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On the holomorphic equivalence of bounded domains in complete Kähler manifolds of nonpositive curvature

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1. Introduction.

Suppose D_1 and D_2 are two bounded domains in the complex n-space $\boldsymbol{C}^n$, $n \geqq 2$, with C^∞ boundaries ∂D_1 and ∂D_2, respectively. One of the fundamental problems in several complex variables is to determine geometric conditions which imply that D_1 and D_2 are biholomorphically equivalent. It has been known from Bochner-Hartogs' theorem (Bochner [1]) that if ∂D_1 and ∂D_2 are connected and CR-diffeomorphic, then D_1 and D_2 are biholomorphically equivalent; and moreover by the second author [7] that the same is the case even for those domains in Stein manifolds.

In this paper we are concerned with the problem for domains in complete Kähler manifolds of nonpositive curvature. Our result is stated as follows.

THEOREM. *Let M and N be complete Kähler manifolds of complex dimension $n \geqq 2$. Let $D_1 \subset M$ and $D_2 \subset N$ be relatively compact subdomains in M and N with C^∞ boundaries ∂D_1 and ∂D_2, respectively. Suppose that* (i) *N has adequate negative curvature in the sense of Siu* [8], (ii) *the boundary ∂D_1 is pseudoconvex, and* (iii) *there exists a* CR-*diffeomorphism $f: \partial D_1 \to \partial D_2$ which extends to a homotopy equivalence of D_1 to D_2. Then D_1 and D_2 are biholomorphically equivalent. In fact, f extends to a biholomorphic diffeomorphism of D_1 to D_2.*

The adequate negativity of curvature, assumed in the hypothesis (i), is in fact stronger than requiring nonpositive sectional curvature. It is, however, known by Siu [8] that the classical bounded symmetric domains with their invariant metrics and their quotient Kähler manifolds have adequate negative curvature. It should be remarked that the curvature hypothesis (i) is assumed only on the target manifold N.

Some results related to ours can be seen in Wood [9]. We wish to thank him for making his manuscript available during the preparation of this paper.

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2. Preliminaries.

First we fix the terminology in the theorem. Let $D_1 \subset M$ and $D_2 \subset N$ be relatively compact subdomains as in the theorem. The boundary ∂D_1 of D_1 is called *pseudoconvex* if the Levi form of ∂D_1 is positive semidefinite everywhere. Let J denote the complex structure of M. A C^∞ mapping $f: \partial D_1 \to \partial D_2$ is said to be a CR-*mapping* if the differential df of f restricted to the maximal complex subspace $H_p(\partial D_1) = T_p(\partial D_1) \cap JT_p(\partial D_1)$ of the real tangent space $T_p(\partial D_1)$ is complex linear at each point $p \in \partial D_1$. Note that $f: \partial D_1 \to \partial D_2$ is a CR-mapping if and only if it satisfies the *tangential Cauchy-Riemann equation* $\bar{\partial}_b f = 0$, where $\bar{\partial}_b f = \bar{\partial} f \circ \pi$, π being the orthogonal projection $\pi: T_p(\partial D_1) \to H_p(\partial D_1)$ for each $p \in \partial D_1$ (cf. [2]). A diffeomorphism $f: \partial D_1 \to \partial D_2$ is called a CR-*diffeomorphism* if f and f^{-1} are CR-mappings.

We need the notion of adequate negativity, defined by Siu [8], of the curvature of a Kähler manifold. The curvature tensor of a Kähler n-manifold N is said to be *adequately negative at* $q \in N$ if the following hold: Let $h: U \to N$ be a C^∞ mapping of an open neighborhood U of $0 \in \boldsymbol{C}^n$ to N with $h(0) = q$. Let (z^i) denote a local complex coordinates of $\boldsymbol{C}^n$ around 0 and (w^α) that of N around q. Then the curvature tensor $(R_{\alpha\bar{\beta}\gamma\bar{\delta}})$ of N has the properties that (a) $\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha\bar{\beta}}_{\bar{i}\bar{j}} \overline{\xi^{\delta\bar{\gamma}}_{\bar{i}\bar{j}}} \geqq 0$ for all $1 \leqq i,\ j \leqq n$, where $\xi^{\alpha\bar{\beta}}_{\bar{i}\bar{j}} = (\partial_{\bar{i}} h^\alpha)(0) \overline{(\partial_j h^\beta)(0)} - (\partial_{\bar{j}} h^\alpha)(0) \cdot \overline{(\partial_i h^\beta)(0)}$, $\partial_{\bar{i}} h^\alpha = \partial h^\alpha / \partial \overline{z^i}$ etc., and (b) if h is a local diffeomorphism around 0 and $\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha\bar{\beta}}_{\bar{i}\bar{j}} \overline{\xi^{\delta\bar{\gamma}}_{\bar{i}\bar{j}}} = 0$ at q, then either $\partial h = 0$ or $\bar{\partial} h = 0$ at 0. If the curvature tensor of N is adequately negative everywhere, we simply say that N *has adequate negative curvature*. Note that the property (a) of adequate negative curvature implies nonpositivity of the sectional curvature. For examples of Kähler manifolds having adequate negative curvature, see Siu [8] and Mostow-Siu [4].

