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On the class of polar sets for a certain class of Lévy processes on the line

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Let X be a Lévy process (a process with stationary independent increments) on the line having the exponent Ψ and λ-capacity function C^{λ}. We assume that X satisfies the following conditions:

(A_1) *the λ-resolvent is absolutely continuous with respect to Lebesgue measure,*

(A_2) *every point is polar,*

(D_α) *for a fixed $\lambda>0$ there exist α $(1>\alpha>0)$, and a continuous function F on $(0, \infty)$ such that*

$$F(z)\asymp \mathcal{R}e([\lambda+\Psi(z)]^{-1}), \qquad z\to\infty,$$

and $z^{\alpha}F(z)$ is decreasing on $(0, \infty)$, and

(I) *for a fixed $\lambda>0$ there exists a constant $M>0$ such that*

$$\mathcal{R}e([\lambda+\Psi(2z)]^{-1})/\mathcal{R}e([\lambda+\Psi(z)]^{-1})\geqq M$$

for every $z>0$.

Throughout this article we use the notation $f(z)\prec g(z)$, $z\to a$, if $\limsup_{z\to a} f(z)/g(z)<\infty$ and $f(z)\asymp g(z)$, $z\to a$, if $f(z)\prec g(z)$, $z\to a$ and $g(z)\prec f(z)$, $z\to a$. We write $f(z)\ll g(z)$, $z\to a$, if $\lim_{z\to a} f(z)/g(z)=0$.

Then we have

THEOREM 1. *Suppose that X satisfies* (A_1), (A_2), (D_α) *and* (I). *Put*

$$\phi(x)=\int_0^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})\,dz, \qquad x>0.$$

Then $C^{\lambda}(K)=0$ if and only if $C^{\phi}(K)=0$, where $C^{\phi}(K)$ denotes the Frostman's ϕ-capacity of K.

For general class of Lévy processes, if, for $\lambda>0$ and $M_1>0$

$$(0.1) \qquad \mathcal{R}e([\lambda+\Psi_1(z)]^{-1})\leqq M_1\mathcal{R}e([\lambda+\Psi_2(z)]^{-1}) \qquad \text{for all } z,$$

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then

$$C_1^\lambda(K) \geqq M_2 C_2^\lambda(K) \qquad (M_2>0) \tag{0.2}$$

for all compact sets K, where C_i^λ, $i=1, 2$, are λ-capacity functions of X_i, $i=1, 2$, with the exponents Ψ_i, $i=1, 2$, respectively. See Hawkes [3] for general case and Orey [9] and Kanda [5] for a restricted class. Especially

$$\mathcal{R}e([\lambda+\Psi_1(z)]^{-1}) \asymp \mathcal{R}e([\lambda+\Psi_2(z)]^{-1}),\ z\to\infty \Longrightarrow P_{X_1}=P_{X_2}, \tag{0.3}$$

where P_{X_i}, $i=1, 2$, are the classes of essentially polar sets of X_i, $i=1, 2$, respectively, that is, $P_{X_i}=(A; C_i^\lambda(A)=0)$. In this article we improve the above as follows.

THEOREM 2. *Let X_i, $i=1, 2$, be Lévy processes on the line having exponents Ψ_i and λ-capacity function C_i^λ, $i=1, 2$, respectively. Assume that both X_i, $i=1, 2$, satisfy* (A_1), (A_2), (D_α) *and* (I). *Put*

$$\phi_i(x)=\int_0^{1/x} \mathcal{R}e([\lambda+\Psi_i(z)]^{-1})dz, \qquad x>0.$$

Then

i) *if*

$$\liminf_{x\to 0} \phi_1(x)/\phi_2(x)=0, \tag{0.4}$$

there exists a compact set K such that $C_1^\lambda(K)>0$ and $C_2^\lambda(K)=0$;

ii) *the following conditions are equivalent to each other:*

ii. 1) $\phi_1(x) \prec \phi_2(x)$, $x\to 0$;

ii. 2) *for each fixed $a>0$, there exists a positive constant M such that $C_2^\lambda(K) \leqq MC_1^\lambda(K)$ for every compact set K in the ball with radius a;*

ii. 3) $P_{X_1} \subset P_{X_2}$.

Especially

$$\phi_1(x) \asymp \phi_2(x),\ x\to 0 \qquad \textit{if and only if } P_{X_1}=P_{X_2}. \tag{0.5}$$

This is really an improvement of (0.3) within the restricted class. Indeed $\phi_1(x)\asymp\phi_2(x)$, $x\to 0$, if $\mathcal{R}e([\lambda+\Psi_1(z)]^{-1}) \asymp \mathcal{R}e([\lambda+\Psi_2(z)]^{-1})$, $z\to\infty$ and there exist examples of pairs of Lévy processes satisfying (A_1), (A_2), (D_α) and (I) for which

$$\mathcal{R}e([\lambda+\Psi_1(z)]^{-1}) \prec \mathcal{R}e([\lambda+\Psi_2(z)]^{-1}), \qquad z\to\infty, \tag{0.6$_1$}$$

$$\liminf_{z\to\infty} \mathcal{R}e([\lambda+\Psi_1(z)]^{-1})/\mathcal{R}e([\lambda+\Psi_2(z)]^{-1})=0, \tag{0.6$_2$}$$

but

$$\phi_1(x)\asymp\phi_2(x), \qquad x\to 0.$$

See Propositions 4.2 and 4.3 in § 4. Indeed, choosing the symmetric Cauchy process as X_2 (that is, $\Psi_2(z)=|z|$), we show in Proposition 4.3 that there exists a symmetric Lévy process X_1 with the exponent Ψ_1 for which (0.6) holds, but $P_{X_1}=P_{X_2}$. It might be interesting to note that such phenomenon never happens within the class of d-dimensional isotropic Lévy processes with density in case $d\geqq 3$. That is, within this class, $P_{X_1}=P_{X_2}$ for the d-dimensional isotropic Cauchy process X_2 if and only if $\Psi_1(z)\asymp|z|$, $z\to\infty$. See [4].

Further it would be worthwhile to recall Kesten's result [7];

$$\lim_{x\to 0}\int_0^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz=\infty\,,\quad \text{if and only if } (\mathrm{A}_2) \text{ holds under } (\mathrm{A}_1)\,.$$

(Kesten's result is a statement whether a point is attainable or not without the condition (A_1). So it is more general than the above.) Then it would be natural that the degree of divergence of functions such as ϕ and ϕ_i reflects the inclusion relation of the class of polar sets. In this respect we compare our result with examples in [6] which satisfy

$$\begin{aligned}&\liminf_{z\to\infty}\mathcal{R}e([\lambda+\Psi_1(z)]^{-1})/\mathcal{R}e([\lambda+\Psi_2(z)]^{-1})\\ &\qquad=\liminf_{z\to\infty}\mathcal{R}e([\lambda+\Psi_2(z)]^{-1})/\mathcal{R}e([\lambda+\Psi_1(z)]^{-1})=0\,.\end{aligned}$$

For the one example, $P_{X_1}\subsetneqq P_{X_2}$ and for the other, $P_{X_1}-P_{X_2}\neq\emptyset$, $P_{X_2}-P_{X_1}\neq\emptyset$. In that paper [6]

$$x^{-2}\int_0^\infty\mathcal{R}e([\lambda+\Psi(t)]^{-1})t^{-2}(1-\cos(xt))dt\equiv x^{-1}\langle 1/\Psi\rangle(x^{-1})$$

plays the similar role as does $\phi(x)=\int_0^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz$ in this article. ($x^{-1}\langle 1/\Psi\rangle(x^{-1})\asymp\phi(x)$, $x\to 0$, under the conditions (A_1), (A_2), (D_α) and (I).) The examples in [6] do not satisfy (D_α). Instead the weaker condition;

(0.7) $$\mathcal{R}e([\lambda+\Psi(z)]^{-1})\leqq Mz^{-\alpha}\qquad\text{for every } z>0\,,$$

holds. But $\langle 1/\Psi_1\rangle(x^{-1})\prec\langle 1/\Psi_2\rangle(x^{-1})$, $x\to 0$, for the example corresponding to $P_{X_1}\subsetneqq P_{X_2}$ and $\liminf_{x\to 0}\langle 1/\Psi_1\rangle(x^{-1})/\langle 1/\Psi_2\rangle(x^{-1})=\liminf_{x\to 0}\langle 1/\Psi_2\rangle(x^{-1})/\langle 1/\Psi_1\rangle(x^{-1})=0$ for the example corresponding to $P_{X_1}-P_{X_2}\neq\emptyset$, $P_{X_2}-P_{X_1}\neq\emptyset$. However it is open whether Theorems hold if we replace (D_α) with (0.7), even if we use $x^{-1}\langle 1/\Psi\rangle(x^{-1})$ instead of ϕ and ϕ_i. (The result in [6] is of a weaker form than the one in this article and is given only for symmetric case.)

