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On some asymptotic properties of systems of entire functions of smooth growth

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1. Introduction.

Let $f=(f_0, f_1, \cdots, f_n)\ (n\geqq 1)$ be a transcendental system in $|z|<\infty$. That is to say, $f_0, f_1, \cdots, f_n$ are entire functions without common zero and the characteristic function of f defined by H. Cartan ([**3**]):

$$T(r, f)=\frac{1}{2\pi}\int_0^{2\pi} U(re^{i\theta})\, d\theta - U(0),$$

where

$$U(re^{i\theta})=\max_{0\leq j\leq n} \log |f_j(re^{i\theta})|,$$

satisfies the condition

$$\lim_{r\to\infty}\frac{T(r, f)}{\log r}=\infty .$$

Let X be a set of linear combinations $(\not\equiv 0)$ of $f_0, f_1, \cdots, f_n$ with coefficients in C in general position; that is, for any $n+1$ elements

$$a_{0j}f_0+a_{1j}f_1+\cdots+a_{nj}f_n \quad (j=1, \cdots, n+1)$$

in X, $n+1$ vectors $(a_{0j}, a_{1j}, \cdots, a_{nj})$ are linearly independent.

In this paper, we shall give some necessary or sufficient conditions for f to satisfy

$$T(r, f)\sim T(2r, f), \tag{1}$$

where "$A(r)\sim B(r)$" means $\lim_{r\to\infty} A(r)/B(r)=1$, and discuss the relations between the Nevanlinna deficiency of F in X and the asymptotic behaviour of f satisfying (1).

We use the standard notation of the Nevanlinna theory (see [**5**]).

2. Cases of meromorphic functions; some lemmas and problems.

In this section, we shall pick up important results concerning transcendental meromorphic functions g in $|z|<\infty$ which satisfy

$$T(r, g) \sim T(2r, g), \tag{2}$$

give some lemmas used in this paper and state some problems which will be settled in this paper.

[I] If

$$T(r, g)=O((\log r)^2) \quad (r \to \infty),$$

then,

$$T(r, g) \sim N(r, a, b),$$

where $N(r, a, b)=\max\{N(r, a), N(r, b)\}$ $(a \neq b \in \bar{C})$ ([**11**]).

We can prove this by using the following well-known

LEMMA 1. *Let $h(z)$ be an entire function of genus 0, then*

$$\log M(r, h) \leqq r\int_0^\infty \frac{n(t)}{t(t+r)}\, dt+O(\log r)=r\int_0^\infty \frac{N(t)}{(t+r)^2}\, dt+O(\log r)$$

$$< N(r)+r\int_r^\infty \frac{n(t)}{t^2}\, dt+O(\log r)=r\int_r^\infty \frac{N(t)}{t^2}\, dt+O(\log r),$$

where $n(t)=n(t, 1/h)-n(0, 1/h)$ and $N(r)=\int_0^r n(t)/t\, dt$ (see [**3**], p. 47-p. 48).

The essential part of the proof of [I] is the fact that, for any $a \in \bar{C}$,

$$r\int_r^\infty \frac{n(t, a)}{t^2}\, dt=o(T(r, g)) \quad (r \to \infty). \tag{3}$$

This is because

$$r\int_r^\infty \frac{n(t, a)}{t^2}\, dt=O(\log r) \quad (r \to \infty).$$

From this point of view, G. Valiron ([**11**]) gave the following

[II] When the order of g is zero, if $T(r, g)$ satisfies (3) for every $a \in \bar{C}$, then,

$$T(r, g) \sim N(r, a, b).$$

He generalized this result to algebroid functions and gave some interesting results in this direction (see [**12**]).

PROBLEM 1. What functions satisfy (3) for every $a \in \bar{C}$?

On the other hand, Y. Kubota ([**8**]) showed that

[III] Suppose that g is of order zero and satisfies (2) and further that there exists a in $\bar{C}$ such that $\delta(a, g)>0$. Then,

$$T(r, g) \sim N(r, b) \quad (b \neq a).$$

Recently, W. K. Hayman ([**7**], Theorem 6) has proved that

[IV] Suppose that g is entire. If g satisfies (2), then g is of order and so genus zero and for every $a \in C$

(4) $$n(r, a)=o(N(r, a)) \quad (r \to \infty).$$

Conversely, if g is of genus zero and there exists a value $a \in C$ satisfying (4), then g satisfies (2).