3. Proof of the theorem.

Let $D_1 \subset M$ and $D_2 \subset N$ be as in the theorem. By hypothesis (iii), we have a CR-diffeomorphism $f: \partial D_1 \to \partial D_2$ which extends to a homotopy equivalence $\tilde{f}: D_1 \to D_2$. We may assume $\tilde{f}$ to be C^∞. Since the sectional curvature of N is nonpositive everywhere by hypothesis (i), it then follows from a theorem of Hamilton [3] and Schoen [6, 2] that there exists a *harmonic mapping* $h: D_1 \to N$ which is homotopic to $\tilde{f}$ relative to ∂D_1, so that $h = f$ on ∂D_1. We refer to Eells-Lemaire [2] for general background material on harmonic mappings. Note that h is C^∞ up to the boundary.

In consequence, we may assume that there exists a harmonic homotopy equivalence $h: D_1 \to N$ such that $h | \partial D_1 : \partial D_1 \to \partial D_2$ is a CR-diffeomorphism. We are going to prove that h is a desired biholomorphic equivalence of D_1 to D_2.

LEMMA 1. *h is holomorphic on D_1.*

PROOF. Let g and ω denote the Kähler metric of N and the Kähler form of M, respectively. Let (z^i) and (w^α) denote respectively the local complex coordinates of M and N, and let $(R_{\alpha\bar{\beta}\gamma\bar{\delta}})$ denote the curvature tensor of N. Denote by $\langle\ ,\ \rangle$ contraction of tensors and consider the (1, 1)-form $\langle g, \bar{\partial}h\wedge\partial\bar{h}\rangle$ on D_1 defined in terms of local coordinates by

$$\langle g, \bar{\partial}h\wedge\partial\bar{h}\rangle=\sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\bar{\partial}h^\alpha\wedge\partial\overline{h^\beta}\,.$$

It is then known in Siu [8] that by harmonicity of h we have, at all points $p\in D_1$,

$$\partial\bar{\partial}\langle g, \bar{\partial}h\wedge\partial\bar{h}\rangle\wedge\omega^{n-2}=\sigma\omega^n-\chi\omega^n\,, \tag{1}$$

where, with respect to a local complex coordinates orthonormal at p,

$$\sigma=\frac{1}{n(n-1)}\sum_{\substack{\alpha,\beta,\gamma,\delta\\ 1\leq i<j\leq n}} R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha\bar{\beta}}_{\bar{i}j}\overline{\xi^{\delta\bar{\gamma}}_{\bar{i}j}}\,, \tag{2}$$

$\xi^{\alpha\bar{\beta}}_{\bar{i}j}=\partial_{\bar{i}}h^\alpha\cdot\overline{\partial_j h^\beta}-\partial_{\bar{j}}h^\alpha\cdot\overline{\partial_i h^\beta}$, and χ is some nonpositive function on D_1. Note that the adequate negativity of the curvature of N implies that $\sigma\geqq 0$. Hence, integrating (1) over D_1, it follows from Stokes' theorem that

$$\int_{\partial D_1}\bar{\partial}\langle g, \bar{\partial}h\wedge\partial\bar{h}\rangle\wedge\omega^{n-2}=\int_{D_1}(\sigma\omega^n-\chi\omega^n)\geqq 0\,. \tag{3}$$

