}_j(u_1, \cdots, u_j), \mathring{B}_j(u_0, v_1, \cdots, v_{j-1})\rangle .$$

Here we remark that by the assumptions of the induction there exists a unique linear isometry A of O_p^{j-1} of φ onto that of $\psi_{n,s}$ such that $AB_{j-1}(u_1, \cdots, u_{j-1})=\mathring{B}_{j-1}(u_1, \cdots, u_{j-1})$. By the above two equations and the assumptions of the induction, we have

$$\langle AD_{j+1}(u_0, u_1, \cdots, u_j), \mathring{B}_{j-1}(v_1, \cdots, v_{j-1})\rangle$$
$$=\langle D_{j+1}(u_0, u_1, \cdots, u_j), B_{j-1}(v_1, \cdots, v_{j-1})\rangle$$
$$=\langle\mathring{D}_{j+1}(u_0, u_1, \cdots, u_j), \mathring{B}_{j-1}(v_1, \cdots, v_{j-1})\rangle ,$$

which implies that $AD_{j+1}(u_0, u_1, \cdots, u_j)=\mathring{D}_{j+1}(u_0, u_1, \cdots, u_j)$. Thus we get

Step 6.

$$\langle D_{j+1}(u_0, \cdots, u_j), D_{j+1}(v_0, \cdots, v_j)\rangle = \langle \mathring{D}_{j+1}(u_0, \cdots, u_j), \mathring{D}_{j+1}(v_0, \cdots, v_j)\rangle .$$

Step 7.

$$\begin{aligned}&\langle B_{j+1}(u_0, u_1, \cdots, u_j), B_{j+1}(v_0, v_1, \cdots, v_j)\rangle \\ &\quad -\langle B_{j+1}(v_0, u_1, \cdots, u_j), B_{j+1}(u_0, v_1, \cdots, v_j)\rangle \\ &= \langle D_{j+1}(v_0, u_1, \cdots, u_j), D_{j+1}(u_0, v_1, \cdots, v_j)\rangle \\ &\quad -\langle D_{j+1}(u_0, u_1, \cdots, u_j), D_{j+1}(v_0, v_1, \cdots, v_j)\rangle \\ &\quad -\sum_{i=1}^{j} \langle B_j(u_1, \cdots, R(u_0, v_0)u_i, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle .\end{aligned}$$

Since $\langle \bar{\nabla} B_j(v_0, u_1, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle = 0$, we see that

$$\begin{aligned}&\langle \bar{\nabla}^2 B_j(u_0, v_0, u_1, \cdots, u_j), B_j(v_1, \cdots, v_j)\rangle \\ &= -\langle \bar{\nabla} B_j(v_0, u_1, \cdots, u_j), \bar{\nabla} B_j(u_0, v_1, \cdots, v_j)\rangle \\ &= -\langle B_{j+1}(v_0, u_1, \cdots, u_j), B_{j+1}(u_0, v_1, \cdots, v_j)\rangle \\ &\quad -\langle D_{j+1}(v_0, u_1, \cdots, u_j), D_{j+1}(u_0, v_1, \cdots, v_j)\rangle .\end{aligned}$$

This, together with Lemma 3.3 (3), gives Step 7.

We remark that the right-hand-side of the equation of Step 7 for the immersion φ is quite equal to that of the standard minimal immersion $\psi_{n,s}$. By the same method as in the case of B_3 we get

Step 8.

$$\begin{aligned}&\langle B_{j+1}(u_0, u_1, \cdots, u_j), B_{j+1}(v_0, v_1, \cdots, v_j)\rangle \\ &\quad = \langle \mathring{B}_{j+1}(u_0, u_1, \cdots, u_j), \mathring{B}_{j+1}(v_0, v_1, \cdots, v_j)\rangle .\end{aligned}$$

Thus our proof of Proposition 3.1 finishes.

In the remainder of this section we show the following property about $\mathring{D}_j$ of the standard minimal immersions of spheres.

PROPOSITION 3.4. *Let $\psi_{n,s}: S^n_{k(s)} \to S^{m(s)}_1$ be the standard minimal immersion of a sphere into a unit sphere ($n \geqq 2$) and for this immersion we use the notations $\mathring{B}_j$ and $\mathring{D}_j$ defined formerly. Then for an arbitrary unit tangent vector x,*

$$\mathring{D}_{j+1}(x, \cdots, x) = -(\lambda_j^2/\lambda_{j-1}^2)\mathring{B}_{j-1}(x, \cdots, x) \quad (j \geqq 3),$$

where

$$\lambda_j = \|\mathring{B}_j(x, \cdots, x)\| \quad \text{and} \quad \lambda_{j-1} = \|\mathring{B}_{j-1}(x, \cdots, x)\| .$$

Before the proof, we prepare the following algebraic lemma.

LEMMA 3.5. *Let F be a symmetric k-linear form on $\boldsymbol{R}^n$ ($k \geqq 2$). Suppose $F(v, \cdots, v)/\|v\|^k$ is constant and $\sum_{i=1}^{n} F(v_1, \cdots, v_{k-2}, e_i, e_i) = 0$ for any $v_1, \cdots, v_{k-2}$,* *where* $\{e_1, \cdots, e_n\}$ *is an orthonormal basis of* $\boldsymbol{R}^n$. *Then* $F\equiv 0$.

PROOF. We set $\lambda=F(v, \cdots, v)$ for a unit vector $v\in\boldsymbol{R}^n$. If k is odd, we have

$$\lambda=F(-v, \cdots, -v)=(-1)^k F(v, \cdots, v)=-F(v, \cdots, v)=-\lambda,$$

so that $\lambda=0$. Thus we get $F\equiv 0$. If k is even,

$$F(u_1, \cdots, u_{2j})=\frac{1}{(2j)!}\lambda S_{2j}\{\langle u_1, u_2\rangle \cdots \langle u_{2j-1}, u_{2j}\rangle\}.$$

Then for a fixed unit vector u,

$$0=\sum_{i=1}^{n} F(u, \cdots, u, e_i, e_i)=K\lambda$$

for some $K>0$. Therefore $\lambda=0$ and we have $F\equiv 0$.

PROOF OF PROPOSITION 3.4. When $n=2$, we can easily prove Proposition 3.4. So we assume that $n\geqq 3$. Since $\psi_{n,s}$ is an equivariant immersion, it is sufficient to show the relation at the origin eK of $S^n_{k(s)}$. Since $\mathring{D}_{j+1}(x, \cdots, x)\in O^{j-1}_{eK}$, it is sufficient to show that

$$\begin{aligned}&\langle \mathring{D}_{j+1}(x, \cdots, x), \mathring{B}_{j-1}(v_1, \cdots v_{j-1})\rangle\\ &\quad=-(\lambda_j^2/\lambda_{j-1}^2)\langle \mathring{B}_{j-1}(x, \cdots, x), \mathring{B}_{j-1}(v_1, \cdots, v_{j-1})\rangle\end{aligned}$$

for any $v_1, \cdots, v_{j-1}$. We recall that

$$\langle \mathring{D}_{j+1}(x, \cdots, x), \mathring{B}_{j-1}(v_1, \cdots, v_{j-1})\rangle=-\langle \mathring{B}_j(x, \cdots, x), \mathring{B}_j(x, v_1, \cdots, v_{j-1})\rangle.$$

Therefore we shall show that

$$\begin{aligned}&\langle \mathring{B}_j(x, \cdots, x), \mathring{B}_j(x, v_1, \cdots, v_{j-1})\rangle\\ &\quad=(\lambda_j^2/\lambda_{j-1}^2)\langle \mathring{B}_{j-1}(x, \cdots, x), \mathring{B}_{j-1}(v_1, \cdots, v_{j-1})\rangle.\end{aligned}$$

