J. Math. Soc. Japan
Vol. 35. No. 2, 1983

Behavior of geodesics in foliated manifolds with bundle-like metrics

By Shinsuke YOROZU

(Received Dec. 22, 1981)

1. Introduction.

Foliated manifolds are studied by C. Ehresmann, A. Haefliger, G. Reeb and many people. Many of works are topological (non-riemannian) cases. The early study of riemannian case was done by B.L. Reinhart [24], that is, he defined foliated manifolds with "bundle-like" metrics with respect to the foliations and proved so-called Reeb stability theorem for this case. The foliated manifolds with bundle-like metrics are studied by R. Hermann [4], A.M. Naveira [19], J.S. Pasternack [22, 23], B.L. Reinhart [24, 25], R. Sacksteder [26], I. Vaisman [28, 29] and others.

The typical examples of foliated manifolds with bundle-like metrics are the followings; (i) each fiber space under a suitable choice of metric, (ii) the foliation of a riemannian manifold by orbits of a group of isometries having all its orbits of the same dimension.

In this paper we discuss the behavior of geodesics in foliated manifolds with bundle-like metrics. As a well-known and fundamental result in this direction, we may state:

THEOREM (B.L. Reinhart [24]). *A geodesic of a bundle-like metric is orthogonal to the leaf at one point if and only if it is orthogonal to the leaf at every point.*

We discuss geodesics making constant angles with leaves, and these are generalizations of [14]. We discuss focal points of leaves along transversal geodesics, and, in the case of codimension 1, we have non-existence of focal points of leaves along transversal geodesics. The relations between the Levi-Civita connection and the second connection defined by I. Vaisman [28] are discussed.

The topological obstructions for the existence of the foliation with a bundle-like metric were studied by H. Kitahara and S. Yorozu [12], J.S. Pasternack [22] and R. Sacksteder [26]. The existence of the complete bundle-like metric

This research was partially supported by Grant-in-Aid for Scientific Research (No. 56540116), Ministry of Education.

was discussed by H. Kitahara [8, 9].

We shall be in C^∞-category. Latin indices run from 1 to p, and Greek indices run from $p+1$ to $p+q$. We use the Einstein's summation convention unless otherwise stated.

2. Foliated manifold.

Let M be an n dimensional connected riemannian manifold with a riemannian metric $\langle\ ,\ \rangle$ and the Levi-Civita connection ∇ with respect to $\langle\ ,\ \rangle$. Let TM denote the tangent bundle (or, its total space) over M. For a subbundle E of TM, $\Gamma(E)$ is a set of all sections of E.

DEFINITION 2.1. A sub-bundle E of TM is called *integrable* if, for any $X, Y\in\Gamma(E)$, $[X, Y]\in\Gamma(E)$ where $[\ ,\ ]$ denotes the bracket operator.

DEFINITION 2.2. If TM admits an integrable sub-bundle E of fiber dimension p $(=n-q, 0<p<n)$, then M is called a *foliated manifold with a foliation E of codimension q*. The maximal connected integral manifolds of E are called *leaves*.

Hereafter we assume that M is a foliated manifold with a foliation E of codimension $q=n-p$. For each point of M, we may find a coordinate neighborhood U with coordinates $(x^1, \cdots, x^p, x^{p+1}, \cdots, x^{p+q})$ such that (i) $|x^i|<1$, $|x^\alpha|<1$, (ii) the integral manifolds of E are given locally by $x^{p+1}=c^{p+1}, \cdots, x^{p+q}=c^{p+q}$ for constants c^α satisfying $|c^\alpha|<1$. Such a coordinate chart $U(x^i, x^\alpha)$ is called *flat*.

If $U(x^i, x^\alpha)$ and $\bar{U}(\bar{x}^i, \bar{x}^\alpha)$ are flat coordinate charts such that $U\cap\bar{U}\neq\emptyset$, then $\partial/\partial x^i$ transforms by coordinate change into a combination of $\partial/\partial\bar{x}^1, \cdots, \partial/\partial\bar{x}^p$, since the tangent space to a leaf goes into the tangent space to the leaf. Thus the coordinate transformation is of the form $\bar{x}^i=\bar{x}^i(x^j, x^\beta)$ and $\bar{x}^\alpha=\bar{x}^\alpha(x^\beta)$.

In each flat coordinate chart $U(x^i, x^\alpha)$, we may choose 1-forms $w^1, \cdots, w^p$ such that $\{w^1, \cdots, w^p, dx^{p+1}, \cdots, dx^{p+q}\}$ is a basis for the cotangent space at each point in U, and vectors $v_{p+1}, \cdots, v_{p+q}$ such that $\{\partial/\partial x^1, \cdots, \partial/\partial x^p, v_{p+1}, \cdots, v_{p+q}\}$ is the dual base for the tangent space. We have $w^i=dx^i+A^i_\alpha dx^\alpha$ and $v_\alpha=\partial/\partial x^\alpha-A^i_\alpha\partial/\partial x^i$ for any functions $A^i_\alpha=A^i_\alpha(x^k, x^\tau)$ on U. If we transform the flat coordinate chart $U(x^i, x^\alpha)$ into $\bar{U}(\bar{x}^i, \bar{x}^\alpha)$ and choose $\bar{w}^i$ and $\bar{v}_\alpha$ in $\bar{U}(\bar{x}^i, \bar{x}^\alpha)$, then $\bar{w}^i$ transforms into a combination of the w^j and $\bar{v}_\alpha$ into a combination of the v_β.

3. Bundle-like metric and examples.

Let Q be the quotient bundle TM/E. The natural projection $\pi: TM\to Q$ induces a map $\pi:\Gamma(TM)\to\Gamma(Q)$.

DEFINITION 3.1. In each flat coordinate chart $U(x^i, x^\alpha)$, a frame $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_{p+q}\}$ is an *adapted frame to* the foliation E if $\{X_1, \cdots, X_p\}$ and

$\{\pi(X_{p+1}), \cdots, \pi(X_{p+q})\}$ span $\Gamma(E|_U)$ and $\Gamma(Q|_U)$ respectively.

In each $U(x^i, x^\alpha)$, frames $\{\partial/\partial x^i, \partial/\partial x^\alpha\}$ and $\{\partial/\partial x^i, v_\alpha\}$ are adapted frames to E (See [**14**], [**22**], [**23**], [**24**]).

DEFINITION 3.2. The adapted frame $\{\partial/\partial x^i, v_\alpha\}$ is called the *basic adapted frame to the foliation E.*

It holds that

(3.1) $$\pi([X, v_\alpha])=0 \quad \text{for any} \quad X\in\Gamma(E|_U).$$

We may identify the quotient bundle Q with the orthogonal complement bundle $E^\perp$ to E in TM with respect to the riemannian metric $\langle\ ,\ \rangle$, and we have

(3.2) $$TM\cong E\oplus Q\cong E\oplus E^\perp$$

where $\oplus$ denotes the Whitney sum.

The riemannian metric $\langle\ ,\ \rangle$ has a local expression $\langle\ ,\ \rangle|_U=h_{ij}dx^i\cdot dx^j+2h_{i\beta}dx^i\cdot dx^\beta+h_{\alpha\beta}dx^\alpha\cdot dx^\beta$ in each flat coordinate chart $U(x^i, x^\alpha)$. We have $\det(h_{ij})>0$, thus we denote by $(\tilde{h}^{ij})$ the inverse matrix of (h_{ij}). If we choose $A^i_\alpha=h_{j\alpha}\tilde{h}^{ij}$, then the frame $\{v_{p+1}, \cdots, v_{p+q}\}$ spans $\Gamma(E^\perp|_U)$. Thus we have following local expression of $\langle\ ,\ \rangle$:

$$\langle\ ,\ \rangle|_U=g_{ij}(x^k, x^\tau)w^i\cdot w^j+g_{\alpha\beta}(x^k, x^\tau)dx^\alpha\cdot dx^\beta$$

where $g_{ij}=h_{ij}$ and $g_{\alpha\beta}=h_{\alpha\beta}-h_{ij}A^i_\alpha A^j_\alpha$.

DEFINITION 3.3. The riemannian metric $\langle\ ,\ \rangle$ is a *bundle-like metric with respect to the foliation E* if, in each flat coordinate chart $U(x^i, x^\alpha)$, it has a local expression

$$\langle\ ,\ \rangle|_U=g_{ij}(x^k, x^\tau)w^i\cdot w^j+g_{\alpha\beta}(x^\tau)dx^\alpha\cdot dx^\beta,$$

that is, $\partial\langle v_\alpha, v_\beta\rangle/\partial x^i=0$ for $1\leqq\forall i\leqq p$ and $p+1\leqq\forall\alpha, \forall\beta\leqq p+q$.

Now, we have the following theorem which will play an important role in the next section.

THEOREM 3.1 (See [**14**]). *The riemannian metric* $\langle\ ,\ \rangle$ *on a foliated manifold M with a foliation E of codimension q is a bundle-like metric with respect to E if and only if, for each flat coordinate chart* $U(x^i, x^\alpha)$, *there exists an orthonormal adapted frame* $\{X_i, X_\alpha\}$ *to E such that*

$$\langle\nabla_{X_\alpha}X_i, X_\beta\rangle+\langle\nabla_{X_\beta}X_i, X_\alpha\rangle=0$$

for $1\leqq\forall i\leqq p$ *and* $p+1\leqq\forall\alpha, \forall\beta\leqq p+q$.

DEFINITION 3.4. A leaf L in M is called *totally geodesic* if $\nabla_X Y|_m\in T_m L$ for each point $m\in L$, any flat coordinate chart U ($m\in U$) and any $X, Y\in\Gamma(E|_U)$, where $T_m L$ denotes the tangent space of L at m.

We remark that an immersed sub-manifold N of a manifold M with the Levi-Civita connection ∇ is totally geodesic (=the second fundamental form of

N identically vanishes) if and only if $\nabla_X Y \in \Gamma(TN)$ for any $X, Y \in \Gamma(TN)$ (See [**15**]).

The foliated manifolds all leaves of which are totally geodesic are studied by many people (See [**2**], [**3**], [**5**], [**16**]).

