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Some statement which implies the existence of Ramsey ultrafilters on ω

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1. Introduction and results.

Throughout this paper, we work in Zelmero-Fraenkel set theory with choice (ZFC). Let $\mathfrak{F}$ be a filter on A and let f be a function from A to B. $f(\mathfrak{F})$ denotes the filter $\{y\subset B; f^{-1}y\in\mathfrak{F}\}$. $\mathfrak{F}$ is said to be *free* if $\emptyset\notin\mathfrak{F}$ and $\bigcap\mathfrak{F}=\emptyset$. $\mathfrak{F}$ is said to be *ample* if there exists an infinite subset A_0 of A such that, for any $x\in\mathfrak{F}$, A_0-x is finite. $\mathfrak{F}$ is said to be *weakly ample* if, for any free ultrafilter (uf) $\mathfrak{U}$ on ω, there exists a function g from ω to A such that $g(\mathfrak{U})\supset\mathfrak{F}$. It is trivial that any free, ample filter is weakly ample. For any infinite cardinal κ, we denote by AN(κ) the statement: "any free, weakly ample filter on κ is ample". It is easy to see that, whenever $\kappa\leqq\lambda$, AN(λ) implies AN(κ). Puritz proved the following Theorem 1.

THEOREM 1 (Puritz [**5**]).

(a) *The continuum hypothesis* (CH) *implies* AN(c), *where c denotes* 2^{ω}.

(b) AN(ω) *implies that there are P-points on ω.*

(c) AN(2^c) *does not hold.*

He asked whether the existence of P-points implies AN(ω). This question is answered negatively by Theorem 5 which appears below. By Theorem 1 (a), (c), under the assumption CH$+2^{\omega_1}=\omega_2$, AN(κ) holds if and only if $\kappa=\omega$ or $\kappa=\omega_1$. Let P be the statement: "any free, κ-generated filter on ω is ample, for all $\kappa<c$". Then, the proof of Theorem 1 (a) (in [**5**; p. 222]) yields a proof of that P implies AN(c). Since Martin's Axiom (MA) implies P (cf. [**4**; Theorem 5]), it holds that MA implies AN(c). By this, Theorem 1 (b) and a result of Shelah that the existence of P-points is unprovable (in ZFC), the negation of CH implies neither AN(c) nor $\neg$AN(c).

We shall consider what cardinals κ satisfy AN(κ) in the cases where CH $+2^{\omega_1}=\omega_2$ fails. Our results are the following Theorems which are proved in Sections 3~6.

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THEOREM 2.

(a) AN(ω) *implies* AN(c).

(b) AN(ω) *implies that there are* c^+ *Ramsey ufs on* ω.

THEOREM 3. *Let* κ *be an infinite cardinal such that* $2^c \leqq \kappa^\omega$. *Then,* AN(κ) *does not hold.*

THEOREM 5. *The existence of* c^+ *Ramsey ufs on* ω *does not imply* AN(ω).

THEOREM 6.

(a) CH$+2^{\omega_1}=\omega_3$ *does not imply* AN(ω_2).

(b) CH$+2^{\omega_1}=\omega_3$ *does not imply* $\neg$AN(ω_2).

2. Notations and definitions.

We adopt the notions and conventions of current set theory. In particular, an ordinal is the set of its predecessors and cardinals are initial ordinals. κ and λ are used to denote cardinals and other lower case Greek letters are used to denote ordinals. For any cardinals κ and λ, κ^λ is the cardinality of the set of all functions from λ to κ. ω is the first infinite cardinal and ω_α is the α-th infinite cardinal. For any set X, $|X|$ denotes the cardinality of X, $P(X)$ denotes the power set of X and $P_\omega(X)$ denotes the set $\{x\subset X;\ |x|=\omega\}$. $A\subset P_\omega(X)$ is said to be *almost disjoint* (mod. finite) if, for any distinct a, b in A, $a\cap b$ is finite. A is said to be a *maximal* almost disjoint subset of $P_\omega(X)$ if A is almost disjoint and, for any $B\subset P_\omega(X)$, whenever A is a proper subset of B, B is not almost disjoint. S is said to be a *partition* of X if $\bigcup S=X$ and, for any distinct s, t in S, $s\cap t=\emptyset$. For any function f and for any set a, $f^{-1}(a)$ (also $f^{-1}a$) denotes the set $\{x\in \mathrm{dom}(f);\ f(x)\in a\}$. Let $\mathfrak{F}$ be a filter on X and T a subset of $P(X)$. $\mathfrak{F}$ is said to be *free* if $\emptyset\notin\mathfrak{F}$ and $\bigcap\mathfrak{F}=\emptyset$. T is a *generator* of $\mathfrak{F}$ if $\mathfrak{F}$ is the smallest filter on X which includes T. $\mathfrak{F}$ is *κ-generated* if there exists a generator T of $\mathfrak{F}$ such that $|T|\leqq\kappa$. For any function f from X to Y, $f(\mathfrak{F})$ denotes the filter $\{y\subset Y;\ f^{-1}y\in\mathfrak{F}\}$ on Y. Let $\mathfrak{U}$ be a free uf on ω. $\mathfrak{U}$ is said to be a *Ramsey* uf (resp. a *P-point*) if, for any partition $\langle x_n \mid n<\omega\rangle$ of ω, whenever $x_n\notin\mathfrak{U}$ for all $n<\omega$, there exists some $y\in\mathfrak{U}$ such that

$$|y\cap x_n|\leqq 1 \quad (\text{resp. } |y\cap x_n|<\omega) \qquad \text{for all} \quad n<\omega.$$

3. Proofs of Theorems 2 and 3.

LEMMA 1. *There exists a free, not ample filter* $\mathfrak{F}$ *on* ω *which satisfies the following* (3.1).