We set

$$\begin{aligned}F_k(v_1, \cdots, v_k)=&\langle \mathring{B}_j(x, \cdots, x), \mathring{B}_j(x, \cdots, x, v_1, \cdots, v_k)\rangle\\ &-(\lambda_j^2/\lambda_{j-1}^2)\langle \mathring{B}_{j-1}(x, \cdots, x), \mathring{B}_{j-1}(x, \cdots, x, v_1, \cdots, v_k)\rangle\end{aligned}$$

for $1\leqq k\leqq j-1$. Then F_k is a symmetric k-linear form on the tangent space $T_{eK}S^n$. We define the subspace V of $T_{eK}S^n$ by $V=\{v\in T_{eK}S^n; \langle v, x\rangle=0\}$ and we use an orthonormal basis $\{e_1, \cdots, e_n\}$ of $T_{eK}S^n$ such that $e_1=x$, and $e_2, \cdots, e_n\in V$. We see that there exists a subgroup K' of $K=SO(n)$ such that $k\cdot x=x$ for any $k\in K'$ and K' acts transitively on the unit sphere of V. So for $k\in K'$ and a unit vector $v\in V$ we have

$$\begin{aligned}&F_i(k\cdot v, \cdots, k\cdot v)\\ &\quad=\langle \mathring{B}_j(x, \cdots, x), \mathring{B}_j(x, \cdots, x, k\cdot v, \cdots, k\cdot v)\rangle\\ &\quad\quad-(\lambda_j^2/\lambda_{j-1}^2)\langle \mathring{B}_{j-1}(x, \cdots, x), \mathring{B}_{j-1}(x, \cdots, x, k\cdot v, \cdots, k\cdot v)\rangle\end{aligned}$$

$$=\langle \mathring{B}_j(k\cdot x, \cdots, k\cdot x), \mathring{B}_j(k\cdot x, \cdots, k\cdot x, k\cdot v, \cdots, k\cdot v)\rangle$$
$$-(\lambda_j^2/\lambda_{j-1}^2)\langle \mathring{B}_{j-1}(k\cdot x, \cdots, k\cdot x), \mathring{B}_{j-1}(k\cdot x, \cdots, k\cdot x, k\cdot v, \cdots, k\cdot v)\rangle$$
$$=\langle \rho(k)\mathring{B}_j(x, \cdots, x), \rho(k)\mathring{B}_j(x, \cdots, x, v, \cdots, v)\rangle$$
$$-(\lambda_j^2/\lambda_{j-1}^2)\langle \rho(k)\mathring{B}_{j-1}(x, \cdots, x), \rho(k)\mathring{B}_{j-1}(x, \cdots, x, v, \cdots, v)\rangle$$
$$=F_i(v, \cdots, v).$$

Therefore $F_i(v, \cdots, v)/\|v\|^i$ is constant on V. Since $\mathring{B}_j$ and $\mathring{B}_{j-1}$ are isotropic, $F_1(x)=\lambda_j^2-(\lambda_j^2/\lambda_{j-1}^2)\times\lambda_{j-1}^2=0$. By Lemma 3.2, it is easy to see that $\langle \mathring{B}_j(x, \cdots, x), \mathring{B}_j(x, \cdots, x, v)\rangle=\langle \mathring{B}_{j-1}(x, \cdots, x), \mathring{B}_{j-1}(x, \cdots, x, v)\rangle=0$ for $v\in V$, which implies that $F_1(v)=0$ for $v\in V$. Therefore we have $F_1\equiv 0$ on $T_{eK}S^n$.

Next we shall show that $F_2\equiv 0$. Since $\psi_{n,s}$ is minimal, we get

$$\sum_{i=1}^n F_2(e_i, e_i)=\langle \mathring{B}_j(x, \cdots, x), \sum_{i=1}^n \mathring{B}_j(x, \cdots, x, e_i, e_i)\rangle$$
$$-(\lambda_j^2/\lambda_{j-1}^2)\langle \mathring{B}_{j-1}(x, \cdots, x), \sum_{i=1}^n \mathring{B}_{j-1}(x, \cdots, x, e_i, e_i)\rangle$$
$$=0.$$

Since $F_2(e_1, e_1)=F_2(x, x)=0$, we have $\sum_{i=2}^n F_2(e_i, e_i)=0$. Using Lemma 3.5 we obtain $F_2\equiv 0$ on V. This, together with $F_1\equiv 0$, implies $F_2\equiv 0$ on $T_{eK}S^n$.

Now we apply a mathematical induction. We assume that $F_i\equiv 0$ on $T_{eK}S^n$ for $1\leqq i\leqq k-1$, where $k\geqq 3$. Since $\sum_{i=1}^n F_k(v_1, \cdots, v_{k-2}, e_i, e_i)=0$ and $F_k(v_1, \cdots, v_{k-2}, x, x)=F_{k-2}(v_1, \cdots, v_{k-2})=0$, we have $\sum_{i=2}^n F_k(v_1, \cdots, v_{k-2}, e_i, e_i)=0$. Again by Lemma 3.5 $F_k\equiv 0$ on V. Combining this with $F_i\equiv 0$ for $1\leqq i\leqq k-1$, we have $F_k\equiv 0$ on $T_{eK}S^n$. In particular $F_{j-1}\equiv 0$ on $T_{eK}S^n$. Then

$$\langle \mathring{B}_j(x, \cdots, x), \mathring{B}_j(x, v_1, \cdots, v_{j-1})\rangle$$
$$=(\lambda_j^2/\lambda_{j-1}^2)\langle \mathring{B}_{j-1}(x, \cdots, x), \mathring{B}_{j-1}(v_1, \cdots, v_{j-1})\rangle.$$

Thus Proposition 3.4 is proved.

4. On the rigidity of isotropic minimal immersions.

In this section we show two theorems.

THEOREM 4.1. *Let $\psi_{n,s}: M_{k(s)}^n\to S_1^{m(s)}$ be the standard minimal immersion of an n-dimensional compact rank 1 symmetric space into a unit sphere corresponding to the s-th eigenvalue. Let $\varphi: M_{k(s)}^n\to S_1^l$ be an another minimal immersion corresponding to the same eigenvalue and assume that φ is full. We assume that* *in the case of* $M=S^n$ *the degree of* $\varphi \geqq [s/2]$ *and the j-th fundamental form* B_j *of* φ *is isotropic at every degree* $j-1$ *regular point* $p \in R_{j-1}$ *for* $2 \leqq j \leqq [s/2]$ *and that in the case of* $M=P_n(\boldsymbol{C})$, $P_n(\boldsymbol{H})$, *or* $P_2(\mathrm{Cay})$ *the degree of* $\varphi \geqq s$ *and the j-th fundamental form* B_j *of* φ *is isotropic at every degree* $j-1$ *regular point* $p \in R_{j-1}$ *for* $2 \leqq j \leqq s$. *Then we see that* φ *is equivalent to the standard minimal immersion* $\phi_{n,s}$ *and in particular* $l=m(s)$.

Furthermore, we get the following result on the non-rigidity for S^n.

THEOREM 4.2. *For* $n \geqq 3$ *and* $s \geqq 6$, *there are many inequivalent minimal immersions of* $S^n_{k(s)}$ *into a unit sphere such that* B_k *is isotropic on* $S^n_{k(s)}$ *for* $2 \leqq k \leqq [s/2]-1$.

We apply the method of do Carmo and Wallach ([**4**]). We formulate the rigidity problem following do Carmo and Wallach.