We note that $N(r, a)$ satisfying (4) is of order zero by the method used in the proof of Theorem 6 ([7]) and so g must be of order zero if g is of genus zero (see Remark 1 in § 3).

In the proof of this theorem, we find the fact that a meromorphic function g satisfying (2) is of order zero. This shows that, in [III], the condition that g is of order zero is unnecessary. But some parts of the proof of [IV] are not applicable to meromorphic functions.

PROBLEM 2. Is it possible to generalize [IV] to meromorphic functions or further to systems?

Next, J.M. Anderson ([**1**]) stated that

[V] If g satisfies (2), then, for each distinct $a, b \in \bar{C}$

$$T(r, g) \sim N(r, a, b).$$

And he says that for a proof, see [**11**], théorème II; that is, [II] in this section. But we cannot find any direct proof of [V] in [**11**]. It is necessary to clarify the relation between (2) and (3).

PROBLEM 3. What relations are there between (2) and (3)?

Concerning the asymptotic values, J.M. Anderson and J. Clunie ([**2**]) proved that

[VI] If

$$T(r, g)=O((\log r)^2) \quad (r \to \infty)$$

and if $\delta(\infty, g)>0$, then

$$\liminf_{\substack{r\to\infty \\ z \notin \varepsilon\text{-set}}} \frac{\log|g(z)|}{T(|z|, g)} \geqq \delta(\infty, g).$$

In [**9**], we proved that

[VII] Suppose that g satisfies (2) and is of order zero. If $\delta(a, f)>0$, then a is an asymptotic value of g.

To this, Hayman ([**7**]) proved that, as is cited above, if g satisfies (2), then g is of order zero and improved [VII]. That is to say,

[VIII] Suppose that g satisfies (2). If $\delta(a, g)>0$, then a is an asymptotic value of g ([**7**], Corollary 2).

J.M. Anderson ([**1**]) improved this result as follows.

[IX] Suppose that g satisfies (2) and $\delta(\infty, g)>0$. Then, for a slim set S

$$\liminf_{\substack{r\to\infty \\ z \notin S}} \frac{\log|g(z)|}{T(|z|, g)} \geqq \delta(\infty, g),$$

where a countable set of circles in the plane is said to form a slim set if the sum of radii of those circles intersecting the annulus $2^k \leqq |z| \leqq 2^{k+1}$ is $o(2^k)$ as $k \to \infty$.

It is known ([**1**]) that if S is a slim set, then there is a receding path Γ from O to ∞ lying eventually outside S such that

$$\text{length of } \Gamma \text{ in } |z| \leqq R = R(1+o(1)) \quad (R \to \infty).$$

To prove [IX], he prepared the following

LEMMA 2. *Let h be an entire function for which*

$$\log M(r, h) \sim \log M(2r, h).$$

Then,

$$\log |h(z)| \sim \log M(r, h)$$

as $z = re^{i\theta} \to \infty$ outside a slim set.

PROBLEM 4. Is it possible to generalize [IX] to systems?

3. Systems f satisfying (1).

Let f and X be as in §1. In this section, we discuss the systems f satisfying (1) and give solutions of Problems 1, 2 and 3.

THEOREM 1. *If f satisfies* (1), *then the order of f is zero and so the integral*

$$\int_1^\infty \frac{T(t, f)}{t^2} dt \tag{5}$$

converges. Further it holds

$$T(r, f) \sim r \int_r^\infty \frac{T(t, f)}{t^2} dt. \tag{6}$$

Conversely, if the integral (5) *converges and* (6) *holds, then f satisfies* (1).

PROOF. Suppose that f satisfies (1). Put

$$V(r) = \int_1^r \frac{T(t, f)}{t} dt, \tag{7}$$

then it is easily seen that

$$V(r) \sim V(2r). \tag{8}$$

From (7),

$$T(r, f) \log 2 \leqq \int_r^{2r} \frac{T(t, f)}{t} dt = V(2r) - V(r)$$

and so by (8)

$$T(r, f) = o(V(r)) \quad (r \to \infty). \tag{9}$$

We apply the method used by Hayman ([7], p. 130) to our case. Let ε be any positive number. Then by (9)

$$T(r, f)<\varepsilon V(r) \quad (r\geqq R_0(\varepsilon)),$$

so that

$$\log\frac{V(r_2)}{V(r_1)}=\int_{r_1}^{r_2}\frac{T(t, f)}{V(t)}\frac{dt}{t}<\varepsilon\log\frac{r_2}{r_1} \quad (r_2>r_1\geqq R_0).$$