(3.1) *For any free uf* $\mathfrak{U}$ *on* ω, (a) *and* (b) *are equivalent.*

(a) $\mathfrak{U}$ *is Ramsey.*

(b) $f(\mathfrak{U})\not\supset\mathfrak{F}$, *for all* $f:\omega\to\omega$.

PROOF. Since there is a bijection from ω to $\omega\times\omega$, it suffices to show that there exists a free, not ample filter $\mathfrak{F}$ on $\omega\times\omega$ which satisfies the following (3.2).

(3.2) For any free uf $\mathfrak{U}$ on ω, (a) and (b)$'$ are equivalent.
(a) $\mathfrak{U}$ is Ramsey.
(b)$'$ $g(\mathfrak{U})\not\supset\mathfrak{F}$, for all $g:\omega\rightarrow\omega\times\omega$.

For any function f on ω, let $a(f)$ be the set $\{(m, n)\in\omega\times\omega;\ f(m)\neq n\}$. For any $k<\omega$, let $b(k)$ be the set $(\omega-k)\times\omega$ $(=\{(m, n)\in\omega\times\omega;\ k\leqq m\})$. Let $\mathfrak{F}$ be the filter on $\omega\times\omega$ which is generated by $\{a(f);\ f$ is a function on $\omega\}\cup\{b(k);\ k<\omega\}$. It is easy to see that $\mathfrak{F}$ is free and not ample. To show (3.2), let $\mathfrak{U}$ be an arbitrary free uf on ω.

(a)$\Rightarrow$(b)$'$. Suppose that $\mathfrak{U}$ is Ramsey. Let g be any function from ω to $\omega\times\omega$. For any $k<\omega$, set $x_k=g^{-1}(\{k\}\times\omega)$.

Case 1. $x_k\in\mathfrak{U}$, for some $k<\omega$.

Since $\{k\}\times\omega$ and $b(k+1)$ are disjoint, we have that

$$b(k+1)\notin g(\mathfrak{U})\,.$$

Case 2. $x_k\notin\mathfrak{U}$, for all $k<\omega$.

Since $\langle x_k \mid k<\omega\rangle$ be a partition of ω and $\mathfrak{U}$ is Ramsey, there exists some $y\in\mathfrak{U}$ such that

$$|y\cap x_k|\leqq 1\,,\qquad \text{for all}\quad k<\omega\,.$$

Since, for any $k<\omega$, $g''y\cap(\{k\}\times\omega)=g''(x_k\cap y)$, $g''y$ is a (graph of) partial function on ω. So, we can choose a function f on ω such that

$$g''y\subset\{(m, n)\in\omega\times\omega;\ f(m)=n\}\,.$$

Since $a(f)\subset\omega\times\omega-g''y$, it holds that $\omega\times\omega-g''y\subset\mathfrak{F}$. So, $\omega\times\omega-g''y\in\mathfrak{F}-g(\mathfrak{U})$.

(b)$'\Rightarrow$(a). Suppose that $\mathfrak{U}$ satisfies (b)$'$. Let h be any function on ω such that, for any $n<\omega$, $h^{-1}\{n\}\notin\mathfrak{U}$. We need to show that h is one to one on some set in $\mathfrak{U}$. Define the function g from ω to $\omega\times\omega$ by

$$g(n)=(h(n), n)\qquad \text{for all}\quad n<\omega\,.$$

Since $g(\mathfrak{U})\not\supset\mathfrak{F}$, there are $k<\omega$ and functions $f_0, \cdots, f_{m-1}$ on ω such that

$$a(f_0)\cap\cdots\cap a(f_{m-1})\cap b(k)\notin g(\mathfrak{U})\,.$$

Since $h^{-1}\{n\}\notin\mathfrak{U}$ for all $n<\omega$, it holds that $b(k)\in g(\mathfrak{U})$. From this, since $g(\mathfrak{U})$ is an uf on $\omega\times\omega$, there exists some $i<m$ such that

$$\omega\times\omega-a(f_i)\in g(\mathfrak{U})\,.$$

Set $y=g^{-1}(\omega\times\omega-a(f_i))\in\mathfrak{U}$. Since $\omega\times\omega-a(f_i)$ is a graph of function, h is one to one on y. $\perp$

LEMMA 2. *Let κ be an infinite cardinal. Then, there exists a subset A of $P_\omega(\kappa)$ which satisfies*

(i) $|A|=\kappa^\omega$,

(ii) *A is almost disjoint (mod. finite).*

PROOF. Set $X=\{x\subset\omega\times\kappa\,;\,x \text{ is finite}\}$. Since $|X|=\kappa$, it suffices to construct a subset A of $P_\omega(X)$ which satisfies (i) and (ii). For any function f from ω to κ, let $a_f=\{f\restriction n\,;\,n<\omega\}$. Then, for any distinct functions $f, g:\omega\to\kappa$, $a_f\cap a_g$ is finite. So, $A=\{a_f\,;\,f \text{ is a function from } \omega \text{ to } \kappa\}$ is as required. $\perp$

LEMMA 3 *Let κ be an infinite cardinal. Then, the following* (a) *and* (b) *are equivalent.*

(a) $\neg\mathrm{AN}(\kappa)$.