We are often able to find out the foliated manifolds with bundle-like metrics in the study of differential geometry: (i) Let M be a riemannian manifold acted on by a group of isometries such that all orbits are of the same dimension. M is a foliated manifold with orbits as its leaves, and the riemannian metric on M is a bundle-like metric with respect to the foliation (See [**5**], [**7**], [**22**], [**23**], [**24**]). (ii) Let M be the tangent bundle TN over a q dimensional riemannian manifold N. Then M is a foliated manifold with fibers as leaves, and the Sasaki metric (See [**27**]) on TN is a bundle-like metric with respect to the foliation. (iii) Let $\varphi: M \to B$ be a riemannian submersion (See [**6**], [**20**]). M is a foliated manifold with fibers $\varphi^{-1}(b)$ $(b \in B)$ as leaves, and the riemannian metric on M is a bundle-like metric with respect to the foliation.

We remark that the canonical metric on S^3 is not bundle-like metric with respect to the Reeb foliation.

4. Geodesic making constant angle with leaves.

Let $\gamma(s)$ (or γ) be a geodesic in M parametrized by arc-length s, that is, $\nabla_{\dot\gamma(s)}\dot\gamma(s)=0$ where $\dot\gamma(s)$ denotes a tangent vector of γ at s.

For any point $\gamma(s)$, we may choose a flat coordinate chart $U(x^i, x^\alpha)$ such that $\gamma(s) \in U$ and an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E in U. Let $\{\theta^i, \theta^\alpha\}$ be its dual adapted frame. Then we define $f=\{f_U\}$ by

$$f_U(s)=f_U(\gamma(s))=\sum_{i=1}^{p}[\theta^i(\dot\gamma(s))]^2. \tag{4.1}$$

LEMMA 4.1. *The function $f=\{f_U\}$ defined by* (4.1) *is independent of the choice of U. f is a differentiable function on I_γ which is a range of parameter s of γ.*

The geometric meaning of $f(s)$ is a square of the length of orthographic vector in $E_{\gamma(s)}$ of a vector $\dot\gamma(s)$ in $T_{\gamma(s)}M$. Let $\alpha(s)$ be an angle between the orthographic vector of $\dot\gamma(s)$ and $\dot\gamma(s)$. Then we have that $f(s)=[\cos\alpha(s)]^2$.

DEFINITION 4.1. A geodesic $\gamma(s)$ parametrized by arc-length s is called a *geodesic making constant angle with leaves* if the function f is a constant, that is, $df(s)/ds=0$ for any $s \in I_\gamma$.

THEOREM 4.1. *Let M be a foliated manifold with a foliation E of codimension q $(=n-p)$ and with a riemannian metric $\langle\ ,\ \rangle$. Suppose that all leaves are totally geodesic.*

(i) *If the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E, then any* *geodesic in M is a geodesic making constant angle with leaves.*

(ii) *If all geodesics in M are of making constant angle with leaves, then the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E.*

PROOF. (i) Let $\gamma(s)$ be a geodesic parametrized by arc-length s. In a flat coordinate chart $U(x^i, x^\alpha)$ such that $\gamma(s)\in U$ for any fixed $s\in I_\gamma$, by Theorem 3.1, we have an orthonormal adapted frame $\{X_i, X_\alpha\}$ to E satisfying $\langle\nabla_{X_\alpha}X_i, X_\beta\rangle+\langle\nabla_{X_\beta}X_i, X_\alpha\rangle=0$, that is,

$$\hat{\Gamma}^\beta_{\alpha i}+\hat{\Gamma}^\alpha_{\beta i}=0 \tag{4.2}$$

where $\nabla_{X_A}X_B=\hat{\Gamma}^C_{AB}X_C$ $(A, B, C=1, 2, \cdots, p, p+1, \cdots, p+q)$. And, by the orthonormality of the frame, we have

$$\hat{\Gamma}^C_{AB}+\hat{\Gamma}^B_{AC}=0\,. \tag{4.3}$$

Then we have

$$df(s)/ds=\frac{d}{ds}\Big(\sum_{i=1}^p [\theta^i(\dot\gamma(s))]^2\Big)$$

$$=2\sum_{i=1}^p (\theta^i(\dot\gamma(s)))\frac{d}{ds}(\theta^i(\dot\gamma(s)))$$

and

$$\begin{aligned}0&=\theta^i(\nabla_{\dot\gamma(s)}\dot\gamma(s))\\&=\frac{d}{ds}(\theta^i(\dot\gamma(s)))\\&\quad+\hat{\Gamma}^i_{jk}\theta^j(\dot\gamma(s))\theta^k(\dot\gamma(s))+\hat{\Gamma}^i_{j\beta}\theta^j(\dot\gamma(s))\theta^\beta(\dot\gamma(s))\\&\quad+\hat{\Gamma}^i_{\beta j}\theta^\beta(\dot\gamma(s))\theta^j(\dot\gamma(s))+\hat{\Gamma}^i_{\alpha\beta}\theta^\alpha(\dot\gamma(s))\theta^\beta(\dot\gamma(s))\end{aligned}$$

where $\{\theta^i, \theta^\alpha\}$ denotes the dual frame of $\{X_i, X_\alpha\}$. Thus we have, by (4.2) and (4.3),

$$df(s)/ds=2\sum_{i,j,\beta}\hat{\Gamma}^\beta_{ji}\theta^i(\dot\gamma(s))\theta^j(\dot\gamma(s))\theta^\beta(\dot\gamma(s))\,.$$

Since all leaves are totally geodesic, we have $\hat{\Gamma}^\beta_{ji}=0$. Therefore we have $df(s)/ds=0$ for any $s\in I_\gamma$.

(ii) For any point $m\in M$, we take a flat coordinate chart $U(x^i, x^\alpha)$ at m and any geodesic $\gamma(s)$ through m making constant angle with leaves. By the method of Schmidt's orthonormalization, we may make the basic adapted frame $\{\partial/\partial x^i, v_\alpha\}$ to E into an adapted frame $\{\tilde{X}_i, \tilde{X}_\alpha\}$ to E such that $\{\tilde{X}_i\}$ is mutually orthonormal and $\tilde{X}_\alpha=v_\alpha$. We set $\nabla_{\tilde{X}_A}\tilde{X}_B=\tilde{\Gamma}^C_{AB}\tilde{X}_C$, we have

$$\tilde{\Gamma}^i_{Ak}+\tilde{\Gamma}^k_{Ai}=0\,,\qquad \tilde{\Gamma}^i_{j\beta}+\tilde{\Gamma}^\tau_{ji}g_{\beta\tau}=0\,,\qquad \tilde{\Gamma}^\beta_{ji}=0\,.$$

Thus we have

$$0=df(s)/ds$$
$$=-2\sum_{i,\alpha,\beta}\tilde{\Gamma}^{i}_{\alpha\beta}\tilde{\theta}^{i}(\dot{\gamma}(s))\tilde{\theta}^{\alpha}(\dot{\gamma}(s))\tilde{\theta}^{\beta}(\dot{\gamma}(s))$$

for any $s\in I_\gamma$, where $\{\tilde{\theta}^i, \tilde{\theta}^\alpha\}$ denotes the dual frame of $\{\tilde{X}_i, \tilde{X}_\alpha\}$. As the choice of a geodesic γ is arbitrary, we have, for each i, $\tilde{\Gamma}^i_{\alpha\beta}\tilde{\theta}^\alpha(\dot{\gamma}(s))\tilde{\theta}^\beta(\dot{\gamma}(s))=0$. We set $\dot{\gamma}(s)=f^i\tilde{X}_i+f^\alpha\tilde{X}_\alpha=f^i\tilde{X}_i+f^\alpha v_\alpha$. Then we have

$$\tilde{\Gamma}^i_{\alpha\beta}f^\alpha f^\beta=0 \quad (\tilde{\theta}^\alpha(\dot{\gamma}(s))=f^\alpha)\,. \tag{4.4}$$

Thus

$$\tilde{X}_i\langle f^\alpha\tilde{X}_\alpha, f^\beta\tilde{X}_\beta\rangle$$
$$=\langle\nabla_{f^\alpha\tilde{X}_\alpha}\tilde{X}_i, f^\beta\tilde{X}_\beta\rangle+\langle[\tilde{X}_i, f^\alpha\tilde{X}_\alpha], f^\beta\tilde{X}_\beta\rangle$$
$$+\langle f^\alpha\tilde{X}_\alpha, \nabla_{f^\beta\tilde{X}_\beta}\tilde{X}_i\rangle+\langle f^\alpha\tilde{X}_\alpha, [\tilde{X}_i, f^\beta\tilde{X}_\beta]\rangle$$
$$=2f^\alpha f^\beta\tilde{\Gamma}^\tau_{\alpha i}g_{\tau\beta}+2f^\alpha\tilde{X}_i(f^\beta)g_{\alpha\beta}\,.$$

Here we note that $[\tilde{X}_i, \tilde{X}_\alpha]\in\Gamma(E|_U)$.

On the other hand, we have

$$\tilde{X}_i\langle f^\alpha\tilde{X}_\alpha, f^\beta\tilde{X}_\beta\rangle=\tilde{X}_i(f^\alpha f^\beta g_{\alpha\beta})$$
$$=2f^\alpha\tilde{X}_i(f^\beta)g_{\alpha\beta}+f^\alpha f^\beta\tilde{X}_i(g_{\alpha\beta})\,.$$

Thus we have

$$2f^\alpha f^\beta\tilde{\Gamma}^\tau_{\alpha i}g_{\tau\beta}=f^\alpha f^\beta\tilde{X}_i(g_{\alpha\beta})\,.$$

Since $\langle\nabla_{\tilde{X}_\alpha}\tilde{X}_\beta, \tilde{X}_i\rangle+\langle\tilde{X}_\beta, \nabla_{\tilde{X}_\alpha}\tilde{X}_i\rangle=0$, that is, $\tilde{\Gamma}^i_{\alpha\beta}+\tilde{\Gamma}^\tau_{\alpha i}g_{\tau\beta}=0$, we have

$$f^\alpha f^\beta\tilde{X}_i(g_{\alpha\beta})=-2f^\alpha f^\beta\tilde{\Gamma}^i_{\alpha\beta}=0 \quad (\text{from } (4.4))\,.$$

As the choice of the geodesic γ is arbitrary, we have $\tilde{X}_i(g_{\alpha\beta})=0$. By the construction of $\tilde{X}_i$, we have $\tilde{X}_i=\sum_{k=1}^{i}h^k_i\partial/\partial x^k$ ($1\leqq\forall i\leqq p$, h^k_i are functions in U), and thus we have

$$\partial g_{\alpha\beta}/\partial x^i=0$$

for $1\leqq\forall i\leqq p$ and $p+1\leqq\forall\alpha, \forall\beta\leqq p+q$. Therefore the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E. Q.E.D.