PROPOSITION 4.3. *Let* $\varphi: M^n_{k(s)} \to S^l_1$ *be a minimal immersion. Then there exists a symmetric positive semi-definite linear map* A *of* $\boldsymbol{R}^{m(s)+1}$ *such that* φ *is equivalent to* $A \circ \phi_{n,s}$. *Furthermore* φ *is equivalent to* $\phi_{n,s}$ *if and only if the associated symmetric linear mapping* A *of* φ *is equal to the identity map.*

PROPOSITION 4.4. *Let* A *be a symmetric positive semi-definite linear map of* $\boldsymbol{R}^{m(s)+1}$. *We assume that* $A \circ \phi_{n,s}$ *is a minimal immersion of* $M^n_{k(s)}$ *into a unit sphere and we denote by* B_j *and* $\mathring{B}_j$ *the j-th fundamental form of* $A \circ \phi_{n,s}$ *and that of* $\phi_{n,s}$ *respectively. (When there is no danger of confusion, we use* ϕ *instead of* $\phi_{n,s}$.) *Then we have* $B_2(u_1, u_2)=A\mathring{B}_2(u_1, u_2)$ *at any point* $p \in M$. *Furthermore if* B_k *is isotropic on* M *for* $2 \leqq k \leqq j$, *then* $B_k(u_1, \cdots, u_k)=A\mathring{B}_k(u_1, \cdots, u_k)$ *for* $3 \leqq k \leqq j+1$. *Under the same assumption, for the orthogonal decomposition* $\boldsymbol{R}^{m(s)+1}=\boldsymbol{R}\cdot\phi(p)+\phi_* T_p M+O^2_p+\cdots+O^d_p$ *at* p *with respect to* ϕ, *where* $d=$ *the degree of* ϕ, A *is an isometric linear mapping on the subspace*

$$\boldsymbol{R}\cdot\phi(p)+\phi_* T_p M+O^2_p+\cdots+O^j_p \qquad \text{at any point} \quad p \in M.$$

We remark that we identify (as usual) $T_p S^{m(s)}_1$ with the subspace of $\boldsymbol{R}^{m(s)+1}$. Under the identification, the above statements make sense.

PROOF OF PROPOSITION 4.4. By the argument in the proof of Proposition 3.1, $\boldsymbol{R}^{m(s)+1}$ admits the following orthogonal decomposition with respect to the immersion $A \circ \phi$:

$$\boldsymbol{R}^{m(s)+1}=\boldsymbol{R}\cdot A\circ\phi(p)+(A\circ\phi)_* T_p M+\tilde{O}^2_p+\cdots+\tilde{O}^j_p+(\tilde{Q}_j)_p\,.$$

Since $A \circ \phi$ is an isometric immersion of M into $S^{m(s)}_1$, it is easily seen that A is an isometric linear map from $\boldsymbol{R}\cdot\phi(p)$ to $\boldsymbol{R}\cdot A\circ\phi(p)$ and also from $\phi_* T_p M$ to $(A\circ\phi)_* T_p M$. We denote by $\tilde{\nabla}$ and ∇' the Riemannian connections on $S^{m(s)}_1$ and $\boldsymbol{R}^{m(s)+1}$ respectively. Then we see that $\tilde{\nabla}_X Y=\nabla'_X Y+\langle X, Y\rangle p$ at $p \in S^{m(s)}_1$ for any vector fields X, Y on $S^{m(s)}_1$. For an arbitrary unit vector x in $T_p M$, we set $\gamma(t)$ to be the geodesic such that $\gamma(0)=p$ and $\dot{\gamma}(0)=x$. We denote by σ the curve in $S^{m(s)}_1$ defined by $\sigma=\phi\circ\gamma$ and naturally consider σ also as a curve in

$\boldsymbol{R}^{m(s)+1}$. Then we have

$$B_2(x, x)=\tilde{\nabla}_{(A\circ\sigma)^{\cdot}}(A\circ\sigma)^{\cdot}=\nabla'_{(A\circ\sigma)^{\cdot}}(A\circ\sigma)^{\cdot}+A\circ\phi(p)$$
$$=\frac{d^2}{dt^2}A\circ\sigma(t)\Big|_{t=0}+A\circ\phi(p)$$
$$=A\frac{d^2}{dt^2}\sigma(t)\Big|_{t=0}+A\circ\phi(p)$$
$$=A\Big(\frac{d^2}{dt^2}\sigma(t)\Big|_{t=0}+\phi(p)\Big)$$
$$=A(\tilde{\nabla}_{\dot\sigma}\dot\sigma)$$
$$=A\mathring{B}_2(x, x).$$

Since $B_2(x, x)$ and $\mathring{B}_2(x, x)$ span linearly $\tilde{O}^2$ and O^2 respectively, we have $A\mathring{B}_2(u_1, u_2)=B_2(u_1, u_2)$ for $u_1, u_2\in T_pM$. If the second fundamental form B_2 of $A\circ\phi$ is isotropic on M, then by Proposition 3.1 we get $\langle B_2(u_1, u_2), B_2(v_1, v_2)\rangle$ $=\langle A\mathring{B}_2(u_1, u_2), A\mathring{B}_2(v_1, v_2)\rangle=\langle\mathring{B}_2(u_1, u_2), \mathring{B}_2(v_1, v_2)\rangle$. Therefore A is an isometric linear mapping from O^2 to $\tilde{O}^2$.

Next we prove Proposition 4.4 for the third fundamental form B_3. Since B_2 is isotropic, by the proof of Proposition 3.1, $B_3(x, x, x)=\overline{\nabla}B_2(x, x, x)$. Then

$$B_3(x, x, x)=\nabla^{\perp}_x(B_2(\dot\gamma, \dot\gamma))$$
$$=\nabla'_{(A\circ\phi)_*x}(B_2(\dot\gamma, \dot\gamma))+A(\phi_*H_{\mathring{B}_2(x,x)}x)$$
$$=\frac{d}{dt}(B_2(\dot\gamma, \dot\gamma))+A(\phi_*H_{\mathring{B}_2(x,x)}x)$$
$$=A\Big(\frac{d}{dt}\mathring{B}_2(\dot\gamma, \dot\gamma)+\phi_*H_{\mathring{B}_2(x,x)}x\Big)$$
$$=A(\nabla'_{\phi_*x}\mathring{B}_2(\dot\gamma, \dot\gamma)+\phi_*H_{\mathring{B}_2(x,x)}x)$$
$$=A\mathring{B}_3(x, x, x).$$

This implies that $B_3(u_1, u_2, u_3)=A\mathring{B}_3(u_1, u_2, u_3)$. If B_3 is isotropic, by Proposition 3.1, A is an isometric linear mapping from O^3 to $\tilde{O}^3$. We assume that $B_i(u_1, \cdots, u_i)$ $=A\mathring{B}_i(u_1, \cdots, u_i)$ for $2\leqq i\leqq k$ $(k\geqq 3)$ at every point of M and that $A: O^i\to\tilde{O}^i$ is an isometric linear mapping for all i. Then by the argument of the proof of Proposition 3.1,

$$B_{k+1}(x, \cdots, x)+D_{k+1}(x, \cdots, x)$$
$$=\nabla^{\perp}_x(B_k(\dot\gamma, \cdots, \dot\gamma))$$
$$=\tilde{\nabla}_{(A\circ\phi)_*x}(B_k(\dot\gamma, \cdots, \dot\gamma))+(A\circ\phi)_*H_{B_k(x,\dots,x)}x$$

$$
\begin{aligned}
&=\tilde{\nabla}_{(A\circ\phi)_*x}(B_k(\dot{\gamma}, \cdots, \dot{\gamma}))\\
&=\nabla'_{(A\circ\phi)_*x}(B_k(\dot{\gamma}, \cdots, \dot{\gamma}))\\
&=\frac{d}{dt}B_k(\dot{\gamma}, \cdots, \dot{\gamma})\\
&=A\frac{d}{dt}\mathring{B}_k(\dot{\gamma}, \cdots, \dot{\gamma})\\
&=A\nabla_x^{\perp}(\mathring{B}_k(\dot{\gamma}, \cdots, \dot{\gamma}))\\
&=A(\mathring{B}_{k+1}(x, \cdots, x)+\mathring{D}_{k+1}(x, \cdots, x)).
\end{aligned}
$$