As a direct consequence of Theorem 2 we get

COROLLARY. *If*

(0.8) $$\lim_{z\to\infty}\mathcal{R}e([\lambda+\Psi_1(z)]^{-1})/\mathcal{R}e([\lambda+\Psi_2(z)]^{-1})=0\,,$$

then $P_{X_1}\subsetneqq P_{X_2}$.

Our example shows that (0.8) cannot be replaced with (0.6).

Throughout this article *λ-capacity function* C^λ of a given Lévy process $X=(X_t, P_x)$ is the one defined as usual. For $\lambda>0$ and a Borel set A we define $\phi_A^\lambda(x)=E_x\exp(-\lambda F_A)$, where $F_A=\inf\{t>0, X_t\in A\}$ and E_x denotes the integral with respect to P_x. Then there exists a unique measure π_A^λ whose support is in the closure $\bar{A}$ of A such that $(f, \phi_A^\lambda)=(\tilde{U}^\lambda f, \pi_A^\lambda)$ for every bounded Borel function f, where $\tilde{U}^\lambda$ is the λ-resolvent of the dual process of X. The λ-capacity of A is defined by

$$C^\lambda(A)=\pi_A^\lambda(\bar{A}).$$

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1. In this section we prepare some preliminary notations and results for a Lévy process X having the exponent Ψ and λ-capacity function C^λ. The conditions (A_1) and (A_2) are always assumed without mentioning. Let $u^\lambda(x, y)$ be λ-resolvent density relative to Lebesgue measure. Put

$$u^\lambda(x)=u^\lambda(0, x), \qquad u_S^\lambda(x)=2^{-1}[u^\lambda(x)+u^\lambda(-x)],$$
(1.1)
$$[u](x)=|x|^{-1}\int_0^{|x|}u_S^\lambda(y)dy, \qquad U_S^\lambda\mu(x)=\int u_S^\lambda(y-x)\mu(dy).$$

The inequality

(1.2) $$\int_0^{x/2}u_S^\lambda(y)dy\leqq\int_0^x u_S^\lambda(y)dy\leqq5\int_0^{x/2}u_S^\lambda(y)dy$$

is proved by Kesten [7] Lemma 3.1 in more detailed form. Set

(1.3) $$\langle 1/\Psi\rangle(z)=|z|\int_0^\infty \mathcal{Re}([\lambda+\Psi(t)]^{-1})t^{-2}(1-\cos(t/|z|))dt.$$

Then we have

PROPOSITION 1.1.

$$\langle 1/\Psi\rangle(z)\asymp z^{-1}[u](z^{-1}), \qquad z\to\infty.$$

PROOF. Set

(1.4) $$E(\mu)=\iint u_S^\lambda(y-x)\mu(dx)\mu(dy), \qquad E(f)=E(f(x)dx).$$

Then it follows from (1.2) and the simple inequality

$$\int_0^{r/2}u_S^\lambda(y)dy\leqq\int_{-r/2}^{r/2}u_S^\lambda(x-y)dy\leqq2\int_0^r u_S^\lambda(y)dy$$

for every $x\in[-r/2, r/2]$ that $E(I_{[-r/2,r/2]})\asymp r^2[u](r)$, $r\to0$, where $I_A(x)=1$ if

$x \in A$, $=0$ if otherwise. On the other hand it holds that $E(I_{[-r/2,r/2]})=$ Const.$\langle 1/\Psi \rangle (r^{-1})r$ for every $r>0$, because the Fourier transform of u_S^λ is $\mathcal{R}e([\lambda+\Psi]^{-1})$. The proof is finished.

Throughout this article we use the notation:

$$\hat{\mu}(z)=\int_{-\infty}^{\infty} \exp(izy)\mu(dy)\,, \tag{1.5}$$
$$J_\Psi(\mu)=\int_{-\infty}^{\infty} |\hat{\mu}(z)|^2 \,\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\,.$$

Set $\mathrm{Pr}(A)=\{\mu$; *a probability measure whose support is contained in* $A\}$.

PROPOSITION 1.2 (Hawkes [3], Theorem 3.1 and Theorem 3.2). *If A is open,* $4^{-1}C^\lambda(A)^{-1} \leqq \inf\{(2\pi)^{-1}J_\Psi(\mu),\ \mu \in \mathrm{Pr}(A)\} \leqq C^\lambda(A)^{-1}$.

The next lemma may be known among those who are interested in this topic. But it does not seem to be explicitly written except the case every semi-polar set is polar.

LEMMA 1. *For a compact set K, $C^\lambda(K)>0$ if and only if $J_\Psi(\mu)<\infty$ for some $\mu \in \mathrm{Pr}(K)$.*

PROOF. Assume that $J_\Psi(\mu)<\infty$ for some $\mu \in \mathrm{Pr}(K)$. For each open neighborhood Q of K, $C^\lambda(Q)^{-1} \leqq 4\inf\{(2\pi)^{-1}J_\Psi(\nu);\ \nu \in \mathrm{Pr}(Q)\} \leqq 4(2\pi)^{-1}J_\Psi(\mu)=M_1<\infty$. Hence $C^\lambda(Q) \geqq M_1^{-1}$, and so $C^\lambda(K) \geqq M_1^{-1}$. Conversely, if $C^\lambda(K)>0$, there exists a capacitary measure π_K^λ. Set $\nu=\pi_K^\lambda/C^\lambda(K)$ and define $\tilde{\nu}$ by $\tilde{\nu}(A)=\nu(-A)$. Then $\nu \in \mathrm{Pr}(K)$ and $U_S^\lambda \nu * \tilde{\nu}$ is a bounded function of class L^1. Since $\widehat{U_S^\lambda \nu * \tilde{\nu}}(z)= \mathcal{R}e([\lambda+\Psi(z)]^{-1})|\hat{\nu}(z)|^2>0$, we see $J_\Psi(\nu) \in L^1$ by Theorem 2.2.1 in Bochner [1]. The proof is complete.

REMARK. For each probability measure μ

$$E(\mu) \leqq (2\pi)^{-1}J_\Psi(\mu)\,, \tag{1.6}$$

where $E(\mu)$ is defined by (1.4). If $J_\Psi(\mu)=\infty$, the inequality is obvious. If $J_\Psi(\mu)<\infty$, $U_S^\lambda \mu * \tilde{\mu}(x)=(2\pi)^{-1}\int \exp(-ixy)\,\mathcal{R}e([\lambda+\Psi(z)]^{-1})|\hat{\mu}(z)|^2 dz$ holds almost everywhere by Theorem 2.1.5 in Bochner [1]. Since $U_S^\lambda \mu * \tilde{\mu}$ is lower semicontinuous, we see that $E(\mu)=U_S^\lambda \mu * \mu(0) \leqq (2\pi)^{-1}J_\Psi(\mu)$.

2. In this section we give a lemma which plays a key role to the proof of Theorems. The positive constants which are independent of variables which appear in the following are denoted by $M_1, M_2, \cdots$.

LEMMA 2. *Let X be a Lévy process with the exponent Ψ satisfying* (A_1), (A_2), (D_α) *and* (I). *Then*

$$(2.1)\qquad (2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[u](x-y)\mu(dx)\mu(dy)$$

$$=\int_{-\infty}^{\infty}\left\{\int_{|z|}^{\infty}u^{-1}\,\mathcal{R}e([\lambda+\Psi(u)]^{-1})du\right\}|\hat{\mu}(z)|^2dz$$

for every probability measure μ *of compact support. Further*

$$(2.2)\qquad [u](x)\asymp\int_0^{1/|x|}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\,,\qquad x\to 0\,,$$

and

$$(2.3)\qquad \int_{|z|}^{\infty}u^{-1}\mathcal{R}e([\lambda+\Psi(u)]^{-1})du\asymp\mathcal{R}e([\lambda+\Psi(z)]^{-1})\,,\qquad z\to\infty\,.$$

PROOF. Set

$$\langle\!\langle 1/\Psi\rangle\!\rangle(z)=\int_{|z|}^{\infty}u^{-1}\,\mathcal{R}e([\lambda+\Psi(u)]^{-1})du\,.$$

We divide the proof into four steps.