The condition that all leaves are totally geodesic is necessary:

EXAMPLE 4.1. Let $\boldsymbol{R}^2$ be an $x-y$ plane with the flat metric. We set $M=\boldsymbol{R}^2-\{\text{the origin point}\}$, then M is considered a foliated manifold whose leaves are $L_r=\{(x, y)\in\boldsymbol{R}^2 \mid x^2+y^2=r^2\}$ for any $r>0$ and a metric $\langle\ ,\ \rangle$ on M is induced from the flat metric on $\boldsymbol{R}^2$. All leaves are not totally geodesic. A geodesic given by $y=\text{constant}=c$ is to be tangent to L_c at $(0, c)$ and make an angle of $\pi/3$ with the leaf L_{2c} at $(\sqrt{3}c, c)$.

For the geodesics orthogonal to the leaves, we may omit the condition that all leaves are totally geodesic.

THEOREM 4.2. *Let M be a foliated manifold with a foliation E of codimension q $(=n-p)$ and with a riemannian metric $\langle\ ,\ \rangle$.*

(i) (B.L. Reinhart [**24**]) *If the riemannian metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E, then any geodesic orthogonal to the leaf at some point on the geodesic is to be orthogonal to the leaves at all points on the geodesic.*

(ii) *If, for any point $m\in M$, all geodesics that are to be orthogonal to the leaf at m are to be orthogonal to the leaves at all points on the geodesics, then the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E.*

Theorem 4.2 (i) is a generalization of the corresponding results of Y. Muto [**17**], B. O'Neill [**21**] and S. Sasaki [**27**].

PROOF. We give a proof of (ii). For any point $m\in M$, we take a flat coordinate chart $U(x^i, x^\alpha)$ of the point m. Let $\gamma(s)$ be any geodesic through m orthogonal to the leaves. We take an adapted frame $\{\bar{X}_i, \bar{X}_\alpha\}$ to E such that $\bar{X}_i$ are mutually orthogonal and are given by the method of Schmidt's orthogonalization from $\partial/\partial x^i$, and $\bar{X}_\alpha=v_\alpha$. Let $\{\bar{\theta}^i, \bar{\theta}^\alpha\}$ denote the dual frame of $\{\bar{X}_i, \bar{X}_\alpha\}$. For each i, we have

$$\begin{aligned}0&=\bar{\theta}^i(\nabla_{\dot{\gamma}(s)}\dot{\gamma}(s))\\&=\frac{d}{ds}(\bar{\theta}^i(\dot{\gamma}(s)))\\&\quad+\bar{\Gamma}^i_{jk}\bar{\theta}^j(\dot{\gamma}(s))\bar{\theta}^k(\dot{\gamma}(s))+\bar{\Gamma}^i_{j\beta}\bar{\theta}^j(\dot{\gamma}(s))\bar{\theta}^\beta(\dot{\gamma}(s))\\&\quad+\bar{\Gamma}^i_{\beta j}\bar{\theta}^\beta(\dot{\gamma}(s))\bar{\theta}^j(\dot{\gamma}(s))+\bar{\Gamma}^i_{\alpha\beta}\bar{\theta}^\alpha(\dot{\gamma}(s))\bar{\theta}^\beta(\dot{\gamma}(s))\\&=\bar{\Gamma}^i_{\alpha\beta}\bar{\theta}^\alpha(\dot{\gamma}(s))\bar{\theta}^\beta(\dot{\gamma}(s))\,.\end{aligned}$$

By the same way as the proof of Theorem 4.1 (ii), we have that the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E. Q.E.D.

Theorems 4.1 and 4.2 are generalizations of [**14**].

DEFINITION 4.2. A geodesic γ on M is called a *transversal geodesic* if γ is to be orthogonal to the leaves at all points on γ.

Even if M admits only one transversal geodesic, then the metric $\langle\ ,\ \rangle$ on M is not necessarily a bundle-like metric with respect to the foliation:

EXAMPLE 4.2. Let $\boldsymbol{R}^2$ be a $u-v$ plane with the flat metric $\langle\ ,\ \rangle$. $\boldsymbol{R}^2$ is a foliated manifold whose leaves are given by $\{(u, v)\in\boldsymbol{R}^2 \mid v=u^2+a\}$ for any $a\in\boldsymbol{R}$. A geodesic given by $u=0$ is only one transversal geodesic. We set

$$f(u)=\frac{1}{2}(2u(4u^2+1)^{1/2}+\log(2u+(4u^2+1)^{1/2}))$$

$$x=f(u)\,,\qquad y=v-u^2\,.$$

Setting $w=dx+2u(4u^2+1)^{-1/2}dy$, we have $\langle\ ,\ \rangle=du\cdot du+dv\cdot dv=w\cdot w+(4u^2+1)^{-1}dy\cdot dy=w\cdot w+(4(f^{-1}(x))^2+1)^{-1}dy\cdot dy$. Thus the metric $\langle\ ,\ \rangle$ is not a bundle-like metric with respect to the foliation.

5. Focal point of a leaf.

We recall that the bundle-like metric $\langle\ ,\ \rangle$ on M is locally expressed by

$$\langle\ ,\ \rangle|_U=g_{ij}(x^k, x^\tau)w^i\cdot w^j+g_{\alpha\beta}(x^\tau)dx^\alpha\cdot dx^\beta$$

in each flat coordinate chart $U(x^i, x^\alpha)$. Here and hereafter, vector fields, forms, tensor fields etc. are locally expressed by the basic adapted frame $\{\partial/\partial x^i, v_\alpha\}$ to E and its dual $\{w^i, dx^\alpha\}$, where $w^i=dx^i+A^i_\alpha dx^\alpha$ and $v_\alpha=\partial/\partial x^\alpha-A^i_\alpha\partial/\partial x^i$. We set, in U,

$$\nabla_{\partial/\partial x^i}\partial/\partial x^j=\Gamma^k_{ij}\partial/\partial x^k+\Gamma^\tau_{ij}v_\tau$$

$$\nabla_{\partial/\partial x^i}v_\beta=\Gamma^k_{i\beta}\partial/\partial x^k+\Gamma^\tau_{i\beta}v_\tau$$

$$\nabla_{v_\alpha}\partial/\partial x^j=\Gamma^k_{\alpha j}\partial/\partial x^k+\Gamma^\tau_{\alpha j}v_\tau$$

$$\nabla_{v_\alpha}v_\beta=\Gamma^k_{\alpha\beta}\partial/\partial x^k+\Gamma^\tau_{\alpha\beta}v_\tau$$

and

$$\begin{aligned}[v_\alpha, v_\beta]&=(\partial A^i_\alpha/\partial x^\beta-\partial A^i_\beta/\partial x^\alpha+A^j_\alpha\partial A^i_\beta/\partial x^j-A^j_\beta\partial A^i_\alpha/\partial x^j)\partial/\partial x^i\\&=B^i_{\alpha\beta}\partial/\partial x^i\,.\end{aligned}$$

LEMMA 5.1. *Suppose that the metric $\langle\ ,\ \rangle$ is a bundle-like metric with respect to E, then*

$$\Gamma^k_{ij}=\frac{1}{2}g^{kh}(\partial g_{hj}/\partial x^i+\partial g_{ih}/\partial x^j-\partial g_{ij}/\partial x^h)$$

$$\Gamma^\tau_{ij}=\frac{1}{2}g^{\tau\varepsilon}(g_{hj}\partial A^h_\varepsilon/\partial x^i+g_{ih}\partial A^h_\varepsilon/\partial x^j-v_\varepsilon(g_{ij}))$$

$$\Gamma^k_{\alpha j}=\frac{1}{2}g^{kh}(v_\alpha(g_{hj})+g_{hl}\partial A^l_\alpha/\partial x^j-g_{jl}\partial A^l_\alpha/\partial x^h)$$

$$\Gamma^k_{j\alpha}=\Gamma^k_{\alpha j}-\partial A^k_\alpha/\partial x^j \qquad \Gamma^k_{\alpha\beta}=-\Gamma^k_{\beta\alpha}=\frac{1}{2}B^k_{\alpha\beta}$$

$$\Gamma^\tau_{\alpha j}=\Gamma^\tau_{j\alpha}=-\frac{1}{2}g^{\tau\varepsilon}B^h_{\alpha\varepsilon}g_{hj}$$

$$\Gamma^\tau_{\alpha\beta}=\frac{1}{2}g^{\tau\varepsilon}(\partial g_{\varepsilon\beta}/\partial x^\alpha+\partial g_{\alpha\varepsilon}/\partial x^\beta-\partial g_{\alpha\beta}/\partial x^\varepsilon)\,.$$

By the decomposition (3.2), $TM\cong E\oplus E^\perp$, any $Y\in\Gamma(TM)$ is decomposed as $Y=Y_E+Y_{E^\perp}$, where Y_E (resp. $Y_{E^\perp}$) denotes a $\Gamma(E)$- (resp. $\Gamma(E^\perp)$-) component of

Y. In a flat coordinate chart $U(x^i, x^\alpha)$, Y_E and $Y_{E^\perp}$ are locally expressed by $Y_E = Y^i\partial/\partial x^i$ and $Y_{E^\perp} = Y^\alpha v_\alpha$ respectively.

Let $\gamma(t)$ be a transversal geodesic in M parametrized proportionally to arc-length, then, setting $\dot{\gamma}(t) = X^\alpha v_\alpha$ in U, we have

$$X^\alpha v_\alpha(X^\tau) + X^\alpha X^\beta \Gamma^\tau_{\alpha\beta} = 0 \qquad (p+1 \leqq \forall\tau \leqq p+q). \tag{5.1}$$

According to B. O'Neill [**21**], we have

DEFINITION 5.1. If $Y(t) = Y = Y_E + Y_{E^\perp}$ is a vector field along a transversal geodesic $\gamma(t)$ in M, then

$$\hat{Y}(t) = \hat{Y} = (\nabla_{\dot{\gamma}(t)} Y_E)_E - (\nabla_{Y_E}\dot{\gamma}(t))_E + 2(\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_E$$

is called the *derived vector field of* Y, and $\hat{Y}(t) \in \Gamma(E|_{\gamma(t)})$.

Hereafter, we assume that M has a bundle-like metric $\langle\ ,\ \rangle$ with respect to E.