(b) *There exists a partition $\langle X_\alpha\,|\,\alpha<\kappa^\omega\rangle$ of the set of all Ramsey ufs on ω such that, for any $\alpha<\kappa^\omega$,*

$$X_\alpha=\varnothing \quad \textit{or} \quad \bigcap X_\alpha \quad \textit{is not ample.}$$

PROOF. (a)$\Rightarrow$(b). Suppose that $\neg\mathrm{AN}(\kappa)$. Let $\mathfrak{F}$ be a free, weakly ample, not ample filter on κ. Since $\mathfrak{F}$ is weakly ample, for any free uf $\mathfrak{U}$ on ω, there exists a function f from ω to κ such that $f(\mathfrak{U})\supset\mathfrak{F}$. Moreover, since $\mathfrak{F}$ is free, if $\mathfrak{U}$ is Ramsey, then f can be chosen to be one to one. So, for each Ramsey uf $\mathfrak{U}$ on ω, let $f_\mathfrak{U}$ be a one to one function from ω to κ such that $f_\mathfrak{U}(\mathfrak{U})\supset\mathfrak{F}$. Let $\langle g_\alpha\,|\,\alpha<\kappa^\omega\rangle$ be an enumeration of the set of all functions from ω to κ. For each $\alpha<\kappa^\omega$, let

$$X_\alpha=\{\mathfrak{U}\,;\,\mathfrak{U} \text{ is a Ramsey uf on } \omega \ \&\ f_\mathfrak{U}=g_\alpha\}\,.$$

Then, $\langle X_\alpha\,|\,\alpha<\kappa^\omega\rangle$ satisfies (b).

(b)$\Rightarrow$(a). Suppose that $\langle X_\alpha\,|\,\alpha<\kappa^\omega\rangle$ satisfies (b). Let $\mathfrak{F}$ be a free, not ample filter on ω which satisfies the condition (3.1) in Lemma 1. Let A be a subset of $P_\omega(\kappa)$ which satisfies (i) and (ii) in Lemma 2. Let $\langle a_\alpha\,|\,\alpha<\kappa^\omega\rangle$ be a one to one enumeration of A. Define the κ^ω-sequence $\langle\mathfrak{F}_\alpha\,|\,\alpha<\kappa^\omega\rangle$ of filter on ω by

$$\mathfrak{F}_{\alpha+1}=\bigcap X_\alpha\,, \qquad \text{if} \quad \alpha<\kappa^\omega \quad \text{and} \quad X_\alpha\neq\varnothing\,,$$

$$\mathfrak{F}_\xi=\mathfrak{F}\,, \quad \text{otherwise.}$$

For any $\alpha<\kappa^\omega$, $\mathfrak{F}_\alpha$ is free and not ample. Let $\langle h_\alpha\,|\,\alpha<\kappa^\omega\rangle$ be such that, for any $\alpha<\kappa^\omega$, h_α is a bijection from ω to a_α. Define the κ^ω-sequence $\langle\mathfrak{G}_\alpha\,|\,\alpha<\kappa^\omega\rangle$ of filters on κ by

$$\mathfrak{G}_\alpha=h_\alpha(\mathfrak{F}_\alpha) \qquad \text{for each} \quad \alpha<\kappa^\omega\,.$$

For any $\alpha<\kappa^\omega$, since h_α is one to one, $\mathfrak{G}_\alpha$ is a free, not ample filter on κ. Define the filter $\mathfrak{G}$ on κ by, for any $x\subset\kappa$,

$$x\in\mathfrak{G} \qquad \text{if and only if,} \quad \text{for all } \alpha<\kappa^\omega,\ x\cap a_\alpha\in\mathfrak{G}_\alpha\,.$$

The following Claim completes the proof.

Claim. $\mathfrak{G}$ is a free, weakly ample, not ample filter on κ.

Proof of Claim. Since $\mathfrak{G}_\alpha$ is free, for all $\alpha<\kappa^\omega$, $\mathfrak{G}$ is free. Since A is almost disjoint (mod. finite) and, for all $\alpha<\kappa^\omega$, $\mathfrak{G}_\alpha$ is not ample, $\mathfrak{G}$ is not ample. To show that $\mathfrak{G}$ is weakly ample, let $\mathfrak{U}$ be any free uf on ω.

Case 1. $\mathfrak{U}$ is Ramsey.

Let $\alpha<\kappa^\omega$ be such that $\mathfrak{U}\in X_\alpha$. Then, it holds that $\mathfrak{U}\supset\bigcap X_\alpha=\mathfrak{F}_{\alpha+1}$. So, $h_{\alpha+1}(\mathfrak{U})\supset\mathfrak{G}_{\alpha+1}\supset\mathfrak{G}$.

Case 2. $\mathfrak{U}$ is not Ramsey.

Since $\mathfrak{F}$ satisfies the condition (3.1) in Lemma 1, there exists a function g on ω such that $g(\mathfrak{U})\supset\mathfrak{F}$. Set $f=h_0\circ g$. Then, it holds that

$$f(\mathfrak{U})=h_0(g(\mathfrak{U}))\supset h_0(\mathfrak{F})=\mathfrak{G}_0\supset\mathfrak{G}.$$ Q. E. D. of Claim. ⊥

COROLLARY 1. *For any infinite cardinal* κ, AN(κ) *holds if and only if* AN(κ^ω) *holds.*

PROOF. This corollary follows immediately from Lemma 3. ⊥

Theorems 2 and 3 follow immediately from Lemma 3 and Corollary 1.

4. Theorem 4.

The purpose of this section is to state Theorem 4 which is used in the proof of Theorem 6 (a). First, we need some definitions.

DEFINITION. Let f and g be functions on ω_1. f *dominates* g (denoted by $g\prec f$) if there is an ordinal $\alpha<\omega_1$ such that, whenever $\alpha\leqq\xi<\omega_1$, $g(\xi)<f(\xi)$.

DEFINITION. Let $\langle f_\delta|\delta<\omega_2\rangle$ be an ω_2-sequence of functions on ω_1. $\langle f_\delta|\delta<\omega_2\rangle$ is said to be an *ω_2-scale* on ω_1 if the following (i) and (ii) are satisfied.