Step 1. We first prove (2.3). Using (D_α) and (I),

$$\langle\!\langle 1/\Psi\rangle\!\rangle(z)\leqq M_1\int_{|z|}^{\infty}u^{-(1+\alpha)}u^{\alpha}F(u)du\leqq M_1\alpha^{-1}F(|z|)$$

$$\leqq M_2\,\mathcal{R}e([\lambda+\Psi(z)]^{-1})\,,$$

and

$$\langle\!\langle 1/\Psi\rangle\!\rangle(z)\geqq M_3\int_{|z|}^{|2z|}u^{-(1+\alpha)}u^{\alpha}F(u)du\geqq M_4F(|2z|)$$

$$\geqq M_5\,\mathcal{R}e([\lambda+\Psi(2z)]^{-1})\geqq MM_5\,\mathcal{R}e([\lambda+\Psi(z)]^{-1})$$

for every large $|z|$. We have proved (2.3). Further

$$(2.4)\qquad \lim_{z\to 0}|z|\langle\!\langle 1/\Psi\rangle\!\rangle(z)=0\,,$$

because $z\langle\!\langle 1/\Psi\rangle\!\rangle(z)\leqq z\int_z^1 u^{-1}du+M_6\int_1^{\infty}u^{-(1+\alpha)}du\cdot z\leqq z\log z+M_7z$ for $z>0$. In the next we show

$$(2.5)\qquad \left|\int_0^B\cos(xz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)\,dz\right|\leqq M_7\int_0^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz$$

for every B and every small $x>0$. Since $\langle\!\langle 1/\Psi\rangle\!\rangle(z)$ is continuous and monotone decreasing on $(0,\infty)$, it holds

$$\int_{1/x}^{B}\cos(xz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz=\langle\!\langle 1/\Psi\rangle\!\rangle(1/x)x^{-1}[\sin(x\xi)-\sin(1)]$$

by Bonnet's theorem, where ξ may depend on B and x, and for $\varepsilon>0$,

$$\int_{\varepsilon}^{1/x}\cos(xz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz=[x^{-1}\sin(xz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)]_{\varepsilon}^{1/x}$$

$$+x^{-1}\int_{\varepsilon}^{1/x}z^{-1}\sin(xz)\,\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz$$

by integration by parts. Letting $\varepsilon\downarrow 0$ and using (2.4), we have

$$\left|\int_{0}^{B}\cos(xz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\right|\leqq x^{-1}\langle\!\langle 1/\Psi\rangle\!\rangle(1/x)$$

$$+x^{-1}\int_{0}^{1/x}z^{-1}\sin(xz)\,\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\,.$$

On the other hand

$$\int_{0}^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\asymp x^{-1}\int_{0}^{1/x}z^{-1}\sin(xz)\,\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz,\ \ x\to 0\,,$$

and

$$\int_{0}^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})ds\geqq M_8\int_{1}^{1/x}z^{-\alpha}z^{\alpha}F(z)dz\geqq M_9x^{-1}F(1/x)$$

$$\geqq M_{10}x^{-1}\mathcal{R}e([\lambda+\Psi(1/x)]^{-1})$$

for every $x>0$. Combining the above estimate with (2.3) proven at the first step, we see that (2.5) holds.

Step 2. At this step we prove

$$\lim_{A\uparrow\infty}\int_{-A}^{A}\exp(ixz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz=(2\pi)[u](x) \tag{2.6}$$

locally uniformly on $R^1-\{0\}$. The limit indeed exists locally uniformly on $R^1-\{0\}$, because

$$\left|\int_{A}^{B}\cos(xz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\right|\leqq 2x^{-1}\langle\!\langle 1/\Psi\rangle\!\rangle(1/A)$$

for $0<A<B$ by Bonnet's theorem. We denote this limit by $\Phi(x)$. It is clear that Φ is continuous on $R^1-\{0\}$ and

$$\int_{-\infty}^{\infty}f(x)\Phi(x)dx=\lim_{A\uparrow\infty}\int_{-A}^{A}\hat{f}(z)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz \tag{2.7}$$

$$=\int_{-\infty}^{\infty}\hat{f}(z)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz$$

for each C^{∞}-function f of compact support in $R^1-\{0\}$. We next show

(2.8) $$\int_{-\infty}^{\infty} f(x)[u](x)dx=(2\pi)^{-1}\int_{-\infty}^{\infty}\hat{f}(z)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz$$

for every nonnegative C^{∞}-function of compact support. Note that

$$\int_{-\infty}^{\infty}\hat{f}(z)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz=\int_{0}^{\infty}\hat{f}(z)\left[\int_{1}^{\infty}t^{-1}\,\mathcal{R}e([\lambda+\Psi(zt)]^{-1})dt\right]dz+\int_{-\infty}^{0}[\,''\,]dz\,,$$

and $\mathcal{R}e([\lambda+\Psi(z)]^{-1})\leq M_{11}z^{-\alpha}$ for every $z>0$ by (D_{α}). Since

$$\int_{1}^{\infty}t^{-1}\int_{0}^{\infty}|\hat{f}(z)|\,\mathcal{R}e([\lambda+\Psi(zt)]^{-1})dzdt<M_{11}\int_{1}^{\infty}t^{-1-\alpha}\int_{0}^{\infty}|\hat{f}(z)|z^{-\alpha}dzdt<\infty$$

we have

$$\int_{0}^{\infty}\hat{f}(z)\left[\int_{1}^{\infty}t^{-1}\,\mathcal{R}e([\lambda+\Psi(zt)]^{-1})dt\right]dz=\int_{1}^{\infty}t^{-1}\left[\int_{0}^{\infty}\hat{f}(z)\,\mathcal{R}e([\lambda+\Psi(zt)]^{-1})dz\right]dt$$

by Fubini's theorem. Consequently

$$\int_{-\infty}^{\infty}\hat{f}(z)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz=\int_{1}^{\infty}t^{-1}\left[\int_{-\infty}^{\infty}\hat{f}(z)\,\mathcal{R}e([\lambda+\Psi(zt)]^{-1})dz\right]dt\,.$$

On the other hand, noting

$$\int_{-\infty}^{\infty}\exp(izx)\left[\int_{-\infty}^{\infty}u_{S}^{\lambda}((y-x)/t)f(y)dy\right]dx=t\,\mathcal{R}e([\lambda+\Psi(zt)]^{-1})\hat{f}(z)\in L^{1}\,,$$

it follows from Theorem 2.1.5, Bochner [**1**] that

$$\int_{-\infty}^{\infty}u_{S}^{\lambda}((y-x)/t)f(y)dy=t(2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-ixz)\hat{f}(z)\,\mathcal{R}e([\lambda+\Psi(zt)]^{-1})dz$$

almost everywhere. As the both terms are continuous functions of x, the above equality holds everywhere. Hence

$$\int_{-\infty}^{\infty}\hat{f}(z)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz=(2\pi)\int_{1}^{\infty}t^{-2}\left[\int_{-\infty}^{\infty}u_{S}^{\lambda}(y/t)f(y)dy\right]dt\,.$$

We have proved (2.8) now, because the right term equals to $(2\pi)\int_{-\infty}^{\infty}f(y)[u](y)dy$. As a corollary of (2.8) we see that $[u]\in L^{1}_{\mathrm{loc}}$. Combining (2.7) with (2.8), we see $\Phi(x)=(2\pi)[u](x)$ almost everywhere on $R^{1}-\{0\}$, which shows the equality holds everywhere, because the both terms are continuous on $R^{1}-\{0\}$. The proof of (2.6) is finished.