That is, we have

$$V(r_2)<V(r_1)(r_2/r_1)^{\varepsilon},$$

which shows that the order of $V(r)$ is zero. As the order of $T(r, f)$ is equal to that of $V(r)$, we obtain that the order of f is zero. From this fact, we know that the integral (5) converges. Using that $T(r, f)$ is non-decreasing, we have

$$\text{(10)} \qquad T(r, f)\leqq r\int_r^{\infty}\frac{T(t, f)}{t^2}dt=r\int_r^{2r}\frac{T(t, f)}{t^2}dt+r\int_{2r}^{\infty}\frac{T(t, f)}{t^2}dt$$

$$\leqq T(2r, f)/2+r\int_r^{\infty}\frac{T(2t, f)}{t^2}dt/2.$$

Now, f satisfying (1), we have

$$r\int_r^{\infty}\frac{T(2t, f)}{t^2}dt\sim r\int_r^{\infty}\frac{T(t, f)}{t^2}dt,$$

so that we obtain (6) from (10).

Conversely, suppose that the integral (5) converges and that (6) is satisfied. As $T(r, f)$ is non-decreasing, we have

$$\text{(11)} \qquad r\int_r^{\infty}\frac{T(t, f)}{t^2}dt=r\int_r^{2r}\frac{T(t, f)}{t^2}dt+r\int_{2r}^{\infty}\frac{T(t, f)}{t^2}dt$$

$$\geqq T(r, f)/2+r\int_r^{\infty}\frac{T(2t, f)}{t^2}dt/2$$

so that we obtain (1), since we have

$$T(2r, f)\sim r\int_r^{\infty}\frac{T(2t, f)}{t^2}dt$$

from (6).

THEOREM 2. *If f satisfies* (1), *then the order of f is zero and so the integral* (5) *converges and for any $F_1, \cdots, F_{n+1}$ in X,*

$$T(r, f)\sim N(r, F_1, \cdots, F_{n+1})\sim r\int_r^{\infty}\frac{N(t, F_1, \cdots, F_{n+1})}{t^2}dt.$$

Conversely, if the integral (5) *converges and if there exist $F_1, \cdots, F_{n+1}$ in X such that*

$$N(r, F_1, \cdots, F_{n+1}) \sim r\int_r^\infty \frac{N(t, F_1, \cdots, F_{n+1})}{t^2} dt, \tag{12}$$

then f *satisfies* (1). *Here,*

$$N(r, F_1, \cdots, F_{n+1}) = \max_{1 \le j \le n+1} N(r, 0, F_j).$$

PROOF. Suppose first that f satisfies (1). Then, by Theorem 1, the order of f is zero, so that we may suppose without loss of generality that $f_0, f_1, \cdots, f_n$ are all of order zero. From this, we obtain that any F in X is of order zero. Now, for any $n+1$ elements $F_1, \cdots, F_{n+1}$ in X,

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \max_{1 \le j \le n+1} \log |F_j(re^{i\theta})| d\theta + O(1) \tag{13}$$

(see [4], p. 8). Applying Lemma 1 to F_j in place of h and using $n_j(r)$ and $N_j(r)$ instead of $n(r)$ and $N(r)$ respectively, we have

$$\log M(r, F_j) \leqq N_j(r) + r\int_r^\infty \frac{n_j(t)}{t^2} dt + O(\log r)$$

$$= r\int_r^\infty \frac{N_j(t)}{t^2} dt + O(\log r),$$

so that, by (13), we obtain

$$\begin{aligned} T(r, f) &\leqq \max_{1 \le j \le n+1} \log M(r, F_j) + O(1) \\ &\leqq r\int_r^\infty \frac{N(t, F_1, \cdots, F_{n+1})}{t^2} dt + O(\log r) \\ &< r\int_r^\infty \frac{T(t, f)}{t^2} dt + O(\log r) \end{aligned} \tag{14}$$

since for any F in X

$$N(r, 0, F) < T(r, f) + O(1). \tag{15}$$

Therefore, by Theorem 1, we obtain

$$T(r, f) \sim r\int_r^\infty \frac{N(t, F_1, \cdots, F_{n+1})}{t^2} dt \sim r\int_r^\infty \frac{T(t, f)}{t^2} dt.$$