(i) For any $\delta<\eta<\omega_2$, $f_\delta\prec f_\eta$.

(ii) For any function g on ω_1, there is some $\delta<\omega_2$ such that $g\prec f_\delta$.

THEOREM 4. *Assume that* CH *holds and that there exists an ω_2-scale on ω_1. Then,* AN(ω_2) *does not hold.*

Throughout this section, we assume that CH holds. To show Theorem 4, we need some definitions and lemmas.

DEFINITION. For any $\alpha<\omega_1$, S_α denotes the set of all functions from α to ω_1.

LEMMA 4. *There exists an ω_1-sequence $\langle\langle x_s|s\in S_{\alpha+1}\rangle|\alpha<\omega_1\rangle$ such that, for any $\alpha<\omega_1$,*

(i) *$x_s\neq x_t$ for any distinct $s, t\in S_{\alpha+1}$,*

(ii) *$\{x_s; s\in S_{\alpha+1}\}$ is a maximal almost disjoint subset of $P_\omega(\omega)$,*

(iii) *for any $\xi<\alpha$, $s\in S_{\xi+1}$, $t\in S_{\alpha+1}$, if $s\subset t$, then x_t-x_s is finite,*

(iv) *for any $y\subset\omega$, there exists some $\beta<\omega_1$ such that, for any $s\in S_{\beta+1}$, $y\cap x_s=\emptyset$ or $x_s\subset y$.*

PROOF. Using CH, the lemma is proved by the induction on $\alpha<\omega_1$. $\perp$

In the rest of this section, $\langle\langle x_s | s\in S_{\alpha+1}\rangle | \alpha<\omega_1\rangle$ denotes some fixed ω_1-sequence which satisfies (i)$\sim$(iv) in Lemma 4.

DEFINITION. For any function h on ω_1, $\mathfrak{U}_h$ denotes the filter on ω which is generated by $\{x_{h\restriction(\alpha+1)}; \alpha<\omega_1\}$.

LEMMA 5. *For any function h on ω_1, $\mathfrak{U}_h$ is a free uf on ω.*

PROOF. This is easy. $\perp$

DEFINITION. W denotes the set of all functions on ω_1.

DEFINITION. For any nonempty subset Y of W, $\mathfrak{F}(Y)$ denotes the filter $\bigcap\{\mathfrak{U}_h; h\in Y\}$.

It is easy to see that $\mathfrak{F}(Y)$ is free.

LEMMA 6. *Let Y be a nonempty subset of W. Then,* (a) *and* (b) *are equivalent.*

(a) *$\mathfrak{F}(Y)$ is not ample.*

(b) *For any $\alpha<\omega_1$, $s\in S_{\alpha+1}$, there exists $\beta<\omega_1$ and $t\in S_{\beta+1}$ such that*

$$\alpha\leqq\beta \quad \& \quad s\subset t \quad \& \quad \forall h\in Y\ (t\not\subset h).$$

PROOF. (a)$\Rightarrow$(b). Suppose that $\mathfrak{F}(Y)$ is not ample. Let $\alpha<\omega_1$ and $s\in S_{\alpha+1}$ be any elements. Then, since x_s is an infinite subset of ω, there exists some $y\in\mathfrak{F}(Y)$ such that x_s-y is infinite. Set $a=x_s-y$. Pick an ordinal $\beta<\omega_1$ such that $\alpha\leqq\beta$ and, for any $t\in S_{\beta+1}$,

$$x_t\cap a \text{ is finite or } x_t-a \text{ is finite}.$$

Since $\{x_t; t\in S_{\beta+1}\}$ is a maximal almost disjoint subset of $P_\omega(\omega)$, there is some $t\in S_{\beta+1}$ such that x_t-a is finite. Then, since $x_t-x_s\subset x_t-a$, it holds that $s\subset t$. We claim that t is as required. Suppose not. There is a function $h\in Y$ such that $t\subset h$. Since $t=h\restriction(\beta+1)$, it holds that $x_t\in\mathfrak{U}_h$. So, $\omega-x_t\notin\mathfrak{U}_h$. Since $\mathfrak{F}(Y)\subset\mathfrak{U}_h$, we have that $\omega-x_t\notin\mathfrak{F}(Y)$. This contradicts the facts that $y\in\mathfrak{F}(Y)$ and that $y\subset\omega-x_t$.

(b)$\Rightarrow$(a). Suppose that Y satisfies (b). Let x be any infinite subset of ω. Pick $\alpha<\omega_1$ and $s\in S_{\alpha+1}$ such that x_s-x is finite. By virtue of the fact that Y satisfies (b), there exist $\beta<\omega_1$ and $t\in S_{\beta+1}$ such that

$$\alpha\leqq\beta \quad \& \quad s\subset t \quad \& \quad \forall h\in Y\ (t\not\subset h).$$

Set $y=\omega-x_t$. Then, because $x-y$ is infinite, the following Claim completes the proof.

Claim. $y\in\mathfrak{F}(Y)$.

Proof of Claim. By the definition of $\mathfrak{F}(Y)$, it suffices to show that, for any $h\in Y$, $y\in\mathfrak{U}_h$. Let h be any function in Y. Since $t\not\subset h$, it holds that $x_t\cap x_{h\restriction(\beta+1)}$ is finite. So,

$$x_{h\restriction(\beta+1)}-y \text{ is finite}.$$

By this and the fact that $x_{h\restriction(\beta+1)}\in\mathfrak{U}_h$, we have that $y\in\mathfrak{U}_h$.

Q. E. D. of Claim. $\perp$

DEFINITION. For any $\alpha<\omega_1$, $\mathfrak{G}_\alpha$ denotes the filter on ω which is generated by $\{\omega-x_s\,;\, s\in S_{\alpha+1}\}$.