Step 3. Now we show (2.2). It follows from (2.6) and (2.5) that

$$[u](x)\leq M_{12}\int_{0}^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\,.$$

On the other hand, it follows from Proposition 1.1 that, for $x>0$,

$$[u](x)\geqq M_{13}x^{-2}\int_0^\infty \mathcal{R}e([\lambda+\Psi(t)]^{-1})t^{-2}(1-\cos(tx))dt$$

$$\geqq M_{13}x^{-2}\int_0^{1/x}(\prime\prime)dt\geqq M_{14}\int_0^{1/x}\mathcal{R}e([\lambda+\Psi(t)]^{-1})dt\,.$$

Step 4. Only the proof of (2.1) remains. We give it at this step. First we show

(2.9) $$\int_{-\infty}^\infty |\hat{\mu}(z)|^2\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz<\infty \quad \text{if and only if}$$

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \mu(dx)[u](x-y)\mu(dy)<\infty$$

for a given probability measure μ of compact support. It follows from (2.2) and (2.5) that

$$\int_{-A}^A \exp(i(x-y,\ z))\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\leqq M_{15}[u](x-y)$$

for every $A>0$. Therefore

$$\int_{-A}^A |\hat{\mu}(z)|^2\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\leqq M_{15}\iint \mu(dx)[u](x-y)\mu(dy)\,.$$

The "*if*" part of (2.9) is proved. Suppose in the next $\int_{-\infty}^\infty |\hat{\mu}(z)|^2\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz<\infty$. Define $\tilde{\mu}$ by $\tilde{\mu}(A)=\mu(-A)$ and put $\mu_S=\mu*\tilde{\mu}$. Since it follows from (2.8) that $\int_{-\infty}^\infty f(x)[u](x-y)dx=(2\pi)^{-1}\int_{-\infty}^\infty \exp(-iyz)\hat{f}(z)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz$ for every non-negative C^∞-function of compact support, we have $\int_{-\infty}^\infty f(x)[u]*\mu_S(x)dx=(2\pi)^{-1}\int_{-\infty}^\infty \hat{f}(z)|\hat{\mu}(z)|^2\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz$ and so

$$[u]*\mu_S(x)=(2\pi)^{-1}\int_{-\infty}^\infty \exp(ixz)|\hat{\mu}(z)|^2\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz$$

almost everywhere. Hence $[u]*\mu_S(0)\leqq(2\pi)^{-1}\int_{-\infty}^\infty |\hat{\mu}(z)|^2\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz$ by the lower semicontinuity of $[u]*\mu_S$. We have finished the proof of (2.9). For the proof of (2.1) it is sufficient to consider the case both terms are finite. Such a measure μ has no point mass. Indeed $J_\Psi(\mu)$ is finite by (2.3), and so it follows from Lemma 1 that a point is non-polar if μ has point mass. Now choose a sequence of open neighborhood U_n of $\{0\}$ such that $U_n\downarrow 0$. Then

$$\limsup_{n\uparrow\infty}\limsup_{A\uparrow\infty}\left\{\int_{U_n}\mu_S(dx)\left[\int_{-A}^{A}\exp(ixz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\right]\right\}=0\,. \tag{2.10}$$

Indeed it follows from (2.5) and (2.2) that

$$\left|\int_{U_n}\mu_S(dx)\left[\int_{-A}^{A}\exp(ixz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\right]\right|\leqq M_{16}\int_{U_n}\mu_S(dx)[u](x)\,,$$

and μ_S has no point mass as mentioned just before. On the other hand it follows from (2.6) that

$$\lim_{A\uparrow\infty}\int_{R^1-U_n}\mu_S(dx)\left[\int_{-A}^{A}\exp(ixz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\right] \tag{2.11}$$

$$=(2\pi)\int_{R^1-U_n}\mu_S(dx)[u](x)\,.$$

Combining (2.10) with (2.11) it is easy to show

$$\lim_{A\uparrow\infty}\int_{-\infty}^{\infty}\mu_S(dx)\left[\int_{-A}^{A}\exp(ixz)\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz\right]=(2\pi)\int_{-\infty}^{\infty}\mu_S(dx)[u](x)\,.$$

Since the left term equals $\lim_{A\uparrow\infty}\int_{-A}^{A}|\hat{\mu}(z)|^2\langle\!\langle 1/\Psi\rangle\!\rangle(z)dz$, the equality (2.1) is proved. We have completed the proof of Lemma 2.

3. In this section we prove Theorem 1, Theorem 2 and Corollary. Lévy processes with the exponent Ψ in this section are assumed to satisfy (A_1), (A_2), (D_α) and (I) without special mentioning. Let $J_\Psi(\mu)=\int_{-\infty}^{\infty}|\hat{\mu}(z)|^2\,\mathcal{Re}([\lambda+\Psi(z)]^{-1})dz$ as defined in (1.5). Then we have

PROPOSITION 3.1. i) *There exists a positive constant M such that*

$$MJ_\Psi(\mu)\leqq\int_{-\infty}^{\infty}\left\{\int_{|z|}^{\infty}u^{-1}\,\mathcal{Re}([\lambda+\Psi(u)]^{-1})du\right\}|\hat{\mu}(z)|^2dz$$

for every probability measure μ.

ii) *For each fixed* $\varepsilon>0$ *there exists a positive constant* M_ε *such that*

$$\int_{-\infty}^{\infty}\left\{\int_{|z|}^{\infty}u^{-1}\,\mathcal{Re}([\lambda+\Psi(u)]^{-1})du\right\}|\hat{\mu}(z)|^2dz\leqq M_\varepsilon J_\Psi(\mu)+\varepsilon$$

for every probability measure μ.

iii) *For each probability measure* μ, $J_\Psi(\mu)<\infty$ *if and only if*

$$\int_{-\infty}^{\infty}\left\{\int_{|z|}^{\infty}u^{-1}\,\mathcal{Re}([\lambda+\Psi(u)]^{-1})du\right\}|\hat{\mu}(z)|^2dz<\infty\,.$$

PROOF. Put $\langle\!\langle 1/\Psi\rangle\!\rangle(z)=\int_{|z|}^{\infty}u^{-1}\mathcal{R}e([\lambda+\Psi(u)]^{-1})du$ as in the proof of Lemma 2. Since $\lim_{z\to 0}\langle\!\langle 1/\Psi\rangle\!\rangle(z)=\infty$, the estimate (2.3) together with the continuity and the positivity of $\langle\!\langle 1/\Psi\rangle\!\rangle(z)$ and $\mathcal{R}e([\lambda+\Psi(z)]^{-1})$ implies that $M\mathcal{R}e([\lambda+\Psi(z)]^{-1})\leqq\langle\!\langle 1/\Psi\rangle\!\rangle(z)$, from which the statement i) follows directly. In the next note that there exists a positive constant M_1 such that $\mathcal{R}e([\lambda+\Psi(u)]^{-1})\leqq M_1u^{-\alpha}$, $(0<\alpha<1)$ for every $u>1$ by (D_α). Then we have

$$\langle\!\langle 1/\Psi\rangle\!\rangle(z)=\int_z^1 u^{-1}\mathcal{R}e([\lambda+\Psi(u)]^{-1})du+\int_1^{\infty}u^{-1}\mathcal{R}e([\lambda+\Psi(u)]^{-1})du$$

$$\leqq\lambda^{-1}\log(1/z)+\alpha^{-1}M_1$$

for every z, $0<z<1$. Hence, for δ, $0<\delta<1$, we have

$$0\leqq\int_{-\delta}^{\delta}\langle\!\langle 1/\Psi\rangle\!\rangle(z)|\hat{\mu}(z)|^2dz=2\int_0^{\delta}\langle\!\langle 1/\Psi\rangle\!\rangle(z)|\hat{\mu}(z)|^2dz$$

$$\leqq 2\lambda^{-1}\int_0^{\delta}\log(1/z)dz+2\alpha^{-1}M_1\delta\,.$$

Choose δ so that the last term of this inequality is smaller than ε and fix it and nextly choose a positive constant $M(\delta)$ so that $\langle\!\langle 1/\Psi\rangle\!\rangle(z)\leqq M(\delta)\mathcal{R}e([\lambda+\Psi(z)]^{-1})$ for every z with $z>\delta>0$ by (2.3). Then we have $\int_{-\infty}^{\infty}\langle\!\langle 1/\Psi\rangle\!\rangle(z)|\hat{\mu}(z)|^2dz=\int_{|z|\leqq\delta}+\int_{|z|>\delta}\leqq\varepsilon+M(\delta)J_{\Psi}(\mu)$. Since δ can be determined by ε, we can replace $M(\delta)$ with M_ε. The proof of ii) is finished. As for the proof of iii) we have only to note that both the statement i) and ii) are valid even if $J_{\Psi}(\mu)$ or $\int_{-\infty}^{\infty}\langle\!\langle 1/\Psi\rangle\!\rangle(z)|\hat{\mu}(z)|^2dz$ diverges.

