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On the group of diffeomorphisms commuting with an elliptic operator

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Introduction.

The group of diffeomorphisms preserving a certain structure of a manifold is often a Lie transformation group. For example,

(1) the group of holomorphic transformations of a bounded domain in $\boldsymbol{C}^n$ (or of a compact complex manifold),

(2) the group of isometries of a Riemannian manifold,

(3) the group of affine transformations on a manifold with an affine connection, and

(4) the group of automorphisms of a compact almost complex manifold are all Lie transformation groups.

The purpose of this paper is to give a generalization of the example (2). The main results are Theorem 2 and Theorem 3 which will be stated in Section 1.

Let G be a group of diffeomorphisms of a connected manifold M. To see that G is a Lie transformation group, it is, in general, enough to apply the following Theorem A or B (on the above examples see [**4**]). For our case we apply Theorem A to prove Theorem 2 and Theorem B to prove Theorem 3.

THEOREM A [**6**, p. 208]. *If G is a locally compact topological transformation group of M, then G, with the compact-open topology, is a Lie transformation group of M.*

THEOREM B [**8**, p. 103]. *Let S be the set of all vector fields X on M which generate global one-parameter groups $\{\varphi_{X,t}\}_{t\in\boldsymbol{R}}$ of transformations of M such that $\varphi_{X,t}\in G$ for all $t\in\boldsymbol{R}$. If S generates a finite dimensional Lie algebra, then the group G is a Lie transformation group of M and S is the Lie algebra of G.*

In our case the underlying manifold must be compact, and it is rather easy to show the finite dimensionality of the given group. It seems interesting that the eigenfunction-expansion theorem for elliptic operators can be applied to prove the compactness of the group.

§ 1. Statement of theorems.

Throughout this note M denotes an n-dimensional compact connected smooth manifold without boundary, and $P: C^{\infty}(M)\to C^{\infty}(M)$ an elliptic differential operator of order $m>0$ with smooth coefficients, where $C^{\infty}(M)$ is the space of all complex-valued smooth functions on the manifold M.

Let $G(P)$ be the group of all diffeomorphisms of M commuting with the elliptic operator P, i.e., it consists of diffeomorphisms φ such that $\varphi^{*}\circ P(f)=P\circ\varphi^{*}(f)$ for all $f\in C^{\infty}(M)$, where $(\varphi^{*}f)(x)=f(\varphi(x))$. Let $X(P)$ be the Lie algebra of all smooth vector fields on M commuting with P, i.e., it consists of smooth vector fields X such that $X\circ P=P\circ X$, where we regard the vector field X as a first order differential operator on M.

We prove in this note the following theorems.

THEOREM 1. *If a smooth map $\varphi: M\to M$ commutes with P, then φ must be a diffeomorphism.*

THEOREM 2. (i) *The group $G(P)$, with the compact-open topology, is a compact Lie transformation group of M.*

(ii) *The Lie algebra $X(P)$ is finite dimensional and contains the Lie algebra of $G(P)$.*

(iii) *If the principal symbol $\sigma(P)$ of the operator P is real* (*in this case P must be of even order*), *then the Lie algebra of $G(P)$ is isomorphic with $X(P)$.*

Let M be endowed with a Riemannian metric and P its Laplace operator, then it is well known that P is an elliptic differential operator of order two with real principal symbol. Also in this case it turns out that the group $G(P)$ coincides with the group of isometries and the Lie algebra $X(P)$ consists of all Killing vector fields. These can be proved by a similar way as the proof of Proposition 3 below. The principal symbol in this case is the metric tensor on $T^{*}(M)$. As a corollary of Theorem 2 we have

COROLLARY. *The group of isometries of a compact Riemannian manifold is a compact Lie transformation group.*

Thus our Theorem 2 may be regarded as a generalization of the example (2).

THEOREM 3. *Let $\{Y_i\}$ $(1\leqq i\leqq j)$ be a finite set of smooth vector fields on M such that at any point $x\in M$ the tangent space $T_x(M)$ is spanned by $\{(Y_i)_x\}_{i=1}^{j}$, then the group of diffeomorphisms of M commuting with all Y_i, with the compact-open topology, is a compact Lie transformation group of M. Its Lie algebra consists of all vector fields Y such that $[Y, Y_i]=0$ $(1\leqq i\leqq j)$.*

§ 2. Completeness of eigenfunctions.

In this section we review several properties of elliptic differential operators on compact manifolds, which are needed later.

Let P be as in §1. Put

$$E_\lambda(P)=\{f\in C^\infty(M):(P-\lambda)^l f=0,\quad \text{for some integer } l>0\}.$$

An element $f\in E_\lambda(P)$, $f\neq 0$, is said to be a generalized eigenfunction of P corresponding to the eigenvalue λ, and the space $E_\lambda(P)$, $E_\lambda(P)\neq\{0\}$, the generalized eigenspace corresponding to an eigenvalue λ. All the spaces $E_\lambda(P)$ are finite dimensional because of the compactness of the manifold M.