It is easy to see that $\mathfrak{G}_\alpha$ is free and not ample.

LEMMA 7. *For any free uf* $\mathfrak{U}$ *on* ω, *the following* (a) *or* (b) *holds.*

(a) $\mathfrak{U}=\mathfrak{U}_h$ *for some* $h\in W$.

(b) $\mathfrak{G}_\alpha\subset\mathfrak{U}$ *for some* $\alpha<\omega_1$.

PROOF. Suppose that (b) fails.

Claim. For any $\alpha<\omega_1$, there exists the unique $s\in S_{\alpha+1}$ such that $x_s\in\mathfrak{U}$.

Proof of Claim. Obvious. Q. E. D. of Claim.

For each $\alpha<\omega_1$, let $u(\alpha)$ be the unique $s\in S_{\alpha+1}$ such that $x_s\in\mathfrak{U}$. Then, for any $\alpha<\beta<\omega_1$, since $x_{u(\alpha)}\cap x_{u(\beta)}\in\mathfrak{U}$, it holds that $u(\alpha)\subset u(\beta)$. Set $h=\bigcup_{\alpha<\omega_1} u(\alpha)$. Then, h is a function on ω_1 and $\mathfrak{U}_h\subset\mathfrak{U}$. But, since $\mathfrak{U}_h$ is an uf on ω, we have that $\mathfrak{U}_h=\mathfrak{U}$. $\perp$

LEMMA 8. *Suppose that there exists an* ω_2*-sequence* $\langle Y_\delta\,|\,\delta<\omega_2\rangle$ *such that*

(i) $\bigcup_{\delta<\omega_2} Y_\delta=W$,

(ii) *for any* $\delta<\omega_2$, $\mathfrak{F}(Y_\delta)$ *is not ample.*

Then, $\mathrm{AN}(\omega_2)$ *does not hold.*

PROOF. Let $\langle Y_\delta\,|\,\delta<\omega_2\rangle$ be an ω_2-sequence which satisfies (i) and (ii) in the Lemma. Define the filter $\mathfrak{F}$ on $\omega\times(\omega_2+\omega_1)$ by, for any $x\subset\omega\times(\omega_2+\omega_1)$,

$x\in\mathfrak{F}$ if and only if the following (a) and (b) hold.

(a) $\{n<\omega\,;\,(n,\delta)\in x\}\in\mathfrak{F}(Y_\delta)$ for all $\delta<\omega_2$.

(b) $\{n<\omega\,;\,(n,\omega_2+\alpha)\in x\}\in\mathfrak{G}_\alpha$ for all $\alpha<\omega_1$.

Then, using Lemma 7, it is easy to see that $\mathfrak{F}$ is free, weakly ample and not ample. $\perp$

PROOF OF THEOREM 4. Let $\langle f_\delta\,|\,\delta<\omega_2\rangle$ be an ω_2-scale on ω_1. For any $\delta<\omega_2$, $\alpha<\omega_1$, define the subset $Y_{\delta\alpha}$ of W by, for any $h\in W$,

$$h\in Y_{\delta\alpha} \text{ if and only if, whenever } \alpha\leqq\xi<\omega_1,\ h(\xi)<f_\delta(\xi)\,.$$

Since $\langle f_\delta\,|\,\delta<\omega_2\rangle$ is an ω_2-scale on ω_1, we have that

$$\bigcup\{Y_{\delta\alpha}\,;\,\delta<\omega_2\ \&\ \alpha<\omega_1\}=W\,.$$

To complete the proof of Theorem 4, by Lemma 8, it suffices to show that, for any $\delta<\omega_2$, $\gamma<\omega_1$,

(4.1) $\mathfrak{F}(Y_{\delta\gamma})$ is not ample.

Let $\delta<\omega_2$ and $\gamma<\omega_1$ be any elements. To show (4.1), by Lemma 6, it suffices to show that

(4.2) for any $\alpha<\omega_1$, $s\in S_{\alpha+1}$, there exists $\beta<\omega_1$ and $t\in S_{\beta+1}$ such that

$$\alpha \leqq \beta \quad \& \quad s \subset t \quad \& \quad \forall h \in Y_{\delta\gamma}\ (t \not\subset h).$$

Let $\alpha < \omega_1$ and $s \in S_{\alpha+1}$ be any elements. Pick $\beta < \omega_1$ and $s_1 \in S_{\beta+1}$ such that

$$\alpha \leqq \beta \quad \& \quad \gamma \leqq \beta \quad \& \quad s \subset s_1.$$

Define $t \in S_{\beta+2}$ by

$$t \upharpoonright (\beta+1) = s_1, \qquad t(\beta+1) = f_\delta(\beta+1).$$

Then, it holds that $s \subset t$ and that, for any $h \in Y_{\delta\gamma}$,

$$h(\beta+1) < f_\delta(\beta+1) = t(\beta+1).$$

Thus, (4.2) holds. $\perp$

5. Proofs of Theorems 5 and 6 (b).

In this section, we assume that the reader is familiar with forcing (see [**1**; Chapter 3], [**3**]). Throughout this section, M denotes a countable transitive model of ZFC+GCH. To prove Theorem 5 (resp. 6 (b)), it suffices to show that there is a generic extension M_1 (resp. M_2) of M which satisfies

(5.1) $M_1 \models \neg \mathrm{AN}(\omega) +$"there are $(2^\omega)^+$ Ramsey ufs on ω"

(resp. (5.2) $M_2 \models \mathrm{CH} + 2^{\omega_1} = \omega_3 + \mathrm{AN}(\omega_2)$). We shall exhibit below such generic extensions which are both well known.