We now investigate the boundary integral in (3). Take a point $p\in\partial D_1$ and let ψ be a defining function of ∂D_1, that is, $\psi=\psi(z^1,\cdots,z^n,\overline{z^1},\cdots,\overline{z^n})$ is a real-valued C^∞ function defined in a neighborhood U of p such that $D_1\cap U=\{x\in U\,|\,\psi(x)<0\}$ and $d\psi\neq 0$ on $\partial D_1\cap U$. Without loss of generality we may assume that $dz^1, d\overline{z^1},\cdots, dz^{n-1}, d\overline{z^{n-1}}, dz^n-d\overline{z^n}$ is a basis of $T^*_p(\partial D_1)\otimes \boldsymbol{C}$ and $(\partial_n\psi)(p)=1$. Then, on $\partial D_1\cap U$, the tangential Cauchy-Riemann operator $\bar{\partial}_b f$ is expressed, in terms of local coordinates and ψ, as

$$\bar{\partial}_b f^\alpha=\sum_{i=1}^{n-1}(\partial_{\bar{i}}f^\alpha-\partial_{\bar{n}}f^\alpha\cdot(\partial_{\bar{i}}\psi/\partial_{\bar{n}}\psi))d\overline{z^i}\,,\qquad 1\leqq\alpha\leqq n\,. \tag{4}$$

Note that, since $h|\partial D_1$ is a CR-mapping, it follows from (4) that on $\partial D_1\cap U$

$$\partial_{\bar{i}}h^\alpha=\partial_{\bar{n}}h^\alpha\cdot(\partial_{\bar{i}}\psi/\partial_{\bar{n}}\psi)\,,\qquad 1\leqq\alpha\leqq n\,.$$

Direct computation then yields that at $p\in\partial D_1$

$$\begin{aligned}&\bar{\partial}\langle g, \bar{\partial}h\wedge\partial\bar{h}\rangle\wedge\omega^{n-2}\\ &=-(n-2)!(\sqrt{-1})^{n-2}\sum_{\substack{1\leq\alpha\leq n\\ 1\leq i\leq n-1}}\partial_{\bar{n}}h^\alpha\cdot\partial_{\bar{i}}\partial_i\overline{h^\alpha}d\overline{z^n}\wedge d\overline{z^i}\wedge dz^i\bigwedge_{\substack{1\leq\tau\leq n-1\\ \tau\neq i}}(dz^\tau\wedge d\overline{z^\tau})\end{aligned} \tag{5}$$

$$=-(\sum_{i=1}^{n-1}\partial_{\bar{i}}\partial_i\psi)\cdot(\sum_{\alpha=1}^{n}|\partial_{\bar{n}}h^\alpha|^2)\times\text{positive }(2n-1)\text{-form on }\partial D_1,$$

where we choose a local complex coordinates (w^α) of N such that $g_{\alpha\bar{\beta}}=\delta_{\alpha\beta}$ and $dg_{\alpha\bar{\beta}}=0$ at $h(p)$. The pseudoconvexity of ∂D_1, assumed in the hypothesis (ii), now implies that

$$\sum_{i=1}^{n-1}\partial_{\bar{i}}\partial_i\psi=\text{the trace of the Levi form of }\partial D_1\geqq 0. \tag{6}$$

Since $p\in\partial D_1$ is arbitrary, it follows from (5) and (6) that

$$\int_{\partial D_1}\bar{\partial}\langle g,\ \partial h\wedge\partial\bar{h}\rangle\wedge\omega^{n-2}\leqq 0. \tag{7}$$

Consequently, we obtain from (3) and (7) that

$$\int_{D_1}(\sigma\omega^n-\chi\omega^n)=0,$$

from which we get $\sigma\equiv 0$ and $\chi\equiv 0$, for $\sigma\geqq 0$ and $\chi\leqq 0$ on D_1. Hence it follows from (2) that for all $1\leqq i,\ j\leqq n$

$$\sum_{\alpha,\beta,\gamma,\delta}R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha\bar{\beta}}_{\bar{i}j}\overline{\xi^{\delta\bar{\gamma}}_{\bar{i}j}}=0 \qquad \text{on } D_1.$$

Recall that h is a local diffeomorphism near ∂D_1. Then the adequate negativity of the curvature of N implies that $\partial h=0$ or $\bar{\partial}h=0$ at each point near ∂D_1. Since h is a harmonic mapping, it then follows as in Siu [8] from the unique continuation property that $\partial h\equiv 0$ on D_1 or $\bar{\partial}h\equiv 0$ on D_1. But $\bar{\partial}_b h=0$ on ∂D_1 and the rank of $dh|\partial D_1$ is $2n-1$, so $\partial h\equiv 0$ is impossible. Hence we conclude that $\bar{\partial}h\equiv 0$ on D_1, that is, h is holomorphic on D_1.