PROPOSITION 5.1. *For a vector field Y along a transversal geodesic $\gamma(t)$ in M, it holds that*

$$(\nabla_{\dot{\gamma}(t)}\nabla_{\dot{\gamma}(t)} Y + R(Y, \dot{\gamma}(t))\dot{\gamma}(t))_E = (\nabla_{\dot{\gamma}(t)}\hat{Y})_E + (\nabla_{\hat{Y}}\dot{\gamma}(t))_E, \tag{5.2}$$

$$\begin{aligned}(\nabla_{\dot{\gamma}(t)}\nabla_{\dot{\gamma}(t)} Y &+ R(Y, \dot{\gamma}(t))\dot{\gamma}(t))_{E^\perp} \\ &= (\nabla_{\dot{\gamma}(t)}(\nabla_{\dot{\gamma}(t)} Y_{E^\perp})_{E^\perp})_{E^\perp} - (\nabla_{\dot{\gamma}(t)}(\nabla_{Y_{E^\perp}}\dot{\gamma}(t))_{E^\perp})_{E^\perp} \\ &\quad - (\nabla_{([Y_{E^\perp}, \dot{\gamma}(t)])_{E^\perp}}\dot{\gamma}(t))_{E^\perp} + 2(\nabla_{\dot{\gamma}(t)}\hat{Y})_{E^\perp}\end{aligned} \tag{5.3}$$

where R denotes the curvature tensor of ∇, that is, $R(V, W)Z = \nabla_V\nabla_W Z - \nabla_W\nabla_V Z - \nabla_{[V,W]}Z$.

This is proved by the direct calculation, taking notice of Lemma 5.1 and (5.1).

Let $\gamma: [0, 1] \to M$ be a transversal geodesic in M parametrized proportionally to arc-length. Let $L_{\gamma(t)}$ denote the leaf through a point $\gamma(t)$ and $T_{\gamma(t)}L$ the tangent space to $L_{\gamma(t)}$ at $\gamma(t)$.

A linear space $\mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ (resp. $\mathcal{E}(L_{\gamma(0)}, \gamma(1))$) consists of piece-wise differentiable vector fields $Y(t)$ along $\gamma(t)$ orthogonal to $\gamma(t)$ satisfying $Y(0) \in T_{\gamma(0)}L$ and $Y(1) \in T_{\gamma(1)}L$ (resp. $Y(1) = 0$). Then the index form I on $\mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ is given by

$$\begin{aligned}I(Y, Z) = \frac{1}{L(\gamma)}\Big[&-\int_0^1 \langle \nabla_{\dot{\gamma}(t)}\nabla_{\dot{\gamma}(t)} Y + R(Y, \dot{\gamma}(t))\dot{\gamma}(t), Z\rangle dt \\ &+ \langle \nabla_{\dot{\gamma}(t)} Y - S_{\dot{\gamma}(t)} Y, Z\rangle|_0^1 \\ &+ \sum_{i=1}^{k-1} \langle (\nabla_{\dot{\gamma}(t)} Y)(t_i^-) - (\nabla_{\dot{\gamma}(t)} Y)(t_i^+), Z(t_i)\rangle \Big],\end{aligned}$$

where $L(\gamma)$ denotes the length of γ, S denotes the second fundamental form: $\langle S_{\dot{\gamma}(t)} Y, Z\rangle = -\langle \nabla_Y Z, \dot{\gamma}(t)\rangle$, $0 < t_1 < t_2 < \cdots < t_{k-1} < 1$ are points where Y is not differentiable, and $(\nabla_{\dot{\gamma}(t)} Y)(t_i^-)$ (resp. $(\nabla_{\dot{\gamma}(t)} Y)(t_i^+)$) denotes the left (resp. right) limit of $\nabla_{\dot{\gamma}(t)}Y$ at t_i (See [15], [18], [21]).

The following lemmas are easily proved.

LEMMA 5.2. *Let Y be a vector field along a transversal geodesic $\gamma(t)$ in M. If $Y(0)\in T_{\gamma(0)}L$, then*

$$\hat{Y}(0)=(\nabla_{\dot{\gamma}(t)}Y_E)_E(0)-(\nabla_{Y_E}\dot{\gamma}(t))_E(0),$$

$$(\nabla_{\dot{\gamma}(t)}Y)_E(0)=(\nabla_{\dot{\gamma}(t)}Y_E)_E(0),$$

$$S_{\dot{\gamma}(t)}Y(0)=(\nabla_{Y_E}\dot{\gamma}(t))_E(0).$$

LEMMA 5.3. *Let Y be a piece-wise differentiable vector field along a transversal geodesic $\gamma(t)$ in M and $0<t_1<t_2<\cdots<t_{k-1}<1$ broken points of Y. Then, for each i $(1\leqq i\leqq k-1)$,*

$$(\nabla_{\dot{\gamma}(t)}Y)(t_i^-)-(\nabla_{\dot{\gamma}(t)}Y)(t_i^+)=\hat{Y}(t_i^-)-\hat{Y}(t_i^+)+(\nabla_{\dot{\gamma}(t)}Y_{E^\perp})_{E^\perp}(t_i^-)-(\nabla_{\dot{\gamma}(t)}Y_{E^\perp})_{E^\perp}(t_i^+).$$

From the above two lemmas, the index form I on $\mathcal{E}(L_{\gamma(0)}, L_{\gamma(1)})$ is rewritten:

$$I(Y, Z)=\frac{1}{L(\gamma)}\Big[-\int_0^1\langle\nabla_{\dot{\gamma}(t)}\nabla_{\dot{\gamma}(t)}Y+R(Y, \dot{\gamma}(t))\dot{\gamma}(t), Z\rangle dt +\langle\hat{Y}, Z\rangle\Big|_0^1+\sum_{i=1}^{k-1}\langle\Delta\hat{Y}(t_i), Z(t_i)\rangle +\sum_{i=1}^{k-1}\langle\Delta(\nabla_{\dot{\gamma}(t)}Y_{E^\perp})_{E^\perp}(t_i), Z_{E^\perp}(t_i)\rangle\Big], \tag{5.4}$$

where $\Delta\hat{Y}(t_i)=\hat{Y}(t_i^-)-\hat{Y}(t_i^+)$ and $\Delta(\nabla_{\dot{\gamma}(t)}Y_{E^\perp})_{E^\perp}(t_i)=(\nabla_{\dot{\gamma}(t)}Y_{E^\perp})_{E^\perp}(t_i^-)-(\nabla_{\dot{\gamma}(t)}Y_{E^\perp})_{E^\perp}(t_i^+)$.

DEFINITION 5.2. A vector field Y along a geodesic $\gamma(t)$ is called a *Jacobi field along γ* if Y satisfies the Jacobi equation: $\nabla_{\dot{\gamma}(t)}\nabla_{\dot{\gamma}(t)}Y+R(Y, \dot{\gamma}(t))\dot{\gamma}(t)=0$.

Let γ be a transversal geodesic in M parametrized proportionally to arc-length and $\mathcal{J}(\gamma)$ the linear space of all Jacobi fields along γ orthogonal to γ. Then we consider the following subspaces of $\mathcal{J}(\gamma)$:

$$\mathcal{J}_L(\gamma)=\{Y\in\mathcal{J}(\gamma);\ \hat{Y}=0\}$$

$$\mathcal{J}(\gamma;L)=\{Y\in\mathcal{J}(\gamma);\ Y(t)\in T_{\gamma(t)}L \quad \text{for any } t\in[0, 1]\}$$

$$\mathcal{J}(\gamma;L_{\gamma(0)}, L_{\gamma(1)})=\{Y\in\mathcal{J}(\gamma);\ Y(0)\in T_{\gamma(0)}L \text{ and } Y(1)\in T_{\gamma(1)}L\}$$

$$\mathcal{J}(\gamma;L_{\gamma(0)}, \gamma(1))=\{Y\in\mathcal{J}(\gamma);\ Y(0)\in T_{\gamma(0)}L \text{ and } Y(1)=0\}$$

$$\mathcal{J}_L(\gamma;L)=\mathcal{J}_L(\gamma)\cap\mathcal{J}(\gamma;L)$$

$$\mathcal{J}_L(\gamma;L_{\gamma(0)}, L_{\gamma(1)})=\mathcal{J}_L(\gamma)\cap\mathcal{J}(\gamma;L_{\gamma(0)}, L_{\gamma(1)})$$

$$\mathcal{J}_L(\gamma;L_{\gamma(0)}, \gamma(1))=\mathcal{J}_L(\gamma)\cap\mathcal{J}(\gamma;L_{\gamma(0)}, \gamma(1)).$$

LEMMA 5.4. *The space $\mathcal{J}_L(\gamma;L)$ consists of all solutions Y of*

$$\nabla_{\dot{\gamma}(t)}Y=(\nabla_{\dot{\gamma}(t)}Y_E)_{E^\perp}+(\nabla_{\dot{\gamma}(t)}Y_{E^\perp})_E+(\nabla_{Y_E}\dot{\gamma}(t))_E \tag{5.5}$$

on γ *such that* $Y(0)\in T_{\gamma(0)}L$. *Moreover* $\dim \mathcal{J}_L(\gamma\,;\,L)=p$.

PROOF. If $Y=Y_E+Y_{E^\perp}$ satisfies (5.5) and $Y_{E^\perp}(0)=0$, then we have $(\nabla_{\dot\gamma(t)}Y_{E^\perp})_{E^\perp}=0$, since $(\nabla_{\dot\gamma(t)}Y)_{E^\perp}=(\nabla_{\dot\gamma(t)}Y_E)_{E^\perp}+(\nabla_{\dot\gamma(t)}Y_{E^\perp})_{E^\perp}$. Then we have $Y_{E^\perp}=0$. Thus $Y=Y_E$. Since $(\nabla_{\dot\gamma(t)}Y_E)_E=(\nabla_{Y_E}\dot\gamma(t))_E$ and $\hat{Y}_E=(\nabla_{\dot\gamma(t)}Y_E)_E-(\nabla_{Y_E}\dot\gamma(t))_E$, we have $\hat{Y}=0$. By Proposition 5.1, we have $\nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)}Y+R(Y,\,\dot\gamma(t))\dot\gamma(t)=0$. And we have $\langle Y,\dot\gamma(t)\rangle=0$. Therefore $Y\in\mathcal{J}_L(\gamma\,;\,L)$.

Conversely, if $Y\in\mathcal{J}_L(\gamma\,;\,L)$, then it is trivial that Y satisfies (5.5).

And we easily have $\dim\mathcal{J}_L(\gamma\,;\,L)=p$. Q.E.D.