I. Model M_1. Let M_1 be the generic extension of M adding ω_2^M Cohen generic reals. Then, M_1 satisfies (5.1). To show this, let P be the notion of forcing in M such that, in M,

$$P = \{p\ ;\ p \text{ is a function } \&\ \mathrm{dom}(p) \text{ is a finite subset of } \omega \times \omega_2 \\ \&\ \mathrm{rang}(p) \subset \{0, 1\}\}.$$

Let G be an M-generic filter on P such that $M_1 = M[G]$. Then, the following facts seem to be folklores. We omit the proofs.

FACT 1. *The following statement holds in* $M[G]$.

"*Let A be a subset of $P_\omega(\omega)$ such that*

(i) $|A| \leqq \omega_1$,

(ii) $\forall x, y \in A\ (x \cap y \in A)$.

Let $\langle x_n | n < \omega \rangle$ be an ω-sequence of elements in A such that

(iii) $x_{n+1} \subset x_n$ *for all* $n < \omega$.

Then, there exist infinite subsets y and z of ω such that

(iv) $y \cap z = \emptyset$,

(v) *both $A \cup \{y\}$ and $A \cup \{z\}$ have the finite intersection property,*

(vi) $|y \cap (x_n - x_{n+1})| \leqq 1$ *and* $|z \cap (x_n - x_{n+1})| \leqq 1$, *for all* $n < \omega$."

FACT 2. *Let x be an infinite subset of ω in $M[G]$. Then, there are y and*

z in M such that

(i) $y \cap z = \emptyset$,

(ii) $y \cap x$ *and* $z \cap x$ *are infinite.*

By Fact 1, it is easy to see that, in $M[G]$, there exist $(2^{\omega})^{+}$ Ramsey ufs on ω.[1)] To show that AN(ω) does not hold in $M[G]$, by Theorem 5, it suffices to show that AN(ω_2) does not hold in $M[G]$. From now on, we work in $M[G]$. Let Γ be the set of free, not ample, ω_1-generated filters on ω. Since $2^{\omega_1}=\omega_2$, it holds that $|\Gamma| \leqq \omega_2$. Let $\langle \mathfrak{G}_\alpha | \alpha<\omega_2 \rangle$ be an enumeration of Γ. Define the filter $\mathfrak{F}$ on $\omega \times \omega_2$ by

$$x \in \mathfrak{F} \text{ if and only if } \{n<\omega;\ (n, \alpha) \in x\} \in \mathfrak{G}_\alpha \text{ for all } \alpha<\omega_2.$$

It is easy to check that $\mathfrak{F}$ is free and not ample. To show that $\mathfrak{F}$ is weakly ample, let $\mathfrak{U}$ be any free uf on ω. Set $\mathfrak{B}$ be the filter on ω which is generated by $M \cap \mathfrak{U}$. By Fact 2, $\mathfrak{B}$ is a free, not ample, ω_1-generated filter. Let $\alpha<\omega_2$ be such that $\mathfrak{B}=\mathfrak{G}_\alpha$. Define the function h from ω to $\omega \times \omega_2$ by

$$h(n)=(n, \alpha) \qquad \text{for all} \quad n<\omega.$$

Then, $h(\mathfrak{U}) \supset h(\mathfrak{B}) \supset \mathfrak{F}$.

II. Model M_2. Let Q be the notion of forcing in M such that, in M,

$$Q=\{q;\ q \text{ is a function \& dom}(q) \text{ is a countable subset of } \omega_1 \times \omega_3 \\ \text{\& rang}(q) \subset \{0, 1\}\}.$$

Let H be an M-generic filter on Q. Then, $M_2=M[H]$ satisfies (5.2). To show this, since it is clear that $M[H] \models \mathrm{CH}+2^{\omega_1}=\omega_3$, it suffices to show that

(5.3) $$M[H] \models \mathrm{AN}(\omega_2).$$

First, we shall state Lemma 9 which is used later. Let I be the ideal $\{x \subset \omega;\ |x|<\omega\}$ on ω, and B the quotient algebra $P(\omega)/I$.

LEMMA 9. *Assume* CH. *Let C be the algebra of regular open sets in* $\{q \in Q;\ \mathrm{dom}(q) \subset \omega_1 \times \{0\}\}$. *Then, the completion of B is isomorphic to C.*

PROOF. It suffices to show that there exist X and Y which satisfy

(5.4) X is a dense subset of B,

(5.5) Y is a dense subset of $\{q \in Q;\ \mathrm{dom}(q) \subset \omega_1 \times \{0\}\}$,

(5.6) X and Y are order isomorphic.

Let $\langle\langle x_s | s \in S_{\alpha+1}\rangle | \alpha<\omega_1\rangle$ be an ω_1-sequence which satisfies (i)~(iv) in Lemma 4. Set

1) Kunen [2; p. 397] remarked that, in $M[G]$, there are Ramsey ufs on ω.

$$X=\{x_s/I\,;\,\exists\alpha<\omega_1(s\in S_{\alpha+1})\}\ ,$$

$$Y=\{q\in Q\,;\,\exists\alpha<\omega_1(\mathrm{dom}(q)=(\omega\alpha+\omega)\times\{0\})\}\ .$$