Here again recalling the proof of Proposition 3.1, we have

$$
\begin{aligned}
&\langle A\mathring{D}_{k+1}(x, \cdots, x), B_{k-1}(v_1, \cdots, v_{k-1})\rangle\\
&\quad=\langle A\mathring{D}_{k+1}(x, \cdots, x), A\mathring{B}_{k-1}(v_1, \cdots, v_{k-1})\rangle\\
&\quad=\langle \mathring{D}_{k+1}(x, \cdots, x), \mathring{B}_{k-1}(v_1, \cdots, v_{k-1})\rangle\\
&\quad=\langle D_{k+1}(x, \cdots, x), B_{k-1}(v_1, \cdots, v_{k-1})\rangle.
\end{aligned}
$$

Therefore $A\mathring{D}_{k+1}(x, \cdots, x)=D_{k+1}(x, \cdots, x)$. Thus we get $B_{k+1}(x, \cdots, x)=A\mathring{B}_{k+1}(x, \cdots, x)$. This implies that $B_{k+1}(u_1, \cdots, u_{k+1})=A\mathring{B}_{k+1}(u_1, \cdots, u_{k+1})$. If B_{k+1} is isotropic on M, A is an isometric linear mapping from O^{k+1} to $\tilde{O}^{k+1}$. Thus Proposition 4.4 is proved.

Now we identify the space of all symmetric linear mappings of $\boldsymbol{R}^{m(s)+1}$ with $S^2(\boldsymbol{R}^{m(s)+1})$, the symmetric square of $\boldsymbol{R}^{m(s)+1}$ as follows: if $u, v\in\boldsymbol{R}^{m(s)+1}$, $u\cdot v\in S^2(\boldsymbol{R}^{m(s)+1})$ (the symmetric product of two vectors will be denoted by $u\cdot v$), and if $t\in\boldsymbol{R}^{m(s)+1}$, we set $u\cdot v(t)=\frac{1}{2}\{\langle u, t\rangle v+\langle v, t\rangle u\}$, where $\langle\ ,\ \rangle$ is the inner product on $\boldsymbol{R}^{m(s)+1}$. Under this identification, the inner product on $S^2(\boldsymbol{R}^{m(s)+1})$ is given by $(A, B)=\mathrm{tr}AB$. We note that if $A\in S^2(\boldsymbol{R}^{m(s)+1})$ and $u, v\in\boldsymbol{R}^{m(s)+1}$, then $\langle Au, v\rangle=(A, u\cdot v)$.

For the orthogonal decomposition $\boldsymbol{R}^{m(s)+1}=\boldsymbol{R}\cdot\phi(p)+\phi_*T_pM+O_p^2+\cdots+O_p^d$ with respect to the standard minimal immersion $\phi_{n,s}$, let $S^2(\boldsymbol{R}\cdot\phi(p)+\phi_*T_pM+\sum_{i=2}^{j}O_p^i)$ be the symmetric square of $\boldsymbol{R}\cdot\phi(p)+\phi_*T_pM+O_p^2+\cdots+O_p^j$. And let $W^{(j)}$ be the subspace of $S^2(\boldsymbol{R}^{m(s)+1})$ spanned by $\bigcup_{p\in M}S^2(\boldsymbol{R}\cdot\phi(p)+\phi_*T_pM+\sum_{i=2}^{j}O_p^i)$. Let W_0 and W_1 be the subspace of $S^2(\boldsymbol{R}^{m(s)+1})$ spanned by $\bigcup_{p\in M}\phi(p)\cdot\phi(p)$ and $\bigcup_{p\in M}S^2(\phi_*T_pM)$, respectively. Let W_j be the subspace of $S^2(\boldsymbol{R}^{m(s)+1})$ spanned by $\bigcup_{p\in M}S^2(O_p^j)$ for any j.

LEMMA 4.5. (1) *If $A\in S^2(\boldsymbol{R}^{m(s)+1})$ and $A\geqq 0$ (i.e., A is positive semi-definite), then $A\circ\phi_{n,s}$ is a minimal immersion of $M_{k(s)}^n$ in $S_1^{m(s)}$ such that the k-th funda-*

mental form B_k of $A\circ\psi_{n,s}$ is isotropic on M for $2\leqq k\leqq j\leqq$the degree of $\psi_{n,s}$ if and only if $A^2-I\in(W^{(j)})^{\perp}$, where $(W^{(j)})^{\perp}$ denotes the orthogonal complement of $W^{(j)}$ in $S^2(\boldsymbol{R}^{m(s)+1})$ and I denotes the identity transformation of $\boldsymbol{R}^{m(s)+1}$.

(2) $W_0+W_1+\cdots+W_j=W^{(j)}$.

(3) $W_0\subset W_1$.

PROOF. (1) It is known that $A\circ\psi_{n,s}$ is a minimal immersion if and only if $A^2-I\in W_1^{\perp}$, where $W_1^{\perp}$ denotes the orthogonal complement of W_1 in $S^2(\boldsymbol{R}^{m(s)+1})$ (Wallach [**10**]). If $A\circ\psi_{n,s}$ is a minimal immersion such that B_k is isotropic for $2\leqq k\leqq j$, then, by Proposition 4.4, A is an isometric linear mapping on the subspace $\boldsymbol{R}\cdot\psi(p)+\psi_*T_pM+O_p^2+\cdots+O_p^j$ of $\boldsymbol{R}^{m(s)+1}$ at every point $p\in M$. Therefore $\langle Au, Av\rangle=\langle u, v\rangle$ for any two vectors $u, v\in\boldsymbol{R}\cdot\psi(p)+\psi_*T_pM+O_p^2+\cdots+O_p^j$. This implies that $(A^2, u\cdot v)=(I, u\cdot v)$. Thus A^2-I is orthogonal to $S^2(\boldsymbol{R}\cdot\psi(p)+\psi_*T_pM+O_p^2+\cdots+O_p^j)$. Since p is arbitrary in M, A^2-I is orthogonal to $W^{(j)}$.

Conversely if $A^2-I\in(W^{(j)})^{\perp}$, $A\circ\psi_{n,s}$ is a minimal immersion of $M_{k(s)}^n$ in $S_1^{m(s)}$. Furthermore by Proposition 4.4 we have

$$\begin{aligned}0&=(A^2-I, \mathring{B}_2(u_1, u_2)\cdot\mathring{B}_2(v_1, v_2))\\&=\langle A\mathring{B}_2(u_1, u_2), A\mathring{B}_2(v_1, v_2)\rangle-\langle\mathring{B}_2(u_1, u_2), \mathring{B}_2(v_1, v_2)\rangle\\&=\langle B_2(u_1, u_2), B_2(v_1, v_2)\rangle-\langle\mathring{B}_2(u_1, u_2), \mathring{B}_2(v_1, v_2)\rangle \quad \text{at every point } p,\end{aligned}$$

where B_2 as usual denotes the second fundamental form of $A\circ\psi_{n,s}$. Thus B_2 is isotropic. Repeating this process, we can prove Lemma 4.5 (1).