Let $H_k(M)$ be the Sobolev space on M of order $k\geqq 0$ with a suitably chosen inner product, and denote the norm by $\|\cdot\|_k$. For the definition of Sobolev spaces on manifolds, see [7]. In particular, the space $H_0(M)$ consists of all square-integrable complex-valued functions on M with respect to a smooth measure. We denote by $\mathrm{Sp}(P)$ the spectrum of the closed extension of P in $H_0(M)$. The set $\mathrm{Sp}(P)$ does not depend on the choice of an inner product in $H_0(M)$, and is closed in $\boldsymbol{C}$.

PROPOSITION 1. *Let $m>0$ be the order of P. For any integer $k>0$, there exists a constant $C_k>0$ such that*

$$\|u\|_{mk}\leqq C_k(\|P^k u\|_0+\|u\|_0),\qquad u\in H_{mk}(M).$$

PROPOSITION 2. *If the principal symbol of P is real, then*

(i) *the spectrum $\mathrm{Sp}(P)$ consists only of countably many isolated eigenvalues of finite multiplicities: $\mathrm{Sp}(P)=\{\lambda\in\boldsymbol{C}: E_\lambda(P)\neq\{0\}\}$,*

(ii) *the algebraic sum $\sum_{\lambda\in\mathrm{Sp}(P)} E_\lambda(P)$ is dense in the Sobolev space $H_k(M)$ for any $k\geqq 0$. Consequently by means of the Sobolev lemma, the space $\sum_{\lambda\in\mathrm{Sp}(P)} E_\lambda(P)$ is dense in $C^\infty(M)$ with respect to C^∞-topology.*

(The Sobolev lemma says that $H_{k+[n/2]+1}(M)\subset B^k(M)$ and the inclusion map is continuous, where $B^k(M)$ is the Banach space of C^k-functions on M with sup-norm up to k-th derivatives. Especially this implies that $\bigcap_{k\geqq 0} H_k(M)=C^\infty(M)$.)

The proofs of these propositions can be found, for instance, in [2] in a more general framework. In our case the proof of the propositions are simpler, because no boundary conditions are taken into account. It should, however, be noticed that for any integer $l>0$

$$\sum_{\lambda\in\mathrm{Sp}(P)} E_\lambda(P)=\sum_{\lambda\in\mathrm{Sp}(P^l)} E_\lambda(P^l),$$

if the principal symbol of P is real. This equality implies the second part of Proposition 2 for any $k\geqq 0$. (In [2], the second part of Proposition 2 follows in case of $k=0$ or the order of P.)

Let $\bar{P}$ be an elliptic operator defined by $(\bar{P}f)(x)=\overline{(P\bar{f})(x)}$, where $^-$ in the right hand side means the complex conjugate.

Then $A=\bar{P}\circ P$ is also elliptic, of order $2m$, and satisfies the assumption in

Proposition 2. From the definitions of $G(P)$ and $X(P)$ we have at once

LEMMA 1. (i) *$G(P)$ is a subgroup of $G(A)$ and $X(P)$ is a subalgebra of $X(A)$,*

(ii) *each space $E_\lambda(A)$ is invariant under $G(P)$ and $X(P)$.*

§ 3. Smooth mapping and elliptic operator.

We denote the value of the principal symbol $\sigma(P)$ of the differential operator P at a cotangent vector $\xi\in T_x^*(M)$ by $\sigma(P)_x(\xi)$. It is defined by

$$\sigma(P)_x(\xi)=\frac{1}{m!}P(f^m)(x),$$

where m is the order of P and f is a smooth function on M such that $f(x)=0$ and $(df)_x=\xi$.

Before proving our theorems, we give a more general result than Theorem 1:

PROPOSITION 3. *Let M, P be as above. Also let N be a connected manifold, and $Q: C^\infty(N)\to C^\infty(N)$ an elliptic differential operator on N with smooth coefficients. If there exists a smooth map $\varphi: M\to N$ such that for any $f\in C^\infty(N)$*

$$P\circ\varphi^*(f)=\varphi^*\circ Q(f),$$

then,

(i) *the orders of P and Q are equal, and*

(ii) *the map φ is a submersion.*

PROOF. Let m and m' be the order of P and Q, respectively. Given a point $x\in M$ and a cotangent vector $0\neq\xi\in T_{\varphi(x)}^*(N)$, we can take a smooth function $f\in C^\infty(N)$ such that $f(\varphi(x))=0$ and $(df)_{\varphi(x)}=\xi$. Then from the assumption we have

$$0=P((\varphi^*f)^{m+1})(x)=P\circ\varphi^*(f^{m+1})(x)=\varphi^*\circ Q(f^{m+1})(x)=Q(f^{m+1})(\varphi(x)).$$

This shows that $m\geqq m'$.