Then, it is easy to see that X and Y satisfy (5.4)~(5.6). ⊥

Henceforth, ω_α denotes the α-th infinite cardinal in M. Since M and $M[H]$ have the same cardinals, ω_α is the α-th infinite cardinal in $M[H]$. To show (5.3), let $\mathfrak{F}$ be any free, not ample filter on ω_2 in $M[H]$. Set $\mathfrak{G}=\{\omega_2-x\,;\,x\subset\omega_2\ \&\ |x|\leqq\omega\}\cap\mathfrak{F}$. Then, we have that, in $M[H]$, $\mathfrak{G}$ is a free, not ample filter with $|\mathfrak{G}|=\omega_2$. Since $|\mathfrak{G}|=\omega_2$, we can choose $\beta<\omega_3$ such that $\mathfrak{G}\in M[H\cap\{q\in Q\,;\,\mathrm{dom}(q)\subset\omega_1\times\beta\}]$. Set

$$Q_\beta=\{q\in Q\,;\,\mathrm{dom}(q)\subset\omega_1\times\beta\}\ ,$$

$$\bar{Q}=\{q\in Q\,;\,\mathrm{dom}(q)\subset\omega_1\times\{\beta\}\}\ ,$$

$$H_\beta=H\cap Q_\beta\ ,$$

$$\bar{H}=H\cap\bar{Q}\ ,$$

$$N=M[H_\beta]\ .$$

Then, it hold that $\bar{H}$ is an N-generic filter on $\bar{Q}$ and that $M[H]$ is a generic extension of $N[\bar{H}]$. To show that $\mathfrak{F}$ is not weakly ample in $M[H]$, since Q is σ-closed, it suffices to prove the following Lemma 10.

LEMMA 10. *In $N[\bar{H}]$, there exists a free uf $\mathfrak{U}$ on ω such that, for any $f:\omega\to\omega_2$, $f(\mathfrak{U})\not\supset\mathfrak{G}$.*

PROOF. Since $N\models\mathrm{CH}$, by Lemma 9, we identify $\bar{H}$ with some N-generic set on B $(=P^N(\omega)/I)$. Define $\mathfrak{U}$ by, in $N[\bar{H}]$,

$$\mathfrak{U}=\{x\subset\omega\,;\,x/I\in\bar{H}\}\ .$$

It is easy to see that $\mathfrak{U}$ is a free uf on ω in $N[\bar{H}]$. We claim that $\mathfrak{U}$ is as required. Let f be any function from ω to ω_2 in $N[\bar{H}]$. Then, f is in M. Define D by, in N,

$$D=\{x/I\in B\,;\,x/I>0\ \&\ \exists y\in\mathfrak{G}(y\cap f''x=\emptyset)\}\ .$$

Since $\mathfrak{G}$ is free and not ample in N, it holds that, in N, D is a dense subset of B. So, there exists some x/I in $\bar{H}\cap D$. Then, x is in $\mathfrak{U}$ and $y\cap f''x$ is finite, for some $y\in\mathfrak{G}$. Thus, $\omega_2-f''x$ is in $\mathfrak{G}-f(\mathfrak{U})$. ⊥

REMARK 1. Let M' be a countable transitive model of ZFC+MA+$2^\omega=\omega_2+2^{\omega_2}=\omega_3$, and let Q' be the notion of forcing in M' which is defined by, in M',

$$Q'=\{q\,;\,q\text{ is a function }\&\ \mathrm{dom}(q)\subset\omega_4\ \&\ |q|\leqq\omega_1\ \&\ \mathrm{rang}(q)\subset\{0,1\}\}\ .$$

Let H' be an M'-generic filter on Q'. Then, by a similar argument in II, it holds that

$$M'[H']\models 2^{\omega}=\omega_2+2^{\omega_2}=\omega_4+\mathrm{AN}(\omega_3).$$

So, $\mathrm{ZFC}+2^{\omega}>\omega_1+2^{c}>c^{+}+\forall\kappa<2^{c}\mathrm{AN}(\kappa)$ is consistent.

6. Proof of Theorem 6 (a).

To show Theorem 6 (a), similarly to the proofs of Theorems 5 and 6 (b), it suffices to show that there exists a generic model N such that

$$N\models \mathrm{CH}+2^{\omega_1}=\omega_3+\neg\mathrm{AN}(\omega_2). \tag{6.1}$$

By Theorem 4, if

$$N\models \mathrm{CH}+2^{\omega_1}=\omega_3+\text{"there is an } \omega_2\text{-scale on } \omega_1\text{"}, \tag{6.2}$$

then N satisfies (6.1). Throughout this section, M denotes an arbitrary but fixed countable transitive model of $\mathrm{ZFC}+\mathrm{CH}+2^{\omega_1}=\omega_3$. We shall construct a generic extension N of M satisfying (6.2). Our method is so called countable support iterated forcing. We assume that the reader is familiar with this iterated forcing (see [**3**; Chapter 8, Section 7]). From now on, we work in M till after Corollary 2 except Lemma 11.

DEFINITION. S denotes the set of all functions from a countable ordinal to ω_1.

DEFINITION. For any complete Boolean algebra B, define $\boldsymbol{Q}=\boldsymbol{Q}(B)\in V^{B}$ by

$$\mathrm{dom}(\boldsymbol{Q})=\{(\check{s},\boldsymbol{J})^{B}\,;\, s\in S \;\&\; \boldsymbol{J}\in V^{B} \;\&\; \|\boldsymbol{J} \text{ is a set of functions on } \omega_1 \text{ with } |\boldsymbol{J}|\leqq\omega\|=1\},$$

$$\boldsymbol{Q}(x)=1 \quad \text{for all} \quad x\in\mathrm{dom}(\boldsymbol{Q}).$$

And, we regard $\boldsymbol{Q}$ as the notion of forcing in V^{B} whose order is defined by, for any $(\check{s},\boldsymbol{J})^{B}$, $(\check{t},\boldsymbol{K})^{B}\in\mathrm{dom}(\boldsymbol{Q})$,

$$\|(\check{s},\boldsymbol{J})^{B}\leqq(\check{t},\boldsymbol{K})^{B}\| =\|\check{s}\supset\check{t} \;\&\; \boldsymbol{J}\supset\boldsymbol{K} \;\&\; \forall\alpha\in\mathrm{dom}(\check{s}-\check{t})\forall f\in\boldsymbol{K}(f(\alpha)<\check{s}(\alpha))\|.$$