(2) It is trivial that $W_0+W_1+\cdots+W_j\subset W^{(j)}$. We shall prove $W_0+W_1+\cdots+W_j\supset W^{(j)}$. We assume that $C\in S^2(\boldsymbol{R}^{m(s)+1})$ is orthogonal to $W_0+W_1+\cdots+W_j$. Let $t>0$ be such that $I+tC\geqq 0$ and let A be the positive square root of $I+tC$. Then $A^2-I=tC$ is orthogonal to W_1. Therefore $A\circ\psi_{n,s}$ is a minimal immersion of $M_{k(s)}^n$ to $S_1^{m(s)}$. Since $A^2-I\in W_2^{\perp}$, by the argument in (1) the second fundamental form B_2 of $A\circ\psi_{n,s}$ is isotropic on M. Then Proposition 4.4 implies that $B_3=A\mathring{B}_3$. Similarly, since $A^2-I\in W_3^{\perp}$, B_3 is isotropic on M. Repeating this argument, we see that the k-th fundamental form B_k of $A\circ\psi_{n,s}$ is isotropic for $2\leqq k\leqq j$. Then (1) implies that $A^2-I=tC\perp W^{(j)}$. Thus (2) is proved.

(3) It is proved in Wallach [**10**].

LEMMA 4.6. *If the degree of the standard minimal immersion $\psi_{n,s}$ is $d\geqq 2$, then $W^{(j)}=S^2(\boldsymbol{R}^{m(s)+1})$ for $j=[d/2]$.*

PROOF. We use again the method of Wallach ([**10**]). We show that if $x\in T_pM$ and if $\sigma:(-\varepsilon, \varepsilon)\to M$ is the geodesic through p in M with tangent vector x, then $p^2, p\cdot\dot{\sigma}(0), p\cdot\ddot{\sigma}(0), \cdots, p\cdot\sigma^{(d)}(0)\in W^{(j)}$, where we identify $\sigma(t)$ with the curve $\psi\circ\sigma(t)$ in $\boldsymbol{R}^{m(s)+1}$ and $\sigma^{(k)}(t)$ denotes $\dfrac{d^k}{dt^k}\sigma(t)$ in $\boldsymbol{R}^{m(s)+1}$. We shall show only that $p\cdot\sigma^{(d)}(0)\in W^{(j)}$. We assume that d=odd, i.e., $d=2j+1$. When d=even, the proof is quite similar. We easily see that $\sigma^{(j)}(t)\in \boldsymbol{R}\cdot\psi(\sigma(t))+\psi_* T_{\sigma(t)}M+O^2_{\sigma(t)}+\cdots+O^j_{\sigma(t)}$, which implies $\sigma^{(j)}(t)\cdot\sigma^{(j)}(t)\in W^{(j)}$. First we have $(\sigma^{(j)}\cdot\sigma^{(j)})'(0)=2\sigma^{(j)}(0)\cdot\sigma^{(j+1)}(0)\in W^{(j)}$. Similarly

$$(\sigma^{(j-1)}\cdot\sigma^{(j)})^{(2)}(0)=\sigma^{(j+1)}(0)\cdot\sigma^{(j)}(0)+2\sigma^{(j)}(0)\cdot\sigma^{(j+1)}(0)+\sigma^{(j-1)}(0)\cdot\sigma^{(j+2)}(0)$$
$$=3\sigma^{(j+1)}(0)\cdot\sigma^{(j)}(0)+\sigma^{(j-1)}(0)\cdot\sigma^{(j+2)}(0)\in W^{(j)}\,.$$

Therefore we get $\sigma^{(j-1)}(0)\cdot\sigma^{(j+2)}(0)\in W^{(j)}$. Similarly

$$(\sigma^{(j-2)}\cdot\sigma^{(j)})^{(3)}(0)=\sigma^{(j+1)}(0)\cdot\sigma^{(j)}(0)+3\sigma^{(j)}(0)\cdot\sigma^{(j+1)}(0)$$
$$+3\sigma^{(j-1)}(0)\cdot\sigma^{(j+2)}(0)+\sigma^{(j-2)}(0)\cdot\sigma^{(j+3)}(0)\in W^{(j)}\,.$$

These, together with the former results, imply $\sigma^{(j-2)}(0)\cdot\sigma^{(j+3)}(0)\in W^{(j)}$. Repeating these calculations, we see that $p\cdot\sigma^{(2j+1)}(0)=p\cdot\sigma^{(d)}(0)\in W^{(j)}$. By the same method we can prove that $p\cdot\sigma^{(k)}(0)\in W^{(j)}$ for $0\leqq k\leqq d$. If $C\in S^2(\boldsymbol{R}^{m(s)+1})$ and C is orthogonal to $W^{(j)}$, then $0=(C,\,p\cdot\sigma^{(k)}(0))=\langle Cp,\,\sigma^{(k)}(0)\rangle$ for $0\leqq k\leqq d$. The O^k_p-component of $\sigma^{(k)}(0)$ is just equal to $\mathring{B}_k(x,\cdots,x)$ and O^k_p is linearly spanned by $\mathring{B}_k(x,\cdots,x)$, $x\in T_pM$. Using $\langle Cp,\,\sigma^{(k)}(0)\rangle=0$ for $0\leqq k\leqq d$ and for an arbitrary vector $x\in T_pM$, we can prove inductively that $\langle Cp,\,u\rangle=0$ for any vector $u\in\boldsymbol{R}\cdot\psi(p)+\psi_*T_pM+O^2_p+\cdots+O^d_p=\boldsymbol{R}^{m(s)+1}$. Thus we get $Cp=0$. As the standard minimal immersion $\psi_{n,s}$ is full, we have $C\equiv 0$. Thus Lemma 4.6 is proved.

PROOF OF THEOREM 4.1. If $A\circ\psi_{n,s}$ is a minimal immersion and the k-th fundamental form B_k of $A\circ\psi_{n,s}$ is isotropic on M for $2\leqq k\leqq[d/2]$ (d is the degree of $\psi_{n,s}$), then by Lemma 4.5 (1) $A^2-I\in(W^{([d/2])})^\perp$. Lemma 4.6 implies that $A^2-I=0$ and then $A=I$. By Remark 2.4, Theorem 4.1 is proved.

To prove Theorem 4.2, we review do Carmo and Wallach's results. For the remainder of this section we assume that $M=S^n_{k(s)}=G/K$, where $G=SO(n+1)$ and $K=SO(n)$.

LEMMA 4.7 (Do Carmo and Wallach [4]). *Let* $\psi_{n,s}\colon S^n_{k(s)}\to S^{m(s)}_1\subset V^s$ *be the standard minimal immersion described in the second way* (cf. § 2). *If* V^s *is orthogonally decomposed as* $V^s=\boldsymbol{R}\cdot\psi(eK)+\psi_*T_{eK}M+O^2_{eK}+\cdots+O^s_{eK}$ *associated with* $\psi_{n,s}$, *then* O^j_{eK} *is the* $SO(n)$*-module of spherical harmonics of order* j *on the* $(n-1)$ *unit sphere.*

From now on we denote the above decomposition by $V^s=V_0+V_1+\cdots+V_s$, where V_i is the K $(=SO(n))$-module of spherical harmonics of order i on S^{n-1}_1.

Now we prepare some results about representation theory of $SO(n+1)$. We first give the classification of representations of $SO(n+1)=G$. Let $T\subset G$ be the subgroup of matrices of the form;

$$\begin{pmatrix} A_1 & & & & \\ & A_2 & & 0 & \\ & & \ddots & & \\ & 0 & & A_p & \\ & & & & 1 \end{pmatrix} \quad \text{if} \quad n=2p$$

or

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & 0 \\ & & \ddots & \\ & 0 & & A_p \end{pmatrix} \quad \text{if} \quad n=2p-1$$

where

$$A_i=\begin{pmatrix} \cos t_i & \sin t_i \\ -\sin t_i & \cos t_i \end{pmatrix}.$$

Let $\mathfrak{g}$ be the linear Lie algebra of G and $\mathfrak{h}$ be the Lie algebra of T in $\mathfrak{g}$. Let E_{ij} be the $(n+1)\times(n+1)$-matrix whose (i, j)-entry is 1 and all other entries are zero. Set $h_i=E_{2i-1,2i}-E_{2i,2i-1}$, $i=1, \cdots, p$. Then $h_1, \cdots, h_p$ is a basis of $\mathfrak{h}$. Let $\mathfrak{h}^*$ be the (real) dual of $\mathfrak{h}$, and let $\lambda_1, \cdots, \lambda_p$ be the dual basis for $h_1, \cdots, h_p$. Order the elements of $\mathfrak{h}^*$ lexicographically relative to $\lambda_1, \cdots, \lambda_p$. Then the highest weights of (complex) irreducible finite dimensional representations of G are of the form $\lambda_{(m)}=\sum_i m_i\lambda_i$, where $m_1, \cdots, m_p$ are integers satisfying

$$\begin{cases} m_1\geqq m_2\geqq \cdots \geqq m_p\geqq 0 & n=2p \\ m_1\geqq m_2\geqq \cdots \geqq |m_p| & n=2p-1\,. \end{cases}$$

We shall denote the representation of $SO(n+1)$ with highest weight $\lambda_{(m)}$ by ${}_nV^{(m)}$.