For a positive and continuous decreasing function ϕ on $(0,\infty)$ such that $\lim_{r\downarrow 0}\phi(r)=\infty$, *$\phi$-capacity* $C^{\phi}(K)$ of a compact set K is defined as follows:

$$C^{\phi}(K)=\phi^{-1}(E(K))\quad\text{if } E(K)<\infty,\qquad C^{\phi}(K)=0\quad\text{if otherwise},$$

where $E(K)=\inf(E_{\phi}(\mu),\ \mu\in\Pr(K))$ and

$$E_{\phi}(\mu)=\iint\mu(dx)\phi(|x-y|)\mu(dy)\,. \tag{3.1}$$

Choose

$$\phi(x)=\int_0^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\,. \tag{3.2}$$

Since $\phi(x)\uparrow\infty$ as $x\downarrow 0$ by (A_2) (Kesten [7]), we can define ϕ-capacity for such

ϕ. Moreover we have

PROPOSITION 3.2. *For every fixed $a>0$, we can choose positive constants M_i, $i=1, 2$ such that*

$$M_2 E_\phi(\mu) \leqq \int_{-\infty}^{\infty} \left\{ \int_{|z|}^{\infty} u^{-1} \mathcal{R}e([\lambda+\Psi(u)]^{-1}) du \right\} |\hat{\mu}(z)|^2 dz \leqq M_1 E_\phi(\mu)$$

for every $\mu \in \Pr(|x| \leqq a)$, where ϕ and $E_\phi(\mu)$ are those defined by (3.2) *and* (3.1) *respectively.*

PROOF. The assertion follows from (2.1) directly, because we can conclude from (2.2) that there exist positive constants M_i, $i=1, 2$ such that $M_2[u](x-y) \leqq \phi(|x-y|) \leqq M_1[u](x-y)$ for every $x, y \in \{|x|<a\}$, where $[u]$ is the one defined in (1.1).

PROOF OF THEOREM 1. $C^\lambda(K)>0$ if and only if $J_\Psi(\mu)<\infty$ for some $\mu \in \Pr(K)$ by Lemma 1, and therefore if and only if

$$\int_{-\infty}^{\infty} \left\{ \int_{|z|}^{\infty} u^{-1} \mathcal{R}e([\lambda+\Psi(u)]^{-1}) du \right\} |\hat{\mu}(z)|^2 dz < \infty \qquad \text{for some } \mu \in \Pr(K)$$

by iii) of Proposition 3.1. This last statement is equivalent to $E_\phi(\mu)<\infty$ for some $\mu \in \Pr(K)$ by Proposition 3.2. Now Theorem 1 has been proved, because it follows from the definition of ϕ-capacity that $C^\phi(K)>0$ if and only if $E_\phi(\mu)<\infty$ for some $\mu \in \Pr(K)$.

Before proving Theorem 2 we refer relations between Hausdorff measures and ϕ-capacity. We denote *h-Hausdorff measure* of K by $\Lambda_h(K)$ for a positive continuous function h on $(0, \infty)$ with $\lim_{r \downarrow 0} h(r)=0$. It is well known that

$$C^{1/h}(K)=0 \qquad \text{if } \Lambda_h(K)=0\,.$$

Furthermore

LEMMA 3 (Taylor [**10**], Theorem 4 and Remark. See also [**6**], Theorem 1). *Let ϕ and $1/h$ be positive, continuous and decreasing functions with $\lim_{x \downarrow 0} \phi(x) = \lim_{x \downarrow 0} 1/h(x)=\infty$. Assume in addition;* a) *$x\phi(x)$ is increasing with $\lim_{x \downarrow 0} x\,\phi(x)=0$, and* b) *$x^{-1}\int_0^x \phi(s)ds \leqq M\phi(x)$ for every small $x>0$. Then if*

$$\liminf_{x \downarrow 0} \phi(x)h(x)=0\,,$$

then there exists a compact set K such that $C^\phi(K)>0$ and $\Lambda_h(K)=0$.

The next proposition shows that Lemma 3 can be applied to our setting.

PROPOSITION 3.3. *Let ϕ be the one defined in Theorem* 1, *and F be the one in the condition* (D_α). *Put $\tilde{F}(u)=F(u)$ for $u \geqq 1$ and $\tilde{F}(u)=u^{-\alpha}F(1)$ for $0<u<1$. Then the function $\tilde{\varphi}$ defined by*

(3.3) $$\tilde{\phi}(x)=\int_0^{1/x}\tilde{F}(u)du$$

is a positive, continuous and decreasing function on $(0, \infty)$ *with* $\lim_{x\downarrow 0}\tilde{\phi}(x)=\infty$ *and satisfies the conditions* a) *and* b) *in Lemma* 3. *Moreover*

(3.4) $$\tilde{\phi}(x)\asymp\phi(x), \qquad x\to 0.$$

PROOF. We first prove (3.4). Noting that $\lim_{x\downarrow 0}\phi(x)=\infty$ as is mentioned in the proof of Theorem 1, we have

$$\phi(x)\asymp\int_1^{1/x}\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\asymp\int_1^{1/x}F(z)dz, \qquad x\downarrow 0,$$

and especially $\lim_{x\downarrow 0}\int_1^{1/x}F(z)dz=\infty$. So $\tilde{\phi}(x)\asymp\int_1^{1/x}\tilde{F}(z)dz$, $x\downarrow 0$, and the relation (3.4) is proved. At the same time we have proved $\lim_{x\downarrow 0}\tilde{\phi}(x)=\infty$. The function $x\tilde{\phi}(x)$ is increasing, because $x\tilde{\phi}(x)=\int_0^1\tilde{F}(t/x)dt$ and $\tilde{F}$ is decreasing. Further $x\tilde{\phi}(x)=x^{\alpha}\int_0^1 t^{-\alpha}(t/x)^{\alpha}\tilde{F}(t/x)dt\leqq x^{\alpha}\int_0^1\tilde{F}(t)dt$ for $0<x\leqq 1$. So $\lim_{x\downarrow 0}x\phi(x)=0$. The condition a) has been proved. The condition b) is proved as follows;

$$\begin{aligned}
x^{-1}\int_0^x\tilde{\phi}(t)dt&=x^{-1}\int_0^x t^{-1}\left[\int_0^1\tilde{F}(u/t)du\right]dt\\
&\leqq x^{-1}\int_0^x t^{-1}\left[\int_0^1(u/t)^{-\alpha}(u/t)^{\alpha}\tilde{F}(u/t)du\right]dt\\
&\leqq x^{-1}\int_0^x t^{-1}\left[\int_0^1(u/t)^{-\alpha}(u/x)^{\alpha}\tilde{F}(u/x)du\right]dt\\
&=x^{-1-\alpha}\int_0^x t^{\alpha-1}dt\left[\int_0^1\tilde{F}(u/x)du\right]\leqq\alpha^{-1}\tilde{\phi}(x).
\end{aligned}$$

PROOF OF THEOREM 2. Define $\tilde{\phi}_i$, $i=1, 2$, as is defined by (3.3). Note that (0.4) is equivalent to $\liminf_{x\downarrow 0}\tilde{\phi}_1(x)/\tilde{\phi}_2(x)=0$, and $C^{\tilde{\phi}_i}(K)=0$ if and only if $C^{\phi_i}(K)=0$ by (3.4). Hence it follows from Lemma 3 that $C^{\varphi_1}(K)>0$ and $\Lambda_{1/\varphi_2}(K)=0$ for some compact set K. So $C^{\varphi_1}(K)>0$ and $C^{\varphi_2}(K)=0$. This implies $C_1^{\lambda}(K)>0$ and $C_2^{\lambda}(K)=0$ by Theorem 1. We next prove ii.1)$\Rightarrow$ii.2)$\Rightarrow$ii.3)$\Rightarrow$ii.1). The assertion ii.2)$\Rightarrow$ii.3) is trivial. The statement ii.3)$\Rightarrow$ii.1) is a consequence of the result i). Assume ii.1). Then we can choose a positive constant M_3 such that $M_3\varphi_1(x)\leqq\varphi_2(x)$ for $0<x<2a$ and therefore $M_4[u_1](x)\leqq[u_2](x)$ for $0<x<2a$ for some positive constant M_4 by (2.2), where $[u_i]$, $i=1, 2$, are those defined in (1.1) for the symmetrized λ-resolvent kernel of X_i, $i=1, 2$, respectively. Then Proposition 3.2 implies