Further, using the following inequalities obtained from (14) as in (10)

$$T(r, f) \leqq N(2r, F_1, \cdots, F_{n+1})/2 + r\int_r^\infty \frac{N(2t, F_1, \cdots, F_{n+1})}{t^2} dt/2 + O(\log r)$$

$$\leqq T(2r, f)/2 + r\int_r^\infty \frac{T(2t, f)}{t^2} dt/2 + O(\log r),$$

we have

$$T(r, f) \sim N(r, F_1, \cdots, F_{n+1}).$$

Conversely, we suppose that the integral (5) converges and that there exist $F_1, \cdots, F_{n+1}$ in X which satisfy (12). Then, for any F in X,

$$\int_1^\infty \frac{N(t, 0, F)}{t^2} dt < \infty$$

by (15), so that for $F_1, \cdots, F_{n+1}$, the integral

$$\int_1^\infty \frac{N(t, F_1, \cdots, F_{n+1})}{t^2} dt$$

converges. Applying Theorem 1 to $N(r, F_1, \cdots, F_{n+1})$ instead of $T(r, f)$, we have

$$N(r, F_1, \cdots, F_{n+1}) \sim N(2r, F_1, \cdots, F_{n+1})$$

by (12), so that $N(r, F_1, \cdots, F_{n+1})$ and so $N(r, 0, F_1), \cdots, N(r, 0, F_{n+1})$ are of order zero. Let Π_j be the canonical product of the zeros ($\neq 0$) of F_j, then Π_j has order and so genus zero ($j=1, \cdots, n+1$). Put

$$F_j = \Pi_j \cdot A_j \cdot z^{d_j} \quad (j=1, \cdots, n+1)$$

where d_j is the multiplicity of zero of F_j at the origin, then A_j is entire without zero. Let

$$g_j = F_j / A_1 \quad (j=1, \cdots, n+1),$$

then $g_1, \cdots, g_{n+1}$ are entire functions and have no common zero, and put

$$g = (g_1, \cdots, g_{n+1}),$$

then g is a system and

$$T(r, \tilde{f}) = T(r, g),$$

where $\tilde{f} = (F_1, \cdots, F_{n+1})$ (see [4], p. 8). Further, as

$$|T(r, f) - T(r, \tilde{f})| < O(1)$$

([4], p. 9), the integral

$$\int_1^\infty \frac{T(r, g)}{t^2} dt$$

converges. As

$$T(r, g_j/g_1) < T(r, g) + O(1) \qquad (j=2, \cdots, n+1)$$

([4], p. 10), the integral

$$\int_1^\infty \frac{T(t, g_j/g_1)}{t^2} dt$$

converges. This shows that A_j/A_1 is constant because it is entire without zero and Π_j/Π_1 is of order zero. Therefore, the order of g is zero, and so that of f is zero. Supposing that $f_0, f_1, \cdots, f_n$ are of order zero as in the former half of this proof, we can obtain (14) and using (15), we have

$$N(r, F_1, \cdots, F_{n+1})-O(1)<T(r, f)<r\int_r^{\infty}\frac{N(t, F_1, \cdots, F_{n+1})}{t^2}dt+O(\log r),$$

so that by (12)

$$T(r, f)\sim N(r, F_1, \cdots, F_{n+1})\sim r\int_r^{\infty}\frac{N(t, F_1, \cdots, F_{n+1})}{t^2}dt.$$

Therefore, we obtain

$$T(r, f)\sim r\int_r^{\infty}\frac{T(t, f)}{t^2}dt$$

and by Theorem 1,

$$T(r, f)\sim T(2r, f).$$

COROLLARY 1. *Suppose that* g *is transcendental meromorphic in* $|z|<\infty$. *If* g *satisfies* (2), *then the order of* g *is zero* (*and so the integral*

$$\int_1^{\infty}\frac{T(t, g)}{t^2}dt \tag{16}$$

converges) *and for any two values* $a, b\in\bar{C}$,

$$T(r, g)\sim N(r, a, b)\sim r\int_r^{\infty}\frac{N(t, a, b)}{t^2}dt.$$

Conversely, if the integral (16) *converges and if, for some two values* $a, b\in\bar{C}$,

$$N(r, a, b)\sim r\int_r^{\infty}\frac{N(t, a, b)}{t^2}dt, \tag{17}$$

then g *satisfies* (2).