LEMMA 11. *Suppose that, in M, B is an ω-distributive, ω_2-saturated, complete Boolean algebra. Let G be an M-generic filter on B, $Q=i_G(\boldsymbol{Q}(B)^{M})$ and H an $M[G]$-generic filter on Q. Set $h=\bigcup\{s\,;\,\exists J((s,J)\in H)\}$. Then,*

(a) *in $M[G]$,*

  (i) *$Q=S\times\{J;\, J$ is a set of functions on ω_1 with $|J|\leqq\omega\}$,*

  (ii) *Q is σ-closed and has the ω_2-chain condition,*

(b) *in $M[G][H]$,*

  (i) *h is a function on ω_1,*

  (ii) *$f\prec h$ for any function f on ω_1 in $M[G]$.*

PROOF. This is well-known. ⊥

Define the sequence $\langle R_\alpha | \alpha \leqq \omega_2 \rangle$ of partially ordered sets by the following induction on α ($\leqq \omega_2$).

For each $\alpha \leqq \omega_2$, set

$$B_\alpha = \text{the algebra of regular open sets in } R_\alpha.$$

Case 1. $\alpha = 0$.

$$R_0 = \{\emptyset(=1)\}.$$

Case 2. $\alpha = \gamma + 1$ for some γ.

Set $\boldsymbol{Q}_\gamma = \boldsymbol{Q}(B_\gamma)$ ($\in V^{B_\gamma}$). For any $p = \langle p_\xi | \xi < \alpha \rangle$,

$$p \in R_\alpha \text{ if and only if } p \restriction \gamma \in R_\gamma \ \& \ p(\gamma) \in \mathrm{dom}(\boldsymbol{Q}_\gamma).$$

Define the order $\leqq$ on R_α by, for any $p, q \in R_\alpha$,

$$p \leqq q \text{ if and only if } p \restriction \gamma \leqq q \restriction \gamma \ \& \ p \restriction \gamma \Vdash p(\gamma) \leqq q(\gamma).$$

(For any $p, q \in R_\alpha$, whenever $p \leqq q$ and $q \leqq p$, we identify p and q.)

Define the function e from R_γ to R_α by, for any $p \in R_\gamma$,

$$e(p) \restriction \gamma = p,$$
$$e(p)(\gamma) = (\check{\emptyset}, \check{\emptyset})^{B_\gamma}.$$

And, for convention, we regard R_γ as the subset $e''R_\gamma$ of R_α.

Case 3. α is limit.

For any $p = \langle p_\xi | \xi < \alpha \rangle$,

$p \in R_\alpha$ if and only if $\forall \xi < \alpha \ (p \restriction \xi \in R_\xi)$ and $\{\xi < \alpha ; \|p(\xi) < 1(=(\emptyset, \emptyset))\| > 0\}$ is at most countable.

Define the order $\leqq$ on R_α by, for any $p, q \in R_\alpha$,

$$p \leqq q \text{ if and only if } \forall \xi < \alpha \ (p \restriction \xi \leqq q \restriction \xi).$$

In the same way in Case 2, for each $\xi < \alpha$, we regard R_ξ as the subset of R_α.

REMARK 2. In Case 3, if $\mathrm{cof}(\alpha) > \omega$, then R_α coincides with the direct limit of $\langle R_\xi | \xi < \alpha \rangle$.

The following Lemma 12 can be proved by the induction on α ($\leqq \omega_2$) using standard arguments (see [3; Lemma 7.2 (p. 282) and Lemma 7.10 (p. 286)]). So, we omit the proof.

LEMMA 12. *For any* $\alpha \leqq \omega_2$,

(a) $|R_\alpha| \leqq \omega_3$,

(b) R_α *is* σ-*closed*,

(c) R_α *has the* ω_2-*chain condition*.

Set $R = R_{\omega_2}$.

COROLLARY 2.
(a) $|R|=\omega_3$.
(b) *R is σ-closed.*
(c) *R has the ω_2-chain condition.*

PROOF. This follows immediately from Lemma 12. $\perp$

Let G be an M-generic filter on R. Set $N=M[G]$. For each $\alpha<\omega_2^M$, set

$$G_\alpha=G\cap R_\alpha ,$$

$$M_\alpha=M[G_\alpha] .$$

Define $\langle h_\alpha|\alpha<\omega_2^M\rangle\in N$ by, for each $\alpha<\omega_2^M$,

$$h_\alpha=\bigcup\{s\,;\,\exists p\in G\exists \boldsymbol{J}((\check{s},\boldsymbol{J})^{B_\alpha}=p(\alpha))\} .$$

Then, by Corollary 2, the following (6.3)∼(6.6) hold.

(6.3) $P^N(\omega)=P^M(\omega)$.

(6.4) N and M have the same cardinals.

(6.5) $2^{\omega_1}=\omega_3$ holds in N.

(6.6) For any function f on ω_1 in N, there is an $\alpha<\omega_2^M$ such that $f\in M_\alpha$.

Now, we shall show (6.2). Let f be any function on ω_1 in N. By (6.6), there is an $\alpha<\omega_2^M$ such that $f\in M_\alpha$. By Lemma 11, since $H=\{(s, i_G(\boldsymbol{J}))\,;\,\exists p\in G((\check{s},\boldsymbol{J})^{B_\alpha}=p(\alpha))\}$ is an $M[G_\alpha]$-generic filter on $i_{G_\alpha}(\boldsymbol{Q}_\alpha)$, it holds that

$$M[G_\alpha][H]\models\text{“}f\prec h_\alpha\text{”} .$$

Thus, in N, $\langle h_\alpha|\alpha<\omega_2\rangle$ is an ω_2-scale on ω_1.

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