THEOREM 4.8 (The Branching Theorem [3]). *Notations being as above, we have as a K-module,*

$${}_nV^{(m)}=\sum_{(m')} {}_{n-1}V^{(m')}\,,$$

where the summation is taken over all integers $m_1', \cdots, m_p'$ *such that*

$$m_1\geqq m_1'\geqq m_2\geqq m_2'\geqq \cdots \geqq m_p\geqq |m_p'| \qquad \textit{if} \quad n=2p\,,$$

$$m_1\geqq m_1'\geqq m_2\geqq m_2'\geqq \cdots \geqq m_{p-1}'\geqq |m_p| \qquad \textit{if} \quad n=2p-1\,.$$

Now we state the formula which gives the $SO(n+1)$-module decomposition of $S^2({}_nV^{(l,0,\cdots,0)})$, where $S^2({}_nV^{(l,0,\cdots,0)})$ is the symmetric product of ${}_nV^{(l,0,\cdots,0)}$ and is naturally an $SO(n+1)$-module.

THEOREM 4.9 (Do Carmo and Wallach [4]). *If* $l\geqq 1$ *and* $n\geqq 3$, *then, as an*

$SO(n+1)$-module,

$$S^2({}_nV^{(l,0,\cdots,0)})=\sum_{j=0}^{[l/2]}{}_nV^{(2l-2j,2j,0,\cdots,0)}+S^2({}_nV^{(l-1,0,\cdots,0)})\,.$$

If $n=2$, as an $SO(3)$-module

$$S^2({}_2V^{(l,0,\cdots,0)})={}_2V^{(2l,0,\cdots,0)}+S^2({}_2V^{(l-1,0,\cdots,0)})\,.$$

Next we state the Frobenius reciprocity. We first need the notion of an induced representation. Let G be a compact topological group and let K be a closed subgroup. Let V be a finite dimensional K-module over $\boldsymbol{C}$. Let $\Gamma(V)$ be the vector space of all continuous functions $f: G\to V$ such that $f(xk)=k^{-1}f(x)$ for all $x\in G$, $k\in K$. Let G act on $\Gamma(V)$ by $L_xf(y)=f(x^{-1}y)$, $x, y\in G$. Then $\Gamma(V)$ is a G-module which is called the G-module induced by V.

LEMMA 4.10 (Frobenius Reciprocity). *Let U be an arbitrary finite dimensional G-module over $\boldsymbol{C}$. Then* $\mathrm{Hom}_G(U, \Gamma(V))$ *is canonically isomorphic to* $\mathrm{Hom}_K(U, V)$.

We return to the standard minimal immersion $\psi_{n,s}$ of $S^n_{k(s)}$ in $S^{m(s)}_1$. We shall complexify the real representations. Let $(V^s)^C$ be the complexification of V^s. It is well-known that $(V^s)^C$ is an irreducible G-module over $\boldsymbol{C}$ with highest weight $s\lambda_1$. We naturally extend G-invariant inner product $\langle\ ,\ \rangle$ of V^s to the G-invariant Hermitian inner product $\langle\ ,\ \rangle$ of $(V^s)^C$. By Lemma 4.7, $(V^s)^C$ is decomposed, as a K-module, as $(V^s)^C=(V_0)^C+(V_1)^C+\cdots+(V_s)^C$, where $(V_i)^C$ is a complexification of V_i. It is also well-known that $(V_i)^C$ is an irreducible K-module over $\boldsymbol{C}$ with highest weight $i\lambda_1$. Next we complexify $S^2(V^s)$ and denote it by $S^2(V^s)^C$. $S^2(V^s)^C$ is naturally isomorphic to $S^2((V^s)^C)$. The Hermitian inner product $(\ ,\ )$ of $S^2(V^s)^C$ extended naturally from the inner product of $S^2(V^s)$ coincides with the Hermitian product of $S^2((V^s)^C)$ induced from the Hermitian product of $(V^s)^C$. And it is G-invariant. Here we recall W_i, $i=0, 1, \cdots, s$, and in this case they are described as follows;

$$W_0=\Big\{\bigcup_{p\in M}\psi(p)\cdot\psi(p)\Big\}_R=\{G\cdot\psi(eK)^2\}_R=\{G\cdot S^2(V_0)\}_R$$

$$W_1=\Big\{\bigcup_{p\in M}S^2(T_pM)\Big\}_R=\{G\cdot S^2(T_{eK}M)\}_R=\{G\cdot S^2(V_1)\}_R$$

and

$$W_i=\Big\{\bigcup_{p\in M}S^2(O^i_p)\Big\}_R=\{G\cdot S^2(O^i_{eK})\}_R=\{G\cdot S^2(V_i)\}_R \qquad i\geqq 2\,,$$

where $\{G\cdot S^2(V_i)\}_R$ denotes the linear span of the orbit of $S^2(V_i)$ in $S^2(V^s)$. Therefore W_i is a G-submodule of $S^2(V^s)$. Similarly we complexify W_j and denote it by W^C_j. Naturally W^C_j is a G-submodule of $S^2(V^s)^C$. We also remark that W^C_j is isomorphic to $\{G\cdot S^2(V^C_j)\}_C$. From now on for simplicity we denote the complexification of a real G-module (or K-module) by the same notation used for the real object. For example we denote $S^2(V^s)$, W_j, V_i, $\cdots$, etc. instead of $S^2(V^s)^C$, W_j^C, V_i^C, $\cdots$, etc.

LEMMA 4.11. *For a fixed positive integer j, let U be the sum of those G-submodules of $S^2(V^s)$ not containing, as K-submodules, ${}_{n-1}V^{(2i-2k,2k,0,\cdots,0)}$ if $n\geqq 4$ and ${}_2V^{(2i,0,\cdots,0)}$ if $n=3$, where i and k are integers satisfying $0\leqq i\leqq j$ and $0\leqq k\leqq[i/2]$. Then U is orthogonal to $W_1+\cdots+W_j$ in $S^2(V^s)$.*