PROOF. Let f_0 and f_1 be two entire functions without common zero for which

$$g=f_0/f_1, \quad \text{order of } f_j\leqq\text{order of } g \qquad (j=0, 1)$$

and put $f=(f_0, f_1)$, then

$$T(r, g)=T(r, f)$$

([4], p. 9). Further, let

$$X=\{f_0-\alpha f_1;\ \alpha\in C\}\cup\{f_1\equiv 0\cdot f_0+1\cdot f_1\},$$

then the elements in X are $\not\equiv 0$ and in general position. And, it holds

$$N(r, \alpha, g)=N(r, 0, f_0-\alpha f_1) \qquad (\alpha\in C),$$

$$N(r, g)=N(r, 0, f_1).$$

By these facts we have this corollary directly by Theorem 2.

REMARK 1. 1) In the proof of the sufficiency of Theorem 6 ([**7**]), Hayman uses that $f(z)$ and $f(z)-a$ have the same genus ([**7**], p. 131). However, let

$$f(z)=\prod_{n=2}^{\infty}\left(1+\frac{z}{n(\log n)^2}\right),$$

then the genus of $f(z)$ is zero but for $a\neq 0$, the genus of $f(z)-a$ is one (see [**3**], p. 34). In addition, his proof is very complicated. Here, we shall give another proof of [IV] applying Corollary 1.

Proof of [IV]. As g is entire now, we take $b=\infty$ in Corollary 1. First, we note that under the condition

$$\int_1^{\infty}\frac{T(t,\,g)}{t^2}dt<\infty,$$

(4) is equivalent to

(18) $$N(r,\,a)\sim r\int_r^{\infty}\frac{N(t,\,a)}{t^2}dt.$$

This is because, (18) is equivalent to

(19) $$N(r,\,a)\sim N(2r,\,a)$$

as in Theorem 1, and (19) is equivalent to (4) from the following inequalities:

$$n(r,\,a)\log 2\leqq\int_r^{2r}\frac{n(t,\,a)}{t}dt=N(2r,\,a)-N(r,\,a)\leqq n(2r,\,a)\log 2.$$

Now, suppose that g satisfies (2). Then, g has order and so genus zero and for any $a\in C$, (4) is satisfied by Corollary 1 and the above note. Conversely, suppose that g is of genus zero and for some $a\in C$, (4) is satisfied. As g has genus zero, it is of order 1 of minimal type at most ([**6**], p. 29) and as $N(r,\,a)$ is of order zero since (4) is equivalent to (19), putting

$$g(z)-a=z^d\Pi(z)A(z),$$

where $\Pi(z)$ is the canonical product of the zeros ($\neq 0$) of $g(z)-a$ and d is the multiplicity of zero of $g(z)-a$ at the origin, then $A(z)$ is entire without zero and so $A(z)$ must be a constant. This shows that $g(z)-a$ and so $g(z)$ has order zero, so that the integral

$$\int_1^{\infty}\frac{T(t,\,g)}{t^2}dt$$

converges. Therefore, (4) deduces (18) and so we have (2) by Corollary 1.

2) This shows that Theorem 2 and Corollary 1 are solutions to Problem 2.

According to Valiron ([11], [12]), we give the following

DEFINITION 1. When the integral (5) converges, we say that f has V-regular growth if and only if for any F in X

$$r\int_r^{\infty}\frac{n(t,0,F)}{t^2}dt=o(T(r,f))\qquad (r\to\infty). \tag{20}$$

THEOREM 3. *If f satisfies (1), then f has V-regular growth. Conversely, if there exist $F_1, \cdots, F_{n+1}$ in X for which (20) holds under the condition that the integral (5) converges, then f satisfies (1).*

PROOF. Suppose first that f satisfies (1). To begin with, we prove that for any F in X

$$n(r,0,F)=o(T(r,f))\qquad (r\to\infty).$$

Indeed, for any positive k,

$$n(r,0,F)k(\log 2)\leqq\int_r^{2^k r}\frac{n(t,0,F)}{t}dt\leqq N(2^k r,0,F)\leqq T(2^k r,f)+O(1),$$

so that we have

$$\limsup_{r\to\infty}\frac{n(r,0,F)}{T(r,f)}\leqq\limsup_{r\to\infty}\frac{T(2^k r,f)}{T(r,f)}\frac{1}{k(\log 2)}=\frac{1}{k(\log 2)}$$

which tends to zero as $k\to\infty$. Therefore, we obtain

$$r\int_r^{\infty}\frac{n(t,0,F)}{t^2}dt=o\Big(r\int_r^{\infty}\frac{T(t,f)}{t^2}dt\Big)=o(T(r,f))\qquad (r\to\infty)$$

by Theorem 1.