(3.5) $$M_5\int_{-\infty}^{\infty}\left\{\int_{|z|}^{\infty}u^{-1}\,\mathcal{R}e([\lambda+\Psi_1(u)]^{-1})du\right\}|\hat{\mu}(z)|^2dz$$

$$\leqq\int_{-\infty}^{\infty}\left\{\int_{|z|}^{\infty}u^{-1}\,\mathcal{R}e([\lambda+\Psi_2(u)]^{-1})du\right\}|\hat{\mu}(z)|^2dz$$

for every probability measure μ whose support is in $\{|z|<a\}$ and some positive constant M_5 independent of μ. Choose ε so that $\varepsilon=\pi 4^{-1}MM_5C_1^{\lambda}(\{|z|\leqq a\})^{-1}$, where M is the constant determined in i) of Proposition 3.1. Then it follows from (3.5), i) and ii) of Proposition 3.1 that $MM_5J_{\Psi_1}(\mu)\leqq M_\varepsilon J_{\Psi_2}(\mu)+\varepsilon$. Using Proposition 1.2, we have $2\pi 4^{-1}MM_5C_1^{\lambda}(A)^{-1}\leqq 2\pi M_\varepsilon C_2^{\lambda}(A)^{-1}+\varepsilon$ for every open set A whose closure is in $\{|z|<a\}$. Noting $C_1^{\lambda}(\{|z|\leqq a\})\geqq C_1(A)$, we have $4^{-1}MM_5C_1^{\lambda}(A)^{-1}\leqq 2MC_2^{\lambda}(A)^{-1}$ for every open set A whose closure is in $\{|z|<a\}$, and so does for every compact set K in $\{|z|<a\}$. We have finished the proof of ii.1), ii.2).

PROOF OF COROLLARY. It is sufficient to prove that the condition (0.8) implies $\lim_{x\downarrow 0}\phi_1(x)/\phi_2(x)=0$. For an arbitrary small $\varepsilon>0$, choose z_0 so that $\mathcal{R}e([\lambda+\Psi_1(z)]^{-1})/\mathcal{R}e([\lambda+\Psi_2(z)]^{-1})<\varepsilon$ for every $z>z_0$. Then $\phi_1(x)/\phi_2(x)\leqq \varepsilon+\mathcal{R}e([\lambda+\Psi_1(z)]^{-1})dz/\phi_2(x)$. Hence $\limsup_{x\downarrow 0}\phi_1(x)/\phi_2(x)<\varepsilon$, because $\lim_{x\downarrow 0}\phi_2(x)=\infty$. As ε is arbitrary, we have finished the proof.

4. In this section we apply our result to the subordinators whose exponents Ψ are of the form below:

(4.1) $$\Psi(z)=\int_0^{\infty}(1-\exp(izy))N(y)dy\,,$$

where $N(y)$ is continuous and positive on $(0, 1)$, $N(y)=0$ for $y>1$ and

$$\int_0^1 yN(y)dy<\infty\,.$$

We study the behaviour of the exponents near infinity under certain regularity conditions. Since $\mathcal{R}e\Psi(z)=\mathcal{R}e\Psi(-z)>0$ and $\mathcal{I}m\Psi(z)=-\mathcal{I}m\Psi(-z)$, we study it near ∞. For convenience we put

(4.2) $N(y)=y^{-2}L(1/y)^{-1}$ *for* $0<y\leqq 1$ *and equals to* 0 *for* $y>1$.

If L satisfies the condition below:

(L_β^1) $z^{\beta}L(z)^{-1}$ *is increasing and* $z^{-\beta}L(z)^{-1}$ *is decreasing on* $[1, \infty)$ *for some* β, $0<\beta<1$,

then

(4.3) $$\mathcal{R}e\Psi(z)\asymp zL(z)^{-1}\,,\qquad z\to\infty\,,$$

and

$$(4.4)_1 \qquad \mathcal{I}m\Psi(z) \leqq z\int_z^\infty u^{-1}L(u)^{-1}du + M_1 z L(z)^{-1},$$

$$(4.4)_2 \qquad \mathcal{I}m\Psi(z) \geqq M_2 z\int_z^\infty u^{-1}L(u)^{-1}du - M_1 z L(z)^{-1}$$

for every large x. For the proof we set

$$R_a^b(x) = \int_a^b f(z)z^{-2}L(x/z)^{-1}dz$$

$$= \int_a^b f(z)z^{-2+\beta'}\{(x/z)^{-\beta'}L(x/z)\}^{-1}dz\, x^{-\beta'}$$

for a nonnegative function f on (a, b), $0 \leqq a < b < \infty$. Then it follows from (L_β^1) that

$$b^{-\beta}c(a, b, \beta)L(x/b)^{-1} \leqq R_a^b(x) \leqq b^\beta c(a, b, -\beta)L(x/b)^{-1}$$

for $0 \leqq a < b < \infty$ and

$$a^\beta c(a, b, -\beta)L(x/a)^{-1} \leqq R_a^b(x) \leqq a^{-\beta}c(a, b, \beta)L(x/a)^{-1}$$

for $0 < a < b < \infty$, where

$$c(a, b, \beta') = \int_a^b f(z)z^{-2+\beta'}dz.$$

Using this estimate for $f(z) = 1 - \cos(z)$, we get

$$\mathcal{R}e\Psi(x) \geqq xR_0^1(x) \geqq c(0, 1, \beta)xL(x)^{-1} \qquad \text{for every } x > 1,$$

and

$$\mathcal{R}e\Psi(x) = xR_0^1(x) + xR_1^x(x) \leqq c(0, 1, -\beta)xL(x)^{-1} + c(1, x, \beta)xL(x)^{-1}.$$

Noting $c(1, x, -\beta) = \int_1^x z^{-2-\beta}(1 - \cos z)dz \leqq M_4$, the proof of (4.3) finishes. In the next we prove (4.4). Applying the estimate for $f(z) = 1$, we have

$$\mathcal{I}m\Psi(x) = x\int_0^x \sin(z)z^{-2}L(x/z)^{-1}dz$$

$$\leqq x\int_0^1 z^{-1}L(x/z)^{-1}dz + xR_1^x(x)$$

$$\leqq x\int_x^\infty u^{-1}L(u)^{-1}du + c(1, x, \beta)xL(x)^{-1}.$$

Since $c(1, x, \beta) = \int_1^x z^{-2+\beta}dz < M_1$, the first inequality of (4.4) is proved. On the other hand

$$\mathcal{I}m\Psi(x)\geqq x\int_0^1(\sin(z))z^{-2}L(x/z)^{-1}dz-xR_1^x(x)$$

$$\geqq M_2x\int_x^\infty u^{-1}L(u)^{-1}du-c(1,\ x,\ \beta)xL(x)^{-1}$$

and $c(1,\ x,\ \beta)<M_1$ as mentioned before. Now the second inequality of (4.4) has been proved.

If we impose the following condition on L in addition to (L_β^1);

(L_γ^2) $\qquad$ *$u^\gamma L(u)^{-1}$ is decreasing on $[1,\ \infty)$ for some $\gamma>0$,*

then

$$|\mathcal{I}m\Psi(x)|\leqq M_3xL(x)^{-1}\asymp\mathcal{R}e\Psi(x),\qquad x\to\infty. \tag{4.5}$$

Indeed, $x\int_x^\infty u^{-1}L(u)^{-1}du=x\int_x^\infty u^{-1-\gamma}\{u^\gamma L(u)^{-1}\}du\leqq\gamma^{-1}xL(x)^{-1}$ for $x>0$.

If L satisfies the following:

(S) $\qquad$ *L is slowly varying at infinity on $[0,\ \infty)$*

then

$$(4.6)_1\qquad \mathcal{I}m\Psi(z)\asymp z\int_z^\infty u^{-1}L(u)^{-1}du,\qquad z\to\infty,$$

$$(4.6)_2\qquad \lim_{z\to\infty}\mathcal{R}e\Psi(z)/\mathcal{I}m\Psi(z)=0.$$

For the proof we first note that

(4.7) *If (S) is satisfied, there exists a continuous function $M(x)$ on $[1,\ \infty)$ such that $M(x)\asymp L(x)$, $x\to\infty$ and M satisfies (L_β^1) for every $\beta>0$.*

Indeed, by Corollary, VIII. 9, Feller [2],

$$L(x)=a(x)\exp\left(\int_1^x\varepsilon(y)/y\ dy\right),$$

where $\lim_{y\to\infty}\varepsilon(y)=0$ and $\lim_{x\to\infty}a(x)=\text{const.}$ So we have only to choose $\exp\left(\int_1^x\varepsilon(y)/ydy\right)$ as $M(x)$. In the next, applying (9.6) of Theorem 1, VIII. 9, Feller [2] to $Z(t)=L(t)^{-1}$ and $p=-1$, $\gamma=0$, (here note that $\int_t^\infty u^{-1}L(u)^{-1}du<\infty$ by the assumption $\int_0^1 yN(y)dy<\infty$), we have

$$\lim_{t\to\infty}L(t)^{-1}\Big/\int_t^\infty u^{-1}L(u)^{-1}du=0.$$

Hence it follows from (4.4) that the first estimate of (4.6) holds and this together with (4.3) implies the second equality of (4.6).