LEMMA 5.5. *Let* Y *be a Jacobi field along a transversal geodesic* γ. *If, for some* $t_1\in[0,\,1]$, $Y(t_1)=0$, *then* $Y\in\mathcal{J}_L(\gamma)$.

PROOF. From (5.2), Y satisfies $(\nabla_{\dot\gamma(t)}\hat{Y})_E+(\nabla_{\hat{Y}}\dot\gamma(t))_E=0$. Thus we have

$$\nabla_{\dot\gamma(t)}\hat{Y}=(\nabla_{\dot\gamma(t)}\hat{Y})_{E^\perp}-(\nabla_{\hat{Y}}\dot\gamma(t))_E\,,\qquad \hat{Y}(t_1)=0\,.$$

Then we have $\hat{Y}=0$. Therefore $Y\in\mathcal{J}_L(\gamma)$. Q.E.D.

Then we have (See [**15**], [**18**])

PROPOSITION 5.2. *A vector field* $Y\in\mathcal{E}(L_{\gamma(0)},\,L_{\gamma(1)})$ *belongs to* $\mathcal{J}_L(\gamma\,;\,L_{\gamma(0)},\,L_{\gamma(1)})$ *if and only if* $I(Y,\,Z)=0$ *for any* $Z\in\mathcal{E}(L_{\gamma(0)},\,L_{\gamma(1)})$.

By the same way, the nullspace of the index form I on $\mathcal{E}(L_{\gamma(0)},\,\gamma(1))$ is $\mathcal{J}_L(\gamma\,;\,L_{\gamma(0)},\,\gamma(1))$. Thus we have

DEFINITION 5.3. Let $\gamma(t)$ be a transversal geodesic in M parametrized proportionally to arc-length. A point $\gamma(1)$ is a *focal point of the leaf* $L_{\gamma(0)}$ *along* γ if there exists a non-zero vector field Y belonging to $\mathcal{J}_L(\gamma\,;\,L_{\gamma(0)},\,\gamma(1))$.

PROPOSITION 5.3. *Let* $Y=Y_E+Y_{E^\perp}$ *be a vector field along a transversal geodesic* γ *in* M. *If* $Y_E\in\mathcal{J}_L(\gamma\,;\,L_{\gamma(0)},\,\gamma(1))$, *then* $Y_E=0$.

DEFINITION 5.4. Let $\gamma(t)$ $(t\in[0,\,1])$ be a transversal geodesic in M and $\alpha\colon[0,\,1]\times(-\varepsilon,\,\varepsilon)\to M$ $(\varepsilon>0)$ a variation of γ, that is, $\alpha(t,\,0)=\gamma(t)$. The variation α of γ is a $(L_{\gamma(0)},\,L_{\gamma(1)})$-*geodesic variation of* γ if (i) for each $u\in(-\varepsilon,\,\varepsilon)$, a curve $\alpha_u(t)$ $(=\alpha(t,\,u))$ is a geodesic, and (ii) two curves $\alpha^0(u)=\alpha(0,\,u)$ and $\alpha^1(u)=\alpha(1,\,u)$ are in $L_{\gamma(0)}$ and $L_{\gamma(1)}$ respectively.

PROPOSITION 5.4. *Let* $\gamma(t)$ $(t\in[0,\,1])$ *be a transversal geodesic in* M *parametrized proportionally to arc-length and* $\alpha\colon[0,1]\times(-\varepsilon,\,\varepsilon)\to M$ $(\varepsilon>0)$ *a* $(L_{\gamma(0)},\,L_{\gamma(1)})$-*geodesic variation of* γ. *Then the variational vector field* $Y(t)=\alpha_*(\partial/\partial u)(t,\,0)$ *along* γ *belongs to* $\mathcal{J}_L(\gamma\,;\,L_{\gamma(0)},\,L_{\gamma(1)})$.

PROOF. We have that Y is a Jacobi field along γ and $\langle Y(t),\,\dot\gamma(t)\rangle=0$ for any $t\in[0,\,1]$. By Lemma 5.2 and $[Y,\,\dot\gamma(t)]|_{t=0}=0$, we have

$$\begin{aligned}\hat{Y}(0)&=(\nabla_{\dot\gamma(t)}Y_E)_E(0)-(\nabla_{Y_E}\dot\gamma(t))_E(0)\\&=([\dot\gamma(t),\,Y_E])_E(0)=([\dot\gamma(t),\,Y])_E(0)\\&=0\,.\end{aligned}$$

By Lemma 5.5, we have $Y\in\mathcal{J}_L(\gamma)$, and $Y\in\mathcal{J}_L(\gamma\,;\,L_{\gamma(0)},\,L_{\gamma(1)})$. Q.E.D.

The following proposition is easily proved.

PROPOSITION 5.5.

$$\mathcal{I}_L(\gamma;\, L_{\gamma(0)},\, \gamma(1)) \oplus \mathcal{I}_L(\gamma;\, L) \subset \mathcal{I}_L(\gamma;\, L_{\gamma(0)},\, L_{\gamma(1)})$$

and $\dim \mathcal{I}_L(\gamma;\, L_{\gamma(0)},\, \gamma(1)) \leqq q-1$, *where* $\oplus$ *denotes the direct sum.*

THEOREM 5.1. *Let M be a foliated manifold with a foliation E of codimension 1 and with a bundle-like metric with respect to E. For any point $m \in M$, there is not a focal point of the leaf L_m through m along every transversal geodesic γ starting from m.*

PROOF. By Proposition 5.5, we have $\dim \mathcal{I}_L(\gamma;\, L_{\gamma(0)},\, \gamma(1))=0$. Q.E.D.

EXAMPLE 5.1. Let $\boldsymbol{R}^3$ be the set of triple (x, y, z) of real numbers. $\boldsymbol{R}^3$ is considered a riemannian manifold with a riemannian metric $\langle\ ,\ \rangle = dx\cdot dx - 2z\,dx\cdot dy + (1+z^2)\,dy\cdot dy + dz\cdot dz$. Then $\boldsymbol{R}^3$ is considered a foliated manifold whose leaves are orbits of a vector field $\partial/\partial x$, and the metric is a bundle-like metric with respect to the foliation, that is, $\langle\ ,\ \rangle = w\cdot w + dy\cdot dy + dz\cdot dz$ where $w = dx - z\,dy$. For any point $(x_0, y_0, z_0) \in \boldsymbol{R}^3$, let γ be an arbitrary transversal geodesic starting from (x_0, y_0, z_0) and $L_{(x_0, y_0, z_0)}$ the leaf through the point (x_0, y_0, z_0). Then there is no focal point of the leaf $L_{(x_0, y_0, z_0)}$ along γ.

EXAMPLE 5.2. Let $\boldsymbol{R}^4$ be identified with the quaternion number field $\boldsymbol{Q}$, and let 3-dimensional sphere $S^3 \subset \boldsymbol{R}^4$ be a set $\{a \in \boldsymbol{Q} \mid \|a\| = 1\}$ where $\|a\|^2 = a\cdot\bar{a}$ and $\bar{a}$ denotes conjugate of a. For any $a \in S^3$, L_a denotes a set given by $\{(\cos\theta)\cdot a + (\sin\theta)\cdot(i\cdot a) \mid 0 \leqq \theta \leqq 2\pi\}$. Then S^3 is a foliated manifold by a family of the set L_a. The metric on S^3 induced from the flat metric on $\boldsymbol{R}^4$ is a bundle-like metric with respect to the foliation (See [**3**], [**12**], [**14**]). For any $a \in S^3$, let L_a be the leaf through a and $\gamma(s)$ a transversal geodesic parametrized by arc-length such that $\gamma(0) = a$. Then a point $\gamma(\pi/2)$ is a focal point of L_a along γ.

6. Clairaut's foliation.

The following Clairaut's theorem is a basic tool for studying geodesics on a surface of revolution.

CLAIRAUT'S THEOREM. *Let $\boldsymbol{r}$ be the distance to the axis of revolution, and let α be the angle between a geodesic and the meridians, viewed as a function of the parameter of the geodesic. Then $\boldsymbol{r}\sin\alpha = constant$.*

Then we have the following definition:

DEFINITION 6.1. Let M be a foliated manifold with a foliation E of codimension q and with a riemannian metric $\langle\ ,\ \rangle$. The foliation E is called the *Clairaut's foliation* if there exists a positive valued function $\boldsymbol{r}: M \to \boldsymbol{R}$ such that, for any geodesic $\gamma(t)$ parametrized proportionally to arc-length,

$$\boldsymbol{r}\sin\alpha = \text{constant}\,,$$

where $\alpha=\alpha(t)$ is defined by $\cos\alpha(t)=\|X_{E^\perp}(t)\|/\|X(t)\|$ $(0\leqq\alpha(t)\leqq\pi/2)$, $\dot\gamma(t)=X(t)=X_E(t)+X_{E^\perp}(t)$ and $\|X(t)\|=\langle X(t), X(t)\rangle^{1/2}$. The function $\boldsymbol{r}$ is called the *girth of* E (See [1], [**10**]).

Let $\gamma(t)$ be a geodesic in M parametrized proportionally to arc-length and $\dot\gamma(t)=X(t)=X_E+X_{E^\perp}$. Setting $\rho^2=\|X(t)\|^2=$constant, we have $\langle X_E, X_E\rangle=\rho^2\sin^2\alpha$ and $\langle X_{E^\perp}, X_{E^\perp}\rangle=\rho^2\cos^2\alpha$.

R. L. Bishop [**1**] defined and studied Clairaut submersions, and H. Kitahara [**10**] discussed the Clairaut's foliations of codimension 1. We will discuss the foliated manifold with a Clairaut's foliation E of codimension q and with a bundle-like metric with respect to E.

Hereafter, let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric with respect to E.

DEFINITION 6.2. A function f on M is called a *foliated function* if f is constant on each leaf of M.