Conversely, suppose that the integral (5) converges and for $F_1, \cdots, F_{n+1}$ in X

$$r\int_r^{\infty}\frac{n(t,0,F_j)}{t^2}dt=o(T(r,f))\qquad (r\to\infty,\ j=1,\cdots,n+1).$$

As

$$r\int_r^{\infty}\frac{N(t,F_1,\cdots,F_{n+1})-N(r,F_1,\cdots,F_{n+1})}{t^2}dt$$

$$\leqq r\int_r^{\infty}\frac{\max_{1\leqq j\leqq n+1}\{N(t,0,F_j)-N(r,0,F_j)\}}{t^2}dt$$

$$\leqq\sum_{j=1}^{n+1}r\int_r^{\infty}\frac{N(t,0,F_j)-N(r,0,F_j)}{t^2}dt=\sum_{j=1}^{n+1}r\int_r^{\infty}\frac{n(t,0,F_j)}{t^2}dt$$

$$=o(T(r,f))\qquad (r\to\infty),$$

we have by using (12)

$$T(r, f)\leqq r\int_r^\infty \frac{N(t, F_1, \cdots, F_{n+1})}{t^2}dt+O(\log r)$$

$$=N(r, F_1, \cdots, F_{n+1})+r\int_r^\infty \frac{N(t, F_1, \cdots, F_{n+1})-N(r, F_1, \cdots, F_{n+1})}{t^2}dt$$

$$+O(\log r)=N(r, F_1, \cdots, F_{n+1})+o(T(r, f)) \qquad (r\to\infty),$$

so that we obtain

$$T(r, f)\sim N(r, F_1, \cdots, F_{n+1})\sim r\int_r^\infty \frac{N(t, F_1, \cdots, F_{n+1})}{t^2}dt.$$

This shows that f satisfies (1) by Theorem 2.

REMARK 2. Applying this result to the case of meromorphic functions as in Corollary 1, we obtain solutions of Problems 1 and 3.

4. Asymptotic points of f satisfying (1).

Let f and X be as in § 1. We recall the following definition of asymptotic points of f at ∞.

DEFINITION 2. We say that $\alpha=\alpha_0:\alpha_1:\cdots:\alpha_n$ belongs to $A(f, \infty)$ if and only if there exists a curve Γ: $z=z(t)$ $(0\leqq t<1)$ in $|z|<\infty$ satisfying the following conditions:

i) $\lim_{t\to 1} z(t)=\infty$,

ii) $\lim_{t\to 1}\|\alpha f(z(t))\|=0$,

where

$$\|\alpha f(z)\|=|(\alpha, f(z))|/|\alpha|\,|f(z)|,\ (\alpha, f(z))=\sum_{j=0}^n \alpha_j f_j(z),$$

$$|\alpha|=(\sum_{j=0}^n |\alpha_j|^2)^{1/2},\ |f(z)|=(\sum_{j=0}^n |f_j(z)|^2)^{1/2} \quad ([\mathbf{10}]).$$

It is easily seen that the concept of asymptotic points for systems in this definition is a natural generalization of "asymptotic values" for meromorphic functions.

Suppose that f satisfies (1). Then, the order of f is zero by Theorem 1 and we may suppose without loss of generality that $f_0, \cdots, f_n$ are all of order zero. In this situation, as a generalization of [IX], we have

THEOREM 4. *If there exist n elements*

$$F_j=\alpha_{0j}f_0+\cdots+\alpha_{nj}f_n$$

in X such that

$$\delta(\alpha_j)>0 \qquad (j=1, \cdots, n),$$

where $\alpha_j=\alpha_{0j}:\cdots:\alpha_{nj}$ *and*

$$\delta(\alpha_j)=1-\limsup_{r\to\infty}\frac{N(r,0,F_j)}{T(r,f)},$$

then it holds that

(i) *for any* F_0 *in* $X-\{F_1,\cdots,F_n\}$,

$$T(r,f)\sim N(r,0,F_0)\sim\log M(r,F_0);$$

(ii)
$$\liminf_{\substack{z\to\infty\\ z\notin S}}\frac{-\log\|\alpha_j f(z)\|}{T(|z|,f)}\geqq\delta(\alpha_j)\qquad(j=1,\cdots,n),$$

where S *is a slim set.*

PROOF. (i) As the order of any element of X is zero, by Lemma 1 and Theorem 1 we have for $j=1,\cdots,n$

$$\log M(r,F_j)\leqq r\int_r^\infty\frac{N(t,0,F_j)}{t^2}dt+O(\log r)$$
$$=(1-\delta(\alpha_j)+o(1))T(r,f)<T(r,f)\qquad(r\geqq r_0).$$