Let us assume either $(L_\beta^1)\cap(L_\gamma^2)$ or (S). Then

$$\mathcal{R}e([\lambda+\Psi(z)]^{-1}) \text{ satisfies } (D_\alpha) \text{ and } (I). \tag{4.8}$$

In case (L_β^1) and (L_γ^2) are satisfied, $\mathcal{R}e([\lambda+\Psi(z)]^{-1})\asymp z^{-1}L(z)$, $z\to\infty$, by (4.3) and (4.5), and it follows from (L_β^1) that $z^{\alpha-1}L(z)$ is decreasing if we choose $\alpha=1-\beta$. Hence (D_α) holds. For (I), we have only to note $z^\beta L(z)<(2z)^\beta L(2z)$ by (L_β^1). In case (S) is satisfied,

$$\begin{aligned}\mathcal{R}e([\lambda+\Psi(z)]^{-1})&\asymp\{|\mathcal{I}m\Psi(z)|^2/\mathcal{R}e\Psi(z)\}^{-1} \\ &\asymp z^{-1}L(z)^{-1}\left[\int_z^\infty u^{-1}L(u)^{-1}du\right]^{-2}, \qquad z\to\infty.\end{aligned} \tag{4.9}$$

Hence

$$\mathcal{R}e([\lambda+\Psi(z)]^{-1})\asymp z^{-1}M(z)^{-1}\left[\int_z^\infty u^{-1}M(u)^{-1}du\right]^{-2}, \qquad z\to\infty,$$

by (4.7). On the other hand $\int_z^\infty u^{-1}M(u)^{-1}du=z^{-\beta}\int_0^1 u^{-1+\beta}(z/u)^\beta M(z/u)du\equiv z^{-\beta}\tilde{M}(z)$ and $\tilde{M}(z)$ is increasing for every $\beta>0$. So $z^{\alpha-1+2\beta}M(z)^{-1}\tilde{M}(z)^{-2}$ is decreasing if we choose α, β so small that $\alpha-1+2\beta<0$. We have proved the property (D_α) for this case. The property (I) is proved similarly.

Now we can get a comparison theorem between the classes of polar sets for subordinators X in the class mentioned above and their symmetrized processes X_S's. Here the symmetrized process X_S of X with the exponent Ψ in (4.1) is defined as the Lévy process on the line whose exponent Ψ_S is of the form below:

$$\begin{aligned}\Psi_S(z)&=\int_0^\infty(1-\exp(izy))N(y)dy+\int_{-\infty}^0(1-\exp(izy))N(-y)dy \\ &=2\mathcal{R}e\Psi(z).\end{aligned}$$

PROPOSITION 4.1. *Let X be a subordinator with the exponent Ψ of the form* (4.1) *and X_S be its symmetrized process.*

i) *If* (L_β^1) *and* (L_γ^2) *hold,* $P_X=P_{X_S}$.

ii) *If* (S) *holds,* $P_X\subsetneqq P_{X_S}$.

PROOF. First note that

$$X \text{ and } X_S \text{ satisfy } (A_1) \text{ and } (A_2). \tag{4.10}$$

The condition (A_1) follows from the estimate; $\mathcal{R}e\Psi(z)\geqq M_5zL(z)^{-1}=M_5z^{1-\beta}z^\beta L(z)^{-1}\geqq M_6z^{1-\beta}$. The condition (A_2) is proved by Kesten's result [7], because $\mathcal{R}e\Psi(z)\leqq M_7\int_0^1(1-\cos(zy))y^{-2}dy\leqq M_8z$, and so $\int_0^\infty\mathcal{R}e([\lambda+\Psi(z)]^{-1})dz\geqq M_8^{-1}\int_1^\infty z^{-1}dz=\infty$.

Combining (4.10) with (4.8), we see that X and X_S satisfy (A_1), (A_2), (D_α) and (I). Under (L_β^1) and (L_γ^2), $\mathcal{R}e([\lambda+\Psi(z)]^{-1})\asymp\mathcal{R}e([\lambda+\Psi_S(z)]^{-1})$, $z\to\infty$, by (4.5), and so $P_X=P_{X_S}$. If (S) hold, the estimate (4.6) implies $P_X\subsetneqq P_{X_S}$ by Corollary of Theorem 2. The proof is finished.

REMARK. Port and Stone [8] proved that every point is nonpolar for the asymmetric Cauchy process X but every point is polar for X_S. So $P_X\subsetneqq P_{X_S}$. Our result ii) shows a similar phenomenon occurs, but every point is polar for both the processes in our case. They also showed that every point is regular for itself relative to X. We can also show that every semipolar set is polar for our X. The proof will be given elsewhere.

Finally we give examples of pairs of Lévy processes with (0.6). We first show an example of a pair of subordinators given in (4.1) satisfying (L_β^1), for which (0.6) holds.

Let X be a subordinator with the exponent Ψ of the form (4.1) satisfying (L_β^1). If

$$zL(z)^{-1}\ll z\int_z^\infty u^{-1}L(u)^{-1}du\,, \qquad z\to\infty\,,$$

we have

$$\int_0^z \mathcal{R}e([\lambda+\Psi(x)]^{-1})dx\asymp\left(\int_z^\infty u^{-1}L(u)^{-1}\,du\right)^{-1} \tag{4.11}$$

$$\asymp z[\mathcal{J}m\Psi(z)]^{-1}, \qquad z\to\infty\,.$$

Indeed, it follows from (4.4) that

$$\int_0^z \mathcal{R}e([\lambda+\Psi(x)]^{-1})dx\asymp\int_1^z\left(\int_x^\infty u^{-1}L(u)^{-1}du\right)^{-2}x^{-1}L(x)^{-1}dx\,, \qquad z\to\infty\,,$$

and, putting $K(x)=\int_x^\infty u^{-1}L(u)^{-1}du$, the right term equals to

$$\int_1^z K(x)^{-2}(-K'(x))dx=\int_{K(z)}^{K(1)}u^{-2}du=K(z)^{-1}-K(1)^{-1}\,.$$

Especially, if (S) holds, then (4.11) is valid by (4.6).

Set

$$L(z)=(\log(z))^2 \quad\text{and}\quad H(z)=(\log(z))^{2+\varepsilon}$$

for a fixed small $\varepsilon>0$. Choose β_0, $0<\beta_0<1$, and sufficiently large z_0 and fix them afterwards. We construct a sequence $\{z_k\}_{k=1,2,\dots}$ with $\lim_{k\to\infty}z_k=\infty$ and a sequence of functions $\{f_k\}_{k=0,1,\dots}$ inductively as follows: We set $f_{3k}(z)=L(z)^{-1}$ for all k. If z_{3k} is determined, then choose $z_{3k+1}=z_{3k}^{16}$ and $f_{3k+1}(z)=a_{3k+1}z^{-\beta_0}$ so that $f_{3k+1}(z_{3k+1})=L(z_{3k+1})^{-1}$ (that is, $a_{3k+1}=z_{3k+1}^{\beta_0}L(z_{3k+1})^{-1}$). Secondly choose $z_{3k+2}=\inf(z>z_{3k+1},\ f_{3k+1}(z)=H(z)^{-1})$ and $f_{3k+2}(z)=a_{3k+2}z^{\beta_0}$ so that $f_{3k+2}(z_{3k+2})=$

$H(z_{3k+2})^{-1}$. Finally choose $z_{3k+3}=\inf(z>z_{3k+2},\ f_{3k+2}(z)=L(z)^{-1})$. This procedure is possible, and

$$H(z)^{-1}\leqq f_{3k+1}(z)\ (\text{resp. } f_{3k+2}(z))\leqq L(z)^{-1} \tag{4.12}$$

for $z\in[z_{3k+1}, z_{3k+2}]$ (resp. $z\in[z_{3k+2}, z_{3k+3}]$), and

$$z_{3k+3}\leqq z_{3k+2}^2\leqq z_{3k+1}^4, \qquad k=0, 1, 2, \cdots. \tag{4.13}$$

Indeed, f_{3k+1} and f_{3k+2} have only one common point with each of $L(z)^{-1}$ and $H(z)^{-1}$, and $f_{3k+1}(z_{3k+1}^2)<H(z_{3k+1}^2)^{-1}$ and $f_{3k+2}(z_{3k+2}^2)>L(z_{3k+2}^2)^{-1}$.