PROOF. We shall show that U is orthogonal to $W_i, i=1, \cdots, j$. We denote by $\Gamma(S^2(V_i))$ the G-module induced by the K-module $S^2(V_i)$ over $\boldsymbol{C}$. Let $W=\{u\in S^2(V^s); (u, S^2(V_i))=0\}$, where (,) is the Hermitian inner product in $S^2(V^s)$. Then as a K-module, $S^2(V^s)$ admits the orthogonal direct sum decomposition $S^2(V^s)=S^2(V_i)+W$. Let $P: S^2(V^s)\to S^2(V_i)$ be the corresponding projection. We claim that W_i is contained in $\Gamma(S^2(V_i))$ as a G-submodule. To see this, define, for each $u\in S^2(V^s)$, a map $f_u: G\to S^2(V_i)$ by $f_u(g)=P(g^{-1}u)$. It is easily verified that $f_u\in\Gamma(S^2(V_i))$. Next define a map $\alpha: S^2(V^s)\to\Gamma(S^2(V_i))$ by $\alpha(u)=f_u$. Since $\alpha(g_0u)(g)=P(g^{-1}(g_0u))=P((g_0^{-1}g)^{-1}u)=\alpha(u)(g_0^{-1}g)=(L_{g_0}\alpha(u))(g)$, we conclude that $\alpha\in\mathrm{Hom}_G(S^2(V^s), \Gamma(S^2(V_i)))$. If $u\in S^2(V^s)$ and $\alpha(u)=0$, then for any $g\in G$, $0=\alpha(u)(g)=P(g^{-1}u)$. Thus $0=(g^{-1}u, S^2(V_i))=(u, g\cdot S^2(V_i))$. Thus we see that $\ker\alpha=W_i^{\perp}$. Therefore $\alpha: W_i\to\Gamma(S^2(V_i))$ is a G-module isomorphism, which proves our claim. Now consider the G-module U. Using the above fact and Frobenius Reciprocity (Lemma 4.10), we obtain $\dim_C\mathrm{Hom}_G(U, W_i)\leqq\dim_C\mathrm{Hom}_G(U, \Gamma(S^2(V_i)))$ $=\dim_C\mathrm{Hom}_K(U, S^2(V_i))$. Since V_i is isomorphic to ${}_{n-1}V^{(i,0,\cdots,0)}$ as a K-module, Theorem 4.9 implies that U does not contain a K-submodule of $S^2(V_i)$. Therefore $\dim_C\mathrm{Hom}_G(U, W_i)=0$. It follows that U is orthogonal to W_i. Thus Lemma 4.11 is proved.

PROOF OF THEOREM 4.2. By Theorem 4.9, there exists the G-submodule ${}_nV^{(2s-2[s/2],2[s/2],0,\cdots,0)}$ of $S^2(V^s)$. Theorem 4.8 implies that ${}_nV^{(2s-2[s/2],2[s/2],0,\cdots,0)}$ does not contain, as K-submodule, ${}_{n-1}V^{(2i-2k,2k,0,\cdots,0)}$ for $0\leqq i\leqq[s/2]-1$ and $0\leqq k\leqq[i/2]$ if $n\geqq 4$ and ${}_2V^{(2i,0,\cdots,0)}$ for $0\leqq i\leqq[s/2]-1$ if $n=3$. Thus by Lemma 4.11 the dimension of the orthogonal complement of $W_1+\cdots+W_{[s/2]-1}$ in $S^2(V^s)$ is positive. This, together with Lemma 4.5, gives Theorem 4.2.

5. Helical geodesic minimal immersions.

First we give the definition of a helical geodesic immersion following Sakamoto ([8]). Let $\gamma: I\to M$ be a C^∞-curve parametrized by the arc-length s. Let $\gamma^{(1)}=\dot\gamma$ be the unit tangent vector and put $\kappa_2=\|\nabla_{\dot\gamma}\dot\gamma\|$. If κ_2 vanishes on I, then γ is said to be of order 1. If κ_2 is not identically zero, then we define $\gamma^{(2)}$ by $\nabla_{\dot\gamma}\gamma^{(1)}=\kappa_2\gamma^{(2)}$ on the set $I_2=\{s\in I; \kappa_2(s)\neq 0\}$. Put $\kappa_3=\|\nabla_{\dot\gamma}\gamma^{(2)}+\kappa_2\gamma^{(1)}\|$. If $\kappa_3\equiv 0$ on I_2, then γ is said to be of order 2. If κ_3 is not identically zero on I_2, then we define $\gamma^{(3)}$ by $\nabla_{\dot\gamma}\gamma^{(2)}=-\kappa_2\gamma^{(1)}+\kappa_3\gamma^{(3)}$. Inductively we put $\kappa_{d+1}=\|\nabla_{\dot\gamma}\gamma^{(d)}+\kappa_d\gamma^{(d-1)}\|$ and if $\kappa_{d+1}\equiv 0$ on I_d, then γ is said to be of order d.

DEFINITION. Let $\varphi: M \to \overline{M}$ be an isometric immersion of a connected complete Riemannian manifold M into a Riemannian manifold $\overline{M}$ and $\sigma: I \to M$ be an arbitrary geodesic in M parametrized by the arc-length. If the curve $\gamma = \varphi \circ \sigma$ in $\overline{M}$ is of order d and has constant curvatures $\kappa_2, \cdots, \kappa_d$ which do not depend on σ, then φ is called a *helical geodesic immersion of order* d.

REMARK. It is known that a strongly harmonic manifold admits a helical geodesic minimal immersion into a sphere (Besse [2]). In particular the standard minimal immersions of compact rank one symmetric spaces into spheres are helical geodesic and minimal.

PROPOSITION 5.1. *Let* $\varphi: S^n_{k(s)} \to S^l_1$ *be a minimal immersion.* φ *is a helical geodesic immersion if and only if the j-th fundamental form* B_j *is isotropic for* $2 \leqq j \leqq$(*degree of* φ). *In particular if* φ *is a helical geodesic minimal immersion of a sphere into a unit sphere, the order of* φ *is equal to the degree of* φ.

PROOF. Suppose that φ is a helical geodesic minimal immersion of order d of $S^n_{k(s)}$ into S^l_1. We use the same notations of covariant differentiations on $S^n_{k(s)}$, S^l_1, the normal bundle of φ etc. as in § 3. Let $\sigma: I \to S^n_{k(s)}$ be an arbitrary geodesic parametrized by the arc-length s. We put $X(s) = \dot{\sigma}(s)$. We denote by $\kappa_2, \cdots, \kappa_d$ the curvatures of $\gamma = \varphi \circ \sigma$ in S^l_1. We shall compute the Frenet frame $\{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(d)}\}$. Since $\tilde{\nabla}_X \gamma^{(1)} = B_2(X, X)$, we have $\gamma^{(2)} = \kappa_2^{-1} B_2(X, X)$. Moreover $\|B_2(X, X)\| = \kappa_2$ for any unit vector X and then the second fundamental form B_2 of φ is isotropic on $S^n_{k(s)}$. Next

$$\tilde{\nabla}_X \gamma^{(2)} = \kappa_2^{-1} \tilde{\nabla}_X (B_2(X, X)) = \kappa_2^{-1} \{-H_{B_2(X,X)} X + \overline{\nabla} B_2(X, X, X)\} .$$

Since B_2 is isotropic, $H_{B_2(X,X)} X = (\kappa_2)^2 X$ and by the proof of Proposition 3.1 we have $\overline{\nabla} B_2(X, X, X) = B_3(X, X, X)$. It follows that $\tilde{\nabla}_X \gamma^{(2)} = -\kappa_2 X + \kappa_2^{-1} B_3(X, X, X)$. Thus $\gamma^{(3)} = (\kappa_2 \kappa_3)^{-1} B_3(X, X, X)$. Similarly we have $\|B_3(X, X, X)\| = \kappa_2 \kappa_3$ for any unit tangent vector X and then B_3 is isotropic on $S^n_{k(s)}$. Here we apply a mathematical induction. Let j be a fixed natural number satisfying $3 \leqq j \leqq d$. We assume that $\gamma^{(k)}$ is described as $\gamma^{(k)} = (\kappa_2 \cdots \kappa_k)^{-1} B_k(X, \cdots, X)$ and B_k is isotropic on $S^n_{k(s)}$ for $2 \leqq k \leqq j$. Under these assumptions, we have

$$\begin{aligned} \tilde{\nabla}_X \gamma^{(j)} &= (\kappa_2 \cdots \kappa_j)^{-1} \tilde{\nabla}_X (B_j(X, \cdots, X)) \\ &= (\kappa_2 \cdots \kappa_j)^{-1} \{\overline{\nabla} B_j(X, \cdots, X)\} \qquad (j \geqq 3) . \end{aligned}$$