PROPOSITION 6.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. If E is a Clairaut's foliation with the girth $\boldsymbol{r}=e^f$, where f is a function on M, then f is a foliated function on M.*

PROOF. Let $\gamma(t)$ be a geodesic parametrized proportionally to arc-length. By assumption, $\boldsymbol{r}\sin\alpha=$constant, thus we have

$$0=\frac{d}{dt}(\boldsymbol{r}\sin\alpha)=\boldsymbol{r}\frac{df}{dt}\sin\alpha+\boldsymbol{r}\cos\alpha\frac{d\alpha}{dt}.$$

Then we have

$$\begin{aligned}0&=\Big(\frac{d}{dt}(\boldsymbol{r}\sin\alpha)\Big)\rho^2\sin\alpha\\&=\boldsymbol{r}\frac{df}{dt}\langle X_E, X_E\rangle+\boldsymbol{r}\langle\nabla_{\dot\gamma(t)}X_E, X_E\rangle\\&=\boldsymbol{r}\langle\dot\gamma(t), \langle X_E, X_E\rangle\operatorname{grad}f\rangle+\boldsymbol{r}\langle\nabla_{\dot\gamma(t)}X_E, X_E\rangle,\end{aligned}$$

since $\dfrac{df}{dt}=\langle\dot\gamma(t), \operatorname{grad}f\rangle$ and $\dot\gamma(t)=X_E+X_{E^\perp}$. Thus we have $\langle\dot\gamma(t), \langle X_E, X_E\rangle\operatorname{grad}f\rangle=-\langle\nabla_{\dot\gamma(t)}X_E, X_E\rangle$. And we have

$$\langle\nabla_{\dot\gamma(t)}X_E, X_E\rangle=\langle X_{E^\perp}, \nabla_{\dot\gamma(t)}X_E\rangle, \tag{6.1}$$

$$\langle X_{E^\perp}, \nabla_{\dot\gamma(t)}X_E\rangle=\langle X_{E^\perp}, \nabla_{X_E}X_E\rangle. \tag{6.2}$$

Thus

$$\langle\dot\gamma(t), \langle X_E, X_E\rangle\operatorname{grad}f\rangle=-\langle X_{E^\perp}, \nabla_{X_E}X_E\rangle. \tag{6.3}$$

For any fixed point $m\in M$ and any non-zero vector $Y\in T_mL$, we take a geodesic $\gamma(t)$ such that $\gamma(0)=m$ and $\dot\gamma(0)=Y$. Then we have, by (6.3) at $t=0$,

$\langle Y, \langle Y, Y\rangle \operatorname{grad} f|_m\rangle=0$. Thus we have $\langle Y, \operatorname{grad} f|_m\rangle=0$.

Therefore, $\operatorname{grad} f$ is orthogonal to the leaf at each point, and f is a foliated function on M. Q.E.D.

DEFINITION 6.3. Let $\{X_i, X_\alpha\}$ be an orthonormal adapted frame to E. The *mean curvature vector* N_m *at* $m\in M$ *of the leaf* L_m is defined by

$$N_m=\frac{1}{n-q}\sum_{i,\alpha}\langle \nabla_{X_i}X_i|_m, X_\alpha|_m\rangle X_\alpha|_m .$$

DEFINITION 6.4. A leaf is called *totally umbilic* if, for each point m of the leaf, it holds

$$\langle X|_m, Y|_m\rangle N_m=(\nabla_X Y)_{E^\perp}|_m$$

for any $X, Y\in\Gamma(E|_U)$ (U: flat coordinate chart at m).

PROPOSITION 6.2. *Let M be a foliated manifold with a foliation E and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. If E is a Clairaut's foliation with the girth $\mathbf{r}=e^f$ where f is a function on M. Then the mean curvature vector N of each leaf is $-\operatorname{grad} f$.*

PROOF. For a geodesic $\gamma(t)$, $\dot\gamma(t)=X_E+X_{E^\perp}$, we have

$$\langle X_{E^\perp}, \langle X_E, X_E\rangle \operatorname{grad} f\rangle=-\langle X_{E^\perp}, \nabla_{X_E}X_E\rangle, \tag{6.4}$$

since $\operatorname{grad} f$ is orthogonal to each leaf and (6.3).

For any fixed point $m\in M$ and any non-zero vector $Y^\alpha X_\alpha|_m$ at m, we may take geodesics $\gamma_i(t)$ ($i=1, 2, \cdots, p$) such that $\gamma_i(0)=m$ and $\dot\gamma_i(0)=X_i|_m+Y^\alpha X_\alpha|_m$, where $\{X_i, X_\alpha\}$ is an orthonormal adapted frame to E. By (6.4), we have, for each i,

$$\langle Y^\alpha X_\alpha|_m, \operatorname{grad} f|_m\rangle=-\langle Y^\alpha X_\alpha|_m, \nabla_{X_i}X_i|_m\rangle .$$

And, for each i and α,

$$\langle X_\alpha|_m, \nabla_{X_i}X_i|_m\rangle=-\langle X_\alpha|_m, \operatorname{grad} f|_m\rangle .$$

Thus we have

$$\begin{aligned}\sum_{i,\alpha}\langle X_\alpha|_m, \nabla_{X_i}X_i|_m\rangle X_\alpha|_m &=-(n-q)\sum_\alpha\langle X_\alpha|_m, \operatorname{grad} f|_m\rangle X_\alpha|_m\\ &=-(n-q)\operatorname{grad} f|_m .\end{aligned}$$

Therefore, by the choice of m, we have $N=-\operatorname{grad} f$. Q.E.D.

THEOREM 6.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. Suppose that all leaves are totally umbilic and the mean curvature vector N of each leaf is $-\operatorname{grad} f$, where f is a function on M. Then E is a Clairaut's foliation with the girth $\mathbf{r}=e^f$.*

PROOF. Let $\gamma(t)$ be an arbitrary geodesic parametrized proportionally to arc-length and $\dot{\gamma}(t)=X=X_E+X_{E^\perp}$. We set

$$\rho=\|\dot{\gamma}(t)\| \ (=\text{constant}), \quad \cos\alpha=\|X_{E^\perp}\|/\|X\|, \quad \boldsymbol{r}=e^f .$$

We have

$$\begin{aligned}
&\langle\dot{\gamma}(t),\ \langle X_E,\ X_E\rangle \operatorname{grad} f\rangle \\
&=\langle X_{E^\perp},\ \langle X_E,\ X_E\rangle \operatorname{grad} f\rangle \\
&=-\langle X_{E^\perp},\ \langle X_E,\ X_E\rangle N\rangle \quad (\text{from that } N=-\operatorname{grad} f) \\
&=-\langle X_{E^\perp},\ \nabla_{X_E} X_E\rangle \quad (\text{from that all leaves are totally umbilic}) \\
&=-\langle X_{E^\perp},\ \nabla_{\dot{\gamma}(t)} X_E\rangle \quad (\text{from (6.2)}) \\
&=-\langle\nabla_{\dot{\gamma}(t)} X_E,\ X_E\rangle \quad (\text{from (6.1)}).
\end{aligned}$$

Thus

$$\langle X_E,\ X_E\rangle\langle\dot{\gamma}(t),\ \operatorname{grad} f\rangle+\langle\nabla_{\dot{\gamma}(t)} X_E,\ X_E\rangle=0 ,$$

that is,

$$2\frac{df}{dt}\rho^2\sin^2\alpha+\frac{d}{dt}(\rho^2\sin^2\alpha)=0 .$$

Then we have

$$2e^f\frac{df}{dt}\rho^2\sin^2\alpha+e^f\frac{d}{dt}(\rho^2\sin^2\alpha)=0 ,$$

and

$$2\rho^2\sin\alpha\Big(\frac{d\boldsymbol{r}}{dt}\sin\alpha+\boldsymbol{r}\frac{d}{dt}(\sin\alpha)\Big)=0 .$$

By the choice of γ, we have $d(\boldsymbol{r}\sin\alpha)/dt=0$. Therefore, E is a Clairaut's foliation with the girth $\boldsymbol{r}=e^f$. Q.E.D.

EXAMPLE 6.1. Let $\boldsymbol{R}^2$ be an $x-y$ plane with the flat metric $\langle\ ,\ \rangle$. We consider $\boldsymbol{R}^2-\{(0, 0)\}$ a foliated manifold whose leaves are sets $L_r=\{(x, y)\in\boldsymbol{R}^2 | x^2+y^2=r^2\}$ $(r>0)$. The metric $\langle\ ,\ \rangle|_{\boldsymbol{R}^2-\{(0,0)\}}$ is a bundle-like metric with respect to the foliation. Then the foliation is a Clairaut's foliation with the girth $\boldsymbol{r}=(x^2+y^2)^{1/2}$.

7. Second connection.

I. Vaisman proved the following theorem:

THEOREM (I. Vaisman [**28**, **29**]). *Let M be a foliated manifold with a foliation E of codimension q and with a riemannian metric $\langle\ ,\ \rangle$. Then there exists a connection D uniquely defined by the conditions:*

(i) *If $Y\in\Gamma(E)$ (resp. $\Gamma(E^\perp)$), then $D_XY\in\Gamma(E)$ (resp. $\Gamma(E^\perp)$) for any $X\in\Gamma(TM)$.*

(ii) *If* $X, Y, Z \in \Gamma(E)$ (*or* $\Gamma(E^{\perp})$), *then* $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$.

(iii) $(T(X, Y))_E = 0$ *if at least one of the arguments is in* $\Gamma(E)$, *and* $(T(X, Y))_{E^{\perp}} = 0$ *if at least one of the arguments is in* $\Gamma(E^{\perp})$. *Here* T *denotes the torsion tensor of* D, *that is,* $T(X, Y) = D_X Y - D_Y X - [X, Y]$.

This is proved by similar way to prove the existence and uniqueness of the Levi-Civita connection on a manifold with a riemannian metric.

DEFINITION 7.1. The connection D of the above theorem is called the *second connection* on a foliated manifold.

The second connection is not metrical with respect to the riemannian metric and has non-zero torsion in general. The foliated manifolds with second connections are studied by H. Kitahara [**11**], H. Kitahara and S. Yorozu [**13**], I. Vaisman [**28**] and others.

Now, we have expressions of the second connection D and its torsion tensor T by using the basic adapted frame $\{\partial/\partial x^i, v_\alpha\}$ to E in a flat coordinate chart $U(x^i, x^\alpha)$.