For $F_0\in X-\{F_1,\cdots,F_n\}$,

$$N(r,0,F_0)-O(1)\leqq T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\max_{0\leq j\leq n}\log|F_j(re^{i\theta})|\,d\theta+O(1)$$
$$\leqq\max_{0\leq j\leq n}\log M(r,F_j)+O(1)$$
$$=\log M(r,F_0)+O(1)\qquad(r\geqq r_0)$$
$$\leqq N(r,0,F_0)+r\int_r^\infty\frac{n(t,0,F_0)}{t^2}dt+O(\log r)\quad(r\geqq r_0)$$
(by Lemma 1)
$$=N(r,0,F_0)+o(T(r,f))\quad(r\to\infty)\quad\text{(by Theorem 3)},$$

which shows that (i) holds.

(ii) Since f satisfies (1), we have

$$\log M(r,F_0)\sim\log M(2r,F_0)$$

for $F_0\in X-\{F_1,\cdots,F_n\}$ from (i), so that by Lemma 2

$$\log|F_0(z)|\sim\log M(r,F_0)\sim T(r,f)$$

as $z=re^{i\theta}\to\infty$ outside a slim set S. Now,

$$-\log\|\alpha_j f(z)\|=\log\frac{|\alpha_j|\,|f(z)|}{|F_j(z)|}=\log|f(z)|-\log|F_j(z)|-O(1)$$

$$\geqq \log|F_0(z)| - r\int_r^\infty \frac{N(t, 0, F_j)}{t^2}dt - O(\log r)$$

$$\geqq (1-o(1))T(r, f) - (1-\delta(\alpha_j)+o(1))T(r, f)$$

$$=(\delta(\alpha_j)-o(1))T(r, f)$$

outside S and $r=|z|\geqq r_0$, so that we have

$$\liminf_{\substack{z\to\infty \\ z\notin S}} \frac{-\log\|\alpha_j f(z)\|}{T(|z|, f)} \geqq \delta(\alpha_j).$$

REMARK 3. $\alpha_j \in A(f, \infty)$ $(j=1, \cdots, n)$.

DEFINITION 3. A countable set of circles in the plane is said to form an E-set if the sum of radii of those circles in

$$r(1-\beta(r)) < |z| < r(1+\beta(r))$$

is at most $Kr\beta(r)^2$, K being constant, where $\beta(r)$ is any function decreasing to zero as $r\to\infty$ with $r\beta(r)>1$ (see [**13**], p. 64).

We note that if E is an E-set, there is a receding path Γ from zero to ∞ lying eventually outside E and an increasing sequence $\{r_n\}$ to ∞ such that $\{|z|=r_n\}\subset E^c (n=1, 2, \cdots)$ and that if $E_1, \cdots, E_m$ are E-sets, then $E_1\cup\cdots\cup E_m$ is also an E-set.

LEMMA 3. *Let h be an entire function of order zero, then*

$$\log|h(re^{i\theta})| = N(r) + \eta r\int_r^\infty \frac{n(t)}{t^2}dt + O(\log r)$$

outside an E-set, where $n(r)$, $N(r)$ are as in Lemma 1 *and*

$$-K'\beta(r)^{-1} < \eta < 1 \qquad (K'\text{: positive constant})$$

(see [**13**], p. 64).