Since B_j is isotropic, the argument in the proof of Proposition 3.1 implies that

$$\overline{\nabla} B_j(X, \cdots, X) = B_{j+1}(X, \cdots, X) + D_{j+1}(X, \cdots, X) .$$

Noticing that we can prove Proposition 3.4 under the only condition that B_k is isotropic for $2 \leqq k \leqq j$, we have

$$D_{j+1}(X, \cdots, X) = -(\lambda_j^2 / \lambda_{j-1}^2) B_{j-1}(X, \cdots, X) ,$$

where $\lambda_j=\|B_j(X, \cdots, X)\|$ and $\lambda_{j-1}=\|B_{j-1}(X, \cdots, X)\|$. So we have $D_{j+1}(X, \cdots, X) =-\kappa_j^2 B_{j-1}(X, \cdots, X)=-\kappa_2 \cdots \kappa_{j-1}(\kappa_j)^2\gamma^{(j-1)}$. It follows that

$$\tilde{\nabla}_X\gamma^{(j)}=-\kappa_j\gamma^{(j-1)}+(\kappa_2 \cdots \kappa_j)^{-1}B_{j+1}(X, \cdots, X).$$

Thus we obtain $\gamma^{(j+1)}=(\kappa_2 \cdots \kappa_{j+1})^{-1}B_{j+1}(X, \cdots, X)$ and that B_{j+1} is isotropic. By the above argument we see that the Frenet frame $\{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(d)}\}$ of γ is given by

$$\gamma^{(j)}=(\kappa_2 \cdots \kappa_j)^{-1}B_j(X, \cdots, X) \qquad \text{for} \quad 2 \leqq j \leqq d.$$

Therefore we see that the order of φ is equal to the degree of φ and that B_j is isotropic for $2 \leqq j \leqq$(degree of φ). Noticing that if the j-th fundamental form B_j of a minimal immersion φ is isotropic, then it is constant isotropic (§3), we can prove the converse similarly.

By Proposition 5.1 we easily get

COROLLARY 5.2. *Let $\psi_{n,s}: S^n_{k(s)} \to S^{m(s)}_1$ be the s-th standard minimal immersion ($n \geqq 2$). Then $\psi_{n,s}$ is a helical geodesic immersion of order s.*

THEOREM 5.3. *Let $\varphi: S^n_{k(s)} \to S^l_1$ be a helical geodesic minimal immersion. Assume that φ is full. Then φ is equivalent to the standard minimal immersion $\psi_{n,s}$ and in particular the order of the helical geodesic immersion φ is s and $l=m(s)$.*

We prepare some lemmas before the proof of Theorem 5.3. Let $X \in T_{x_0}M - \{0\}$ and $\gamma: s \to \exp_{x_0}\frac{s}{\|X\|}X$ be the geodesic. Let $\{Y_i\}_{i=2,\dots,n}$ be Jacobi fields along γ such that $Y_i(0)=0$ for every i and $\{Y_i'(0)\}_{i=2,\dots,n}$ forms an orthonormal basis of the orthogonal complement of X in $T_{x_0}M$. Then we define $\theta: TM \to \boldsymbol{R}$ by

$$\begin{cases} \theta(0)=1 \\ \theta(X)=\|X\|^{-n+1}\det(Y_2(\|X\|), \cdots, Y_n(\|X\|)), \end{cases}$$

where the determinant should be understood with respect to the parallel frame field of $\{Y_i'(0)\}$. It is known that S^n is a globally harmonic manifold, i.e., there exists a C^∞-function $\Theta: \boldsymbol{R}_+ \to \boldsymbol{R}$ such that $\theta(X)=\Theta(\|X\|)$ for every $x \in S^n$ and every $X \in T_xS^n$.

LEMMA 5.4 (Berger-Gauduchon-Mazet [1] p. 134). *Let f be a C^∞-function on S^n of the form $f(x)=F(\delta(x, x_0))$ (i.e., which depends only on the distance to x_0), where δ denotes the distance function. Then we have*

$$\Delta f=-\frac{d^2F}{ds^2}-\left(\frac{\theta'_{x_0}}{\theta_{x_0}}+\frac{n-1}{s}\right)\frac{dF}{ds},$$

where Δ denotes the Laplacian and θ'_{x_0} is the radial derivative of θ_{x_0} in $T_{x_0}S^n$.

LEMMA 5.5 (Sakamoto [8]). *Let $\varphi: M \to S^l_1$ be a helical geodesic immersion. Then there exists a C^∞-function $F: \boldsymbol{R}_+ \to \boldsymbol{R}$ such that the Euclidean inner product* *of position vectors* $\varphi(x)$ *and* $\varphi(y)$ *is given by* $\langle\varphi(x), \varphi(y)\rangle=F(\delta(x, y))$.

LEMMA 5.6 (Besse [2] p. 177, p. 178). *Let* $\varphi: M\rightarrow S_1^l$ *be a helical geodesic immersion of order* d *and let* $\sigma: I\rightarrow M$ *be a geodesic parametrized by the arc-length. Then the curvatures* $\kappa_2, \cdots, \kappa_d$ *of the curve* $\gamma=\varphi\circ\sigma$ *in* S_1^l *are completely determined by* $F^{(k)}(0)$, $k=1, 2, \cdots$, *where* F *is the function introduced in Lemma* 5.5. *In particular the order of* φ *is determined by* $F^{(k)}(0)$, $k=1, 2, \cdots$.

PROOF OF THEOREM 5.3. We denote by F and $\mathring{F}$ the functions introduced in Lemma 5.5 associated with the helical geodesic minimal immersion φ and the standard minimal immersion $\psi_{n,s}$ respectively. For a fixed point $x_0\in S^n$ we define the functions f and $\mathring{f}$ on S^n by

$$f(x)=\langle\varphi(x), \varphi(x_0)\rangle=F(\delta(x, x_0))$$

$$\mathring{f}(x)=\langle\psi(x), \psi(x_0)\rangle=\mathring{F}(\delta(x, x_0)),$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product. By a well-known theorem (Takahashi [9]), $f(x)$ and $\mathring{f}(x)$ are eigenfunctions of the Laplacian on $S_{k(s)}^n$ with eigenvalue n. By Lemma 5.4, we have

$$-\frac{d^2F}{ds^2}-\frac{dF}{ds}\left(\frac{\theta'_{x_0}}{\theta_{x_0}}+\frac{n-1}{s}\right)=nF$$

and

$$-\frac{d^2\mathring{F}}{ds^2}-\frac{d\mathring{F}}{ds}\left(\frac{\theta'_{x_0}}{\theta_{x_0}}+\frac{n-1}{s}\right)=n\mathring{F}.$$

Since $F(0)=\mathring{F}(0)=1$ and $F'(0)=\mathring{F}'(0)=0$, we obtain $F\equiv\mathring{F}$. Then Lemma 5.6 implies that the order of φ is equal to the order of $\psi_{n,s}=s$. By Proposition 5.1, the j-th fundamental form B_j of φ is isotropic for $2\leqq j\leqq s$. This, together with Theorem 4.1, gives Theorem 5.3.

COROLLARY 5.7. *Let* $\varphi: S_{k(s)}^n\rightarrow S_1^l$ *be a minimal immersion. If the j-th fundamental form* B_j *is isotropic for* $2\leqq j\leqq$*(degree of* φ*), then* φ *is equivalent to the standard minimal immersion* $\psi_{n,s}$.

PROOF. By Proposition 5.1, φ is a helical geodesic immersion. By Theorem 5.3 we see that φ is equivalent to $\psi_{n,s}$.

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