LEMMA 7.1. *It holds that*

$$D_{\partial/\partial x^i}\partial/\partial x^j = \Gamma^k_{ij}\partial/\partial x^k \qquad D_{v_\alpha}\partial/\partial x^j = \Gamma^k_{\alpha j}\partial/\partial x^k$$

$$D_{\partial/\partial x^i} v_\beta = 0 \qquad D_{v_\alpha} v_\beta = \Gamma^\tau_{\alpha\beta} v_\tau\,,$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kh}(\partial g_{hj}/\partial x^i + \partial g_{ih}/\partial x^j - \partial g_{ij}/\partial x^h)$$

$$\Gamma^k_{\alpha j} = \partial A^k_\alpha/\partial x^j$$

$$\Gamma^\tau_{\alpha\beta} = \frac{1}{2} g^{\tau\varepsilon}(v_\alpha(g_{\varepsilon\beta}) + v_\beta(g_{\alpha\varepsilon}) - v_\varepsilon(g_{\alpha\beta}))\,.$$

Moreover

$$T(\partial/\partial x^i, \partial/\partial x^j) = 0 \qquad T(\partial/\partial x^i, v_\beta) = 0$$

$$T(v_\alpha, v_\beta) = (\partial A^k_\alpha/\partial x^\beta - \partial A^k_\beta/\partial x^\alpha + A^h_\alpha \partial A^k_\beta/\partial x^h - A^h_\beta \partial A^k_\alpha/\partial x^h)\partial/\partial x^k\,.$$

LEMMA 7.2. *It holds that*

$$(\partial/\partial x^i)\langle\partial/\partial x^j, \partial/\partial x^k\rangle = \langle D_{\partial/\partial x^i}\partial/\partial x^j, \partial/\partial x^k\rangle + \langle\partial/\partial x^j, D_{\partial/\partial x^i}\partial/\partial x^k\rangle$$

$$v_\alpha\langle v_\beta, v_\tau\rangle = \langle D_{v_\alpha} v_\beta, v_\tau\rangle + \langle v_\beta, D_{v_\alpha} v_\tau\rangle\,.$$

Moreover, if the metric $\langle\ ,\ \rangle$ *is a bundle-like metric with respect to* E, *then*

$$(\partial/\partial x^i)\langle v_\alpha, v_\beta\rangle = \langle D_{\partial/\partial x^i} v_\alpha, v_\beta\rangle + \langle v_\alpha, D_{\partial/\partial x^i} v_\beta\rangle = 0\,.$$

We discuss the relation between the second connection D and the Levi-Civita connection ∇.

PROPOSITION 7.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. If all leaves are totally geodesic and $E^{\perp}$ is integrable, then $\nabla=D$.*

PROOF. By the integrability of $E^{\perp}$, we have $[v_{\alpha}, v_{\beta}]=B^{i}_{\alpha\beta}\partial/\partial x^{i}=0$ (See section 5), that is, $B^{i}_{\alpha\beta}=0$ for every i, α, β. Then we have

$$\Gamma^{k}_{\alpha\beta}=\Gamma^{k}_{\beta\alpha}=\Gamma^{\tau}_{\alpha j}=\Gamma^{\tau}_{j\alpha}=0$$

by Lemma 5.1.

Since all leaves are totally geodesic, we have $\Gamma^{\tau}_{ij}=0$ and, by Lemma 5.1,

$$v_{\varepsilon}(g_{ij})=g_{kj}\partial A^{k}_{\varepsilon}/\partial x^{i}+g_{ik}\partial A^{k}_{\varepsilon}/\partial x^{j}\,. \tag{7.1}$$

Substituting above equality to the right side of the third equality in Lemma 5.1, we have $\Gamma^{k}_{\alpha j}=\partial A^{k}_{\alpha}/\partial x^{j}$. Thus we have $\Gamma^{k}_{j\alpha}=0$.

Therefore we have

$$\Gamma^{k}_{ij}=\boldsymbol{\Gamma}^{k}_{ij}\,,\qquad \Gamma^{k}_{\alpha j}=\boldsymbol{\Gamma}^{k}_{\alpha j}\,,\qquad \Gamma^{\tau}_{\alpha\beta}=\boldsymbol{\Gamma}^{\tau}_{\alpha\beta}$$

and others vanish. Q.E.D.

8. Geodesic with respect to the second connection.

Hereafter, M is a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E.

Let $\gamma(t)$ be a curve in M. Locally, $\gamma(t)$ is expressed by $\gamma(t)=(\gamma^{i}(t), \gamma^{\alpha}(t))$ in a flat coordinate chart $U(x^{i}, x^{\alpha})$, and

$$\begin{aligned}\dot\gamma(t)&=\dot\gamma^{i}(t)\partial/\partial x^{i}+\dot\gamma^{\alpha}(t)\partial/\partial x^{\alpha}\\&=(\dot\gamma^{i}(t)+A^{i}_{\alpha}\dot\gamma^{\alpha}(t))\partial/\partial x^{i}+\dot\gamma^{\alpha}(t)v_{\alpha}\,.\end{aligned}$$

DEFINITION 8.1. A curve $\gamma(t)$ in M is called a *D-geodesic* if $D_{\dot\gamma(t)}\dot\gamma(t)=0$. Such a parameter t is called a *D-affine parameter*.

REMARK. To distinguish a geodesic with respect to the Levi-Civita connection ∇ from a D-geodesic, we will use "∇-geodesic" instead of "geodesic with respect to ∇".

Let $\gamma(u)$ be a D-geodesic in M parametrized by a parameter $u=u(t)$ where t is a D-affine parameter. Then we have

$$D_{\gamma'(u)}\gamma'(u)=-((d^{2}u/dt^{2})/(du/dt)^{2})\gamma'(u)$$

where $\gamma'(u)=(d\gamma^{i}/du+A^{i}_{\alpha}d\gamma^{\alpha}/du)\partial/\partial x^{i}+(d\gamma^{\alpha}/du)v_{\alpha}$.

Now, let $\gamma(t)$ be a D-geodesic parametrized by a D-affine parameter t and $\dot\gamma(t)=X=X_{E}+X_{E^{\perp}}$. Let s be the arc-length along γ. Then we have $ds/dt=(\langle X_{E}, X_{E}\rangle+\langle X_{E^{\perp}}, X_{E^{\perp}}\rangle)^{1/2}$ and

(8.1) $$d^2s/dt^2=\frac{1}{2}(\langle X_E, X_E\rangle+\langle X_{E^\perp}, X_{E^\perp}\rangle)^{-1/2}[X(\langle X_E, X_E\rangle+\langle X_{E^\perp}, X_{E^\perp}\rangle)].$$

By Lemma 7.2, we have

$$\begin{aligned}
&X(\langle X_E, X_E\rangle+\langle X_{E^\perp}, X_{E^\perp}\rangle)\\
&=2\langle D_{X_E}X_E, X_E\rangle+X_{E^\perp}\langle X_E, X_E\rangle+2\langle D_{X_E}X_{E^\perp}, X_{E^\perp}\rangle+2\langle D_{X_{E^\perp}}X_{E^\perp}, X_{E^\perp}\rangle\\
&=2\langle D_X X, X_E\rangle+2\langle D_X X, X_{E^\perp}\rangle+X_{E^\perp}\langle X_E, X_E\rangle-2\langle D_{X_{E^\perp}}X_E, X_E\rangle\\
&=X_{E^\perp}\langle X_E, X_E\rangle-2\langle D_{X_{E^\perp}}X_E, X_E\rangle .
\end{aligned}$$

Thus, setting $X_E=X^i\partial/\partial x^i$ and $X_{E^\perp}=X^\alpha v_\alpha$, we have

(8.2) $$X(\langle X_E, X_E\rangle+\langle X_{E^\perp}, X_{E^\perp}\rangle)=X^\alpha X^j(X^i v_\alpha(g_{ij})-2X^k g_{ij}\partial A^i_\alpha/\partial x^k).$$

PROPOSITION 8.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric with respect to E. Suppose that all leaves are totally geodesic. Then the arc-length parameter is a D-affine parameter.*

PROOF. By the assumption, we have $v_\alpha(g_{ij})=g_{kj}\partial A^k_\alpha/\partial x^i+g_{ik}\partial A^k_\alpha/\partial x^j$ (See (7.1)). By (8.2), we have

$$\begin{aligned}
&X(\langle X_E, X_E\rangle+\langle X_{E^\perp}, X_{E^\perp}\rangle)\\
&=X^\alpha X^j(X^i g_{kj}\partial A^k_\alpha/\partial x^i+X^i g_{ik}\partial A^k_\alpha/\partial x^j-2X^k g_{ij}\partial A^i_\alpha/\partial x^k)\\
&=X^\alpha X^j(X^i g_{ik}\partial A^k_\alpha/\partial x^j-X^k g_{ij}\partial A^i_\alpha/\partial x^k)\\
&=X^\alpha X^j X^i g_{ik}\partial A^k_\alpha/\partial x^j-X^\alpha X^j X^k g_{ij}\partial A^i_\alpha/\partial x^k\\
&=0 .
\end{aligned}$$

Thus, by (8.1), $d^2s/dt^2=0$. Therefore we have $D_{\gamma'(s)}\gamma'(s)=0$ where $'$ denotes the derivative with respect to s. Q.E.D.

DEFINITION 8.2. A D-geodesic $\gamma(t)$ in M is called a *transversal D-geodesic* if $\dot\gamma(t)\in\Gamma(E^\perp|_{\gamma(t)})$ for every t.

The following theorem is easily proved.

THEOREM 8.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric with respect to E. A curve $\gamma(t)$ in M is a transversal D-geodesic if and only if $\gamma(t)$ is a transversal ∇-geodesic.*

9. Jacobi field with respect to the second connection.

Let M be as in the above section. We define a D-Jacobi field along a D-geodesic in M.

DEFINITION 9.1. Let $\gamma(t)$ be a D-geodesic in M. A vector field $Y=Y(t)$ along $\gamma(t)$ is called a *D-Jacobi field along $\gamma(t)$* if Y satisfies the Jacobi equation:

$$D_{\dot\gamma(t)}D_{\dot\gamma(t)}Y+R_D(Y,\dot\gamma(t))\dot\gamma(t)+D_{\dot\gamma(t)}(T(Y,\dot\gamma(t)))=0$$

where R_D denotes the curvature tensor of D and T denotes the torsion tensor of D (See [**15**]).

We notice that

$$(D_{\dot\gamma(t)}(T(Y,\dot\gamma(t)))_{E^\perp}=0 \tag{9.1}$$

by Lemma 7.1.

REMARK. We will use "∇-Jacobi field" and "∇-focal point" instead of "Jacobi field" and "focal point" in section 5, respectively.

DEFINITION 9.2. A vector field Y on M is called *transversal* if $Y\in\Gamma(E^\perp)$.