Define

$$\tilde{L}(z)^{-1}=\begin{cases} L(z)^{-1}, & z\leqq z_0, \\ f_{3k}(z), & z_{3k}\leqq z<z_{3k+1}, \\ f_{3k+1}(z), & z_{3k+1}\leqq z<z_{3k+2}, \\ f_{3k+2}(z), & z_{3k+2}\leqq z<z_{3k+3}, \end{cases} \tag{4.14}$$

for $k=0, 1, 2, \cdots$. It is easy to check that $\tilde{L}$ satisfies (L_β^1) for $\beta_0<\beta<1$. Put

$$\tilde{N}(y)=y^{-2}\tilde{L}(1/y)^{-1}, \tag{4.15$_1$}$$

$$\tilde{\Psi}(z)=\int_0^1(1-\exp(izy))\tilde{N}(y)dy, \tag{4.15$_2$}$$

and let $\tilde{X}$ be a subordinator with the exponent $\tilde{\Psi}$. Then

$$\mathcal{R}e\tilde{\Psi}(z)\asymp z\tilde{L}(z)^{-1}, \qquad z\to\infty, \tag{4.16}$$

by (4.3). Especially

$$\mathcal{R}e\tilde{\Psi}(z)\leqq zL(z)^{-1} \tag{4.17}$$

by (4.12).

For the estimate of $\mathcal{I}m\tilde{\Psi}(z)$, we note that, for every given large z, $\tilde{L}(z')^{-1}=L(z')^{-1}$ for every $z'\in[z^8, z^{16}]$ if $z\in[z_{3k+1}^{1/2}, z_{3k+3}]$, $k=0, 1, 2, \cdots$ and $\tilde{L}(z')^{-1}=L(z')^{-1}$ for every $z'\in[z, z^2]$ if $z\in[z_{3k}, z_{3k+1}^{1/2}]$, $k=0, 1, 2, \cdots$. Then

$$z\int_z^\infty u^{-1}\tilde{L}(u)^{-1}du\geqq\min\left(z\int_{z^4}^{z^8}u^{-1}(\log(u))^{-2}du,\ z\int_z^{z^2}u^{-1}(\log(u))^{-2}du\right)\geqq z(8\log(z))^{-1}, \tag{4.18}$$

and

$$z\int_z^\infty u^{-1}\tilde{L}(u)^{-1}du\leqq z\int_z^\infty u^{-1}L(u)^{-1}du\leqq z(\log(z))^{-1}.$$

The estimate (4.4) together with (4.17) and (4.18) implies that

(4.19) $$z\tilde{L}(z)^{-1} \ll \mathcal{I}m\tilde{\Psi}(z) \asymp z\int_z^\infty u^{-1}\tilde{L}(u)^{-1}du \asymp z(\log(z))^{-1}, \qquad z\to\infty .$$

Hence it follows from (4.11) that

(4.20) $$\int_0^z \mathcal{R}e([\lambda+\tilde{\Psi}(x)]^{-1})dx \asymp \log(z), \qquad z\to\infty .$$

On the other hand, let X be a subordinator with the exponent Ψ of the form;

(4.21) $$\Psi(z)=\int_0^1 (1-\exp(izy))y^{-2}L(1/y)^{-1}dy, \ L(x)=(\log(x))^2\vee 1 .$$

Then we have

(4.22) $$\int_0^z \mathcal{R}e([\lambda+\Psi(x)]^{-1})dx \asymp \log(z), \qquad z\to\infty$$

by (4.11), because L satisfies (S). Noting that $\mathcal{I}m\Psi(z)\asymp\mathcal{I}m\tilde{\Psi}(z)\asymp z(\log(z))^{-1}$, $z\to\infty$, and $\mathcal{R}e\tilde{\Psi}(z)\asymp z\tilde{L}(z)^{-1}\prec zL(z)^{-1}\asymp\mathcal{R}e\Psi(z)$, $z\to\infty$ and $\liminf_{z\to\infty} L(z)/\tilde{L}(z)=0$, we get

PROPOSITION 4.2. *Let X be a subordinator with the exponent of the form* (4.21) *and $\tilde{X}$ be a subordinator with the exponent of the form* (4.15). *Then* (0.6) *holds if we put $\Psi_1=\tilde{\Psi}$ and $\Psi_2=\Psi$. However $P_X=P_{\tilde{X}}$.*

Now the proof is clear. Indeed, $P_X=P_{\tilde{X}}$ follows from (4.20) and (4.22) by using Theorem 2.

Using the pair of subordinators in Proposition 4.2, we can give a pair of symmetric Lévy processes on the line for which (0.6) holds. For this purpose we prepare a lemma.

LEMMA 4. *Let G be a positive function on $(0,\infty)$ such that $u^{-\delta}G(u)$ is increasing on (c,∞) and $u^{-\gamma}G(u)$ is decreasing on (c,∞) for some $0<\delta<\gamma<2$ and $c>0$. Then there exists a symmetric Lévy process with the exponent Ψ such that*

$$\Psi(z)\asymp G(z), \qquad z\to\infty .$$

PROOF. We may assume $c=1$ without generality. Set $N(y)=|y|^{-1}G(1/|y|)^{-1}$ and let X be a symmetric Lévy process with the exponent Ψ of the form;

$$\Psi(z)=\int_{-1}^1 (1-\exp(izy))N(y)dy$$

Then, for $z>0$,

$$\begin{aligned}\Psi(z)&=2\int_0^z (1-\cos(y))y^{-1}G(z/y)dy\\ &=2\int_1^z (1-\cos(y))y^{-1}(z/y)^{\delta}(z/y)^{-\delta}G(z/y)dy\\ &\quad+2\int_0^1 (1-\cos(y))y^{-1}(z/y)^{\gamma}(z/y)^{-\gamma}G(z/y)dy\end{aligned}$$

$$\leqq 4G(z)\int_1^z y^{-1-\delta}dy + 2G(z)\int_0^1 (1-\cos(y))y^{-1-\gamma}dy$$
$$\leqq M_1 G(z),$$

and

$$\Psi(z) \geqq 2\int_0^1 (1-\cos(y))y^{-1}(z/y)^{\delta}(z/y)^{-\delta}G(z/y)dy \geqq M_2 G(z).$$

The proof is finished.

Let X and $\tilde{X}$ be subordinators with exponent Ψ and $\tilde{\Psi}$ respectively which are given in Proposition 4.1. Then it follows from (4.9) that

$$\mathcal{R}e([\lambda+\Psi(z)]^{-1}) \asymp z^{-1}, \qquad z\to\infty,$$

and from (4.19) that

$$\mathcal{R}e([\lambda+\tilde{\Psi}(z)]^{-1}) \asymp z^{-1}\tilde{L}(z)^{-1}(\log(z))^2, \qquad z\to\infty.$$

Define

$$(4.23) \qquad G(z) = \begin{cases} z\tilde{L}(z)(\log(z))^{-2}, & z \geqq e, \\ e\tilde{L}(e), & z \leqq e. \end{cases}$$

Then, if we choose $1+\beta_0<\gamma<2$ and $0<\delta<1-\beta_0$, G satisfies the condition in Lemma 4. Hence we have

PROPOSITION 4.3. *Let X_1 be a symmetric Lévy process with the exponent Ψ_1 which is constructed by Lemma 4 so that*

$$\Psi_1(z) \asymp G(z), \qquad z\to\infty,$$

where G is given by (4.23) *and X_2 be the symmetric Cauchy process* (*that is, the exponent $\Psi_2(z)$ equals to $|z|$*). *Then* (0.6) *holds, but $P_{X_1}=P_{X_2}$.*

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