THEOREM 5. *Suppose that f satisfies* (1). *If there exists an element*

$$F_0 = \alpha_0 f_0 + \alpha_1 f_1 + \cdots + \alpha_n f_n$$

in X such that

$$\delta(\alpha) > 0 \qquad (\alpha = \alpha_0 : \alpha_1 : \cdots : \alpha_n),$$

then

$$\liminf_{\substack{z\to\infty \\ z\notin E\text{-set}}} \frac{-\log\|\alpha f(z)\|}{T(|z|, f)} \geqq \delta(\alpha).$$

PROOF. We may suppose without loss of generality that $f_0, f_1, \cdots, f_n$ are all of order zero since the order of f is zero by Theorem 1. Now, using the relation

$$(21)\qquad |\log|f(z)|-\max_{0\leq j\leq n}\log|f_j(z)||\leqq n/2\,,$$

we have from Lemma 1

$$(22)\qquad -\log\|\alpha f(z)\|=\log\frac{|\alpha||f(z)|}{|F_0(z)|}$$

$$\geqq \max_{0\leq j\leq n}\log|f_j(re^{i\theta})|-r\int_r^\infty\frac{N(t,\,0,\,F_0)}{t^2}dt-O(\log r)\quad (z=re^{i\theta})\,.$$

Further, let

$$\beta(r)=\left(\max_{[r,\infty)}\frac{\sum_{j=0}^{n}s\int_s^\infty\frac{n(t,\,0,\,f_j)}{t^2}dt}{T(s,\,f)}\right)^{1/2},$$

then $\beta(r)$ tends to zero as $r\to\infty$ by Theorem 3 and $r\beta(r)>1$ for sufficiently large every r since f is of order zero and at least one of f_j has infinitely many zeros. (For example, apply Theorem 2 to $f_0, f_1, \cdots, f_n$.) By making use of this $\beta(r)$, we apply Lemma 3 to f_j $(j=0, 1, \cdots, n)$. Then, we have for a positive constant K''

$$(23)\qquad \log|f_j(re^{i\theta})|\geqq N_j(r)-\frac{K''}{\beta(r)}r\int_r^\infty\frac{n_j(t)}{t^2}dt-O(\log r)$$

$(z=re^{i\theta}\notin E_j$, an E-set), so that from (22) we have for $z=re^{i\theta}\notin E_0\cup E_1\cup\cdots\cup E_n \equiv E$

$$-\log\|\alpha f(z)\|\geqq N(r,\,f_0,\,\cdots,\,f_n)-\frac{K''}{\beta(r)}\sum_{j=0}^{n}r\int_r^\infty\frac{n(t,\,0,\,f_j)}{t^2}dt$$

$$-r\int_r^\infty\frac{N(t,\,0,\,F_0)}{t^2}dt-O(\log r)$$

and by Theorems 2 and 1

$$\geqq(1-o(1))T(r,\,f)-K''\beta(r)T(r,\,f)-(1-\delta(\alpha)+o(1))T(r,\,f)$$

$(r\to\infty)$. This shows that

$$\liminf_{\substack{z\to\infty\\ z\notin E}}\frac{-\log\|\alpha f(z)\|}{T(|z|,\,f)}\geqq\delta(\alpha)\,,$$

where E is an E-set. This completes the proof.

REMARK 4. $\alpha\in A(f,\,\infty)$.

COROLLARY 2. *Suppose that f satisfies* (1) *and that $f_0, f_1, \cdots, f_n$ are all of order zero. Then for any $F_1, \cdots, F_{n+1}$ in X*

$$\log\Big(\sum_{j=1}^{n+1}|F_j(z)|^2\Big)^{1/2}\sim T(r,\,f)\qquad (z\to\infty,\ z\notin E\text{-set})\,.$$

PROOF. To begin with, we prove

(24) $$\log\left(\sum_{j=0}^{n}|f_j(z)|^2\right)^{1/2}\sim T(r, f) \qquad (z\notin E\text{-set},\ z\to\infty).$$

In fact, from (23) and from the inequality

$$\log|f_j(re^{i\theta})|\leqq N_j(r)+r\int_r^{\infty}\frac{n_j(t)}{t^2}dt+O(\log r),$$

we have for $z\notin E$-set and a positive constant $\tilde{K}'$

$$N(r, f_0, \cdots, f_n)-\tilde{K}'\beta(r)T(r, f)-O(\log r)\leqq \max_{0\leq j\leq n}\log|f_j(re^{i\theta})|$$

$$\leqq N(r, f_0, \cdots, f_n)+\sum_{j=0}^{n}r\int_r^{\infty}\frac{n(t, 0, f_j)}{t^2}dt+O(\log r),$$

so that by Theorems 2 and 3, we obtain (24) by using (21). Here, using the inequality

$$|\log\left(\sum_{j=1}^{n+1}|F_j(z)|^2\right)^{1/2}-\log\left(\sum_{j=0}^{n}|f_j(z)|^2\right)^{1/2}|<O(1)$$

for all z, we have the result.

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