By Lemma 7.1 and (9.1), we have

LEMMA 9.1. *If Y is a transversal D-Jacobi field along a transversal D-geodesic $\gamma(t)$ in M, then*

$$D_{\dot\gamma(t)}D_{\dot\gamma(t)}Y+R_D(Y,\dot\gamma(t))\dot\gamma(t)=0\,.$$

Every transversal D-geodesic $\gamma(t)$ admits two D-Jacobi fields in a natural way. One is given by $\dot\gamma(t)$ and the other is given by $t\dot\gamma(t)$.

PROPOSITION 9.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric with respect to E. Then every D-Jacobi field $Y=Y(t)$ along a transversal D-geodesic $\gamma(t)$ in M is uniquely decomposed in the following form: $Y(t)=(at+b)\dot\gamma(t)+V(t)$, where a and b are real constants, and $V(t)$ is a D-Jacobi field along $\gamma(t)$ orthogonal to $\gamma(t)$.*

PROPOSITION 9.2. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric $\langle\ ,\ \rangle$ with respect to E. Let $\gamma(t)$ ($t\in[0,1]$) be a transversal D-geodesic in M and Y a transversal D-Jacobi field along $\gamma(t)$. If $\langle R_D(Y,\dot\gamma(t))\dot\gamma(t),Y\rangle\leqq 0$ and Y vanishes at two points $\gamma(0)$ and $\gamma(1)$, then Y vanishes identically.*

PROOF. We have

$$\begin{aligned}\frac{d}{dt}\langle D_{\dot\gamma(t)}Y,Y\rangle&=\langle D_{\dot\gamma(t)}D_{\dot\gamma(t)}Y,Y\rangle+\langle D_{\dot\gamma(t)}Y,D_{\dot\gamma(t)}Y\rangle\\&=-\langle R_D(Y,\dot\gamma(t))\dot\gamma(t),Y\rangle+\langle D_{\dot\gamma(t)}Y,D_{\dot\gamma(t)}Y\rangle\,,\end{aligned}$$

thus

$$\begin{aligned}&\int_0^1\{\langle D_{\dot\gamma(t)}Y,D_{\dot\gamma(t)}Y\rangle-\langle R_D(Y,\dot\gamma(t))\dot\gamma(t),Y\rangle\}\,dt\\&=\langle(D_{\dot\gamma(t)}Y)(1),Y(1)\rangle-\langle(D_{\dot\gamma(t)}Y)(0),Y(0)\rangle\\&=0\,.\end{aligned}$$

Since $\langle R_D(Y,\dot\gamma(t))\dot\gamma(t),Y\rangle\leqq 0$, we have $\langle D_{\dot\gamma(t)}Y,D_{\dot\gamma(t)}Y\rangle=0$ for any $t\in[0,1]$. Since Y vanishes at $\gamma(0)$, $D_{\dot\gamma(t)}Y=0$ implies $Y=0$ for any $t\in[0,1]$. Q.E.D.

Now we have the non-existence of ∇-focal points of each leaf under a certain condition of R_D.

For a point $m\in M$, a plane Π in the tangent space T_mM is called a *transversal plane* if Π is spanned by linearly independent vectors X_m, Y_m such that $X_m, Y_m\in E_m^\perp$ (that is, X_m and Y_m are transversal vectors). For each point $m\in M$ and each transversal plane Π in T_mM, the *transversal D-sectional curvature* $K(m, \Pi)$ is defined by

$$K(m, \Pi)=\frac{\langle R_D(X_m, Y_m)Y_m, X_m\rangle}{\langle X_m, X_m\rangle\langle Y_m, Y_m\rangle-\langle X_m, Y_m\rangle^2}$$

where X_m and Y_m are linearly independent vectors and span a transversal plane Π. If $K(m, \Pi)\leqq 0$ for each point $m\in M$ and for all transversal planes Π in T_mM, then M is called to have *non-positive transversal D-sectional curvature.*

THEOREM 9.1. *Let M be a foliated manifold with a foliation E of codimension q and with a bundle-like metric* $\langle\ ,\ \rangle$ *with respect to E. Suppose that M has non-positive transversal D-sectional curvature. Then, for any point* $m\in M$, *there is not a* ∇*-focal point of the leaf through m along every transversal* ∇*-geodesic starting from m.*

PROOF. Let $\gamma(t)$ ($t\in[0, 1]$) be a transversal ∇-geodesic starting from m. We assume that a point $\gamma(1)$ is a ∇-focal point of the leaf L_m through m along γ. That is, we assume that there exists a non-zero ∇-Jacobi field $Y\in\mathcal{T}_L(\gamma\,;L_{\gamma(0)},\gamma(1))$. Then we have $Y_{E^\perp}\neq 0$ by Proposition 5.3. Thus we have

$$\hat{Y}=0\,,\qquad Y(0)\in T_{\gamma(0)}L\,,\qquad Y(1)=0$$

and, by Proposition 5.1,

$$0=(\nabla_{\dot\gamma(t)}(\nabla_{\dot\gamma(t)}Y_{E^\perp})_{E^\perp})_{E^\perp}-(\nabla_{\dot\gamma(t)}(\nabla_{Y_{E^\perp}}\dot\gamma(t))_{E^\perp})_{E^\perp}-(\nabla_{([Y_{E^\perp},\dot\gamma(t)])_{E^\perp}}\dot\gamma(t))_{E^\perp}\,. \tag{9.2}$$

The transversal ∇-geodesic γ is also a transversal D-geodesic by Theorem 8.1. By Lemma 5.1 and Lemma 7.1, (9.2) implies

$$\begin{aligned}0&=D_{\dot\gamma(t)}D_{\dot\gamma(t)}Y_{E^\perp}-D_{\dot\gamma(t)}D_{Y_{E^\perp}}\dot\gamma(t)-D_{([Y_{E^\perp},\dot\gamma(t)])_{E^\perp}}\dot\gamma(t)\\&=D_{\dot\gamma(t)}D_{\dot\gamma(t)}Y_{E^\perp}+R_D(Y_{E^\perp},\dot\gamma(t))\dot\gamma(t)\,.\end{aligned}$$

Thus $Y_{E^\perp}$ is a transversal D-Jacobi field along γ and satisfies $Y_{E^\perp}(0)=Y_{E^\perp}(1)=0$. By $\langle R_D(Y_{E^\perp},\dot\gamma(t))\dot\gamma(t), Y_{E^\perp}\rangle\leqq 0$ and Proposition 9.2, we have $Y_{E^\perp}=0$. This is a contradiction. Q.E.D.

See Example 5.1.

ACKNOWLEDGEMENTS. The author expresses his hearty thanks to Professor Haruo Kitahara for his valuable advices and encouragements.

References

[1] R.L. Bishop, Clairaut submersions, Differential geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 21-31.

[2] P. Dombrowski, Jacobi fields, totally geodesic foliations and geodesic differential forms, Resultate der Math., **1** (1980), 156-194.

[3] R.H. Escobales, Jr., Riemannian submersions with totally geodesic fibers, J. Differential Geometry, **10** (1975), 253-276.

[4] R. Hermann, On the differential geometry of foliations, Ann. of Math., **72** (1960), 445-457.

[5] R. Hermann, Totally geodesic orbits of groups of isometries, Indag. Math., **24** (1962), 291-298.

[6] S. Ishihara and M. Konishi, Differential geometry of fibred spaces, Publications of the Study Group of Geometry, **8**, Tokyo, 1973.

[7] S. Kashiwabara, The structure of a Riemannian manifold admitting a parallel field of one-dimensional tangent vector subspaces, Tôhoku Math. J., **11** (1959), 327-350.

[8] H. Kitahara, The existence of complete bundle-like metrics, Ann. Sci. Kanazawa Univ., **9** (1972), 37-40.

[9] H. Kitahara, The existence of complete bundle-like metrics II, Ann. Sci. Kanazawa Univ., **10** (1973), 51-54.

[10] H. Kitahara, The completeness of a Clairaut's foliation, Ann. Sci. Kanazawa Univ., **11** (1974), 37-40.

[11] H. Kitahara, On a parametrix form in a certain V-submersion, Lecture Notes in Math., **792**, Springer-Verlag, 1980, 264-298.

[12] H. Kitahara and S. Yorozu, On the cohomology groups of a manifold with a non-integrable subbundle, Proc. Amer. Math. Soc., **56** (1976), 201-204.

[13] H. Kitahara and S. Yorozu, A formula for the normal part of the Laplace-Beltrami operator on the foliated manifold, Pacific J. Math., **69** (1977), 425-432.

[14] H. Kitahara and S. Yorozu, On some differential geometric characterizations of a bundle-like metric, Kōdai Math. J., **2** (1979), 130-138.

[15] S. Kobayashi and K. Nomizu, Foundations of differential geometry, II, Interscience, New York, 1969.

[16] R. Maltz, The nullity spaces of the curvature operator, Cahiers Topologie Géom. Différentielle, **8** (1966), 1-20.

[17] Y. Muto, On some properties of a fibred Riemannian manifold, Sci. Rep. Yokohama Nat. Univ. Sect. I, **1** (1952), 1-14.

[18] H. Nakagawa, Riemannian geometry in the large (in Japanese), Kaigai-boeki, Tokyo, 1977.

[19] A.M. Naveira, Variedades foliadas con metrica casifibrada, Collect. Math., **21** (1970), 41-97.

[20] B. O'Neill, The fundamental equations of a submersion, Michigan Math. J., **13** (1966), 459-469.

[21] B. O'Neill, Submersions and geodesics, Duke Math. J., **34** (1967), 363-373.

[22] J.S. Pasternack, Topological obstructions to integrability and the Riemannian geometry of smooth foliations, Thesis, Princeton Univ., 1970.

[23] J.S. Pasternack, Foliations and compact Lie group actions, Comment. Math. Helv., **46** (1971), 467-477.

[24] B.L. Reinhart, Foliated manifolds with bundle-like metrics, Ann. of Math., **69** (1959), 119-132.

[25] B.L. Reinhart, Closed metric foliations, Michigan Math. J., **8** (1961), 7-9.

[26] R. Sacksteder, Foliations and pseudogroups, Amer. J. Math., **87** (1965), 79-102.

[27] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tohoku Math. J., **10** (1958), 338-354.

[28] I. Vaisman, Variétés riemanniennes feuilletées, Czechoslovak Math. J., 21 (1971), 46-75.
[29] I. Vaisman, Cohomology and differentiable forms, Marcel Dekker, New York, 1973.

Shinsuke YOROZU
Department of Mathematics
College of Liberal Arts
Kanazawa University
1-1 Marunouchi, Kanazawa 920
Japan