LEMMA 2. *h maps D_1 onto D_2.*

PROOF. Since $h|\partial D_1\colon \partial D_1\to\partial D_2$ is a diffeomorphism, it suffices to prove that h maps $\bar{D}_1=D_1\cup\partial D_1$ onto $\bar{D}_2=D_2\cup\partial D_2$. First we recall that h, a holomorphic mapping of D_1 into N, is homotopic to $\tilde{f}$, a C^∞ homotopy equivalence of D_1 to D_2, relative to ∂D_1. Then it follows by homotopy theoretic argument that $h(\bar{D}_1)\supset\bar{D}_2$ and $h(\bar{D}_1)$ has the same $2n$-dimensional Lebesgue measure, induced by the metric of N, as that of $\bar{D}_2$. Thus we are left to show that $h(\bar{D}_1)-\bar{D}_2$, which is a set of measure zero, is empty.

Assume the contrary, namely assume that $h(\bar{D}_1)-\bar{D}_2\neq\emptyset$. Then, for each $q\in h(\bar{D}_1)-\bar{D}_2$, the inverse image $h^{-1}(q)$ is contained in the set V of critical points of h. Note that V is a compact complex-analytic subvariety in D_1 of pure complex codimension 1, for locally V is defined by $\det(\partial w^\alpha/\partial z^i)$ and h is locally diffeomorphic near ∂D_1. We now take points $q\in h(\bar{D}_1)-\bar{D}_2$ and $p\in V$ such that $h(p)=q$, and let U be a neighborhood of p in D_1. Then, by continuity of h, choosing U sufficiently small, we see that $h(U)\cap\bar{D}_2=\emptyset$ or $h(U)\subset h(\bar{D}_1)-\bar{D}_2$. Hence $U\subset V$, which is a contradiction.

LEMMA 3. *h is a biholomorphic mapping of D_1 to D_2.*

PROOF. Let V be as in the proof of Lemma 2, that is, V is the set of points of D_1 where h is not locally diffeomorphic. Note that h is of degree 1 and so maps $D_1-h^{-1}(h(V))$ bijectively onto $h(D_1)-h(V)$. Hence it suffices to prove that V is empty. Assume the contrary, namely assume that $V\neq\emptyset$. Then V, a compact complex-analytic subvariety in D_1, defines a nonzero homology class $[V]$ in $H_{2n-2}(D_1; \boldsymbol{R})$. Since $h: D_1\to D_2$ is a proper holomorphic mapping, it follows from a theorem of Remmert [5] that $h(V)$ is a compact complex-analytic subvariety in D_2 of complex codimension at least 2. Hence $[V]$ is mapped by h to the zero element in $H_{2n-2}(D_2; \boldsymbol{R})$, that is, $h_*([V])=0$ in $H_{2n-2}(D_2; \boldsymbol{R})$, contradicting the fact that h is a homotopy equivalence of D_1 to D_2.

The proof of the theorem is now complete.

REMARK 1. In the course of the proof, we in fact prove that the theorem holds under the hypotheses (i), (iii) and, instead of (ii), a somewhat weaker convexity condition (6) in Lemma 1: *The trace of the Levi form of the boundary ∂D_1 is positive semidefinite everywhere.* At present the authors do not know whether the hypothesis (ii) can be further weakened, but the following remark illustrates that a kind of convexity condition is necessary.

REMARK 2. Let M be a compact quotient of the ball in $\boldsymbol{C}^n$, with its invariant metric. Let $p, q\in M$, $p\neq q$, and B_p and B_q be balls with sufficiently small radius centered at p and q, respectively. Let $f: B_p\to B_q$ be a biholomorphic diffeomorphism defined via local complex coordinates in a natural way. Then $f|\partial B_p: \partial B_p\to\partial B_q$ extends to a diffeomorphism of $M-B_p$ to $M-B_q$, but, for some p, q, it can not extend to a biholomorphic one. In fact, if this were the case for arbitrary p, q, then it would follow that M were homogeneous.

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