Assume that $m>m'$. Then the following equality holds:

$$0=Q(f^m)(\varphi(x))=P((\varphi^*f)^m)(x)=m!\,\sigma(P)_x(d(\varphi^*f)_x).$$

From this and the ellipticity of the operator P we see that the map $\varphi^*: T_{\varphi(x)}^*(N)\to T_x^*(M)$ must be identically zero. Hence $d\varphi=0$, so that φ is a constant map. Let $\varphi(x)\equiv y_0$, and take an $f\in C^\infty(N)$ such that $f(y_0)=0$ and $df_{y_0}\neq 0$. Then, we have

$$0=P((\varphi^*f)^{m'})(x)=\varphi^*\circ Q(f^{m'})(x)=m'!\,\sigma(Q)_{y_0}(df_{y_0}),$$

which contradicts the ellipticity of the operator Q. Hence the orders of P and Q must be equal.

Let $x\in M$ and $f\in C^\infty(N)$, $f(\varphi(x))=0$ and $df_{\varphi(x)}\neq 0$. As before we have

$$\sigma(P)_x((d\varphi^* f)_x)=\sigma(Q)_{\varphi(x)}(df_{\varphi(x)})\neq 0\,,$$

which shows together with the ellipticity of the operator P that the map $\varphi^*: T^*_{\varphi(x)}(N)\to T^*_x(M)$ is injective. Hence φ is a submersion.

Concerning this proposition, see [5] and [10]. In these, the case that the operators P and Q are Laplace operators is discussed. The map φ, there, is a Riemannian submersion.

§ 4. Proof of theorems.

4.1. Proof of Theorem 1. By Proposition 3 the map φ is an open mapping, and the compactness and connectedness of M imply that φ is surjective. Hence the map $\varphi^*: C^\infty(M)\to C^\infty(M)$ is injective, and so from Lemma 1 $\varphi^*(C^\infty(M))$ contains all generalized eigenspaces $E_\lambda(A)$ of $A=\bar{P}\circ P$. Consequently by Proposition 2 $\varphi^*(C^\infty(M))$ separates any pair of points of M. This shows that the map φ is injective.

4.2. To prove Theorem 2 we shall here recall some well-known facts about the compact-open topology for a group of homeomorphisms in the form of propositions (see [8, Appendix]):

PROPOSITION 4. *Let X be a locally compact Hausdorff space and G its homeomorphism group, then the compact-open topology for the group G is the weakest topology making the map $(\varphi, p)\mapsto\varphi(p)$ of $G\times X\to X$ continuous. If, furthermore, X is locally connected, then G becomes a topological group with the compact-open topology.*

PROPOSITION 5. *Let X be a compact metric space, then,*

(i) *the compact-open topology for the group of homeomorphisms of X coincides with the topology of the uniform convergence,*

(ii) *a sequence $\{\varphi_n\}_{n\geq 1}$ of homeomorphisms of X converges uniformly to a homeomorphism φ of X, if and only if for every continuous function f on X the sequence $\{\varphi_n^*(f)\}$ converges to the function $\varphi^*(f)$ uniformly on X.*

PROPOSITION 6. *Let M be a compact smooth manifold and $\{\varphi_n\}_{n\geq 1}$ a sequence of diffeomorphisms of M, then the sequence $\{\varphi_n\}$ converges uniformly to a diffeomorphism φ, if and only if for every $f\in C^\infty(M)$ the sequence $\{\varphi_n^*(f)\}$ converges to $\varphi^*(f)$ uniformly on M.*

4.3. Proof of Theorem 2, (i). According to Propositions 4, 5 and 6, the group $G(P)$ becomes a topological transformation group of M with the compact-open topology. If we can conclude that the group $G(P)$ is compact, then the proof of the first part of Theorem 2 reduces to Theorem A in Introduction. So we shall show this below.

4.4. Compactness of $G(P)$. The proof is accomplished by showing the following Lemmas.

Let $\|\circ\|_{k,k'}$ denote the norm of linear operators from $H_k(M)$ to $H_{k'}(M)$.

LEMMA 2. *Let k_0 be an integer such that $2mk_0>[n/2]+1$, where $n=\dim M$ and $m=$order of P. Then for each integer $k\geqq 0$,*

$$\sup_{\varphi\in G(A)}\|\varphi^*\|_{2m(k+k_0),2mk}<+\infty,$$

where φ^* is regarded as an operator from $H_{2m(k+k_0)}(M)$ to $H_{2mk}(M)$, and $A=\bar{P}\circ P$.

PROOF. By Proposition 1 and the Sobolev lemma we have the following inequalities:

$$\begin{aligned}\|\varphi^*(f)\|_{2mk}&\leqq C_1(\|A^k\circ\varphi^*(f)\|_0+\|\varphi^*(f)\|_0)\\&=C_1(\|\varphi^*\circ A^k(f)\|_0+\|\varphi^*(f)\|_0)\\&\leqq C_2(\sup_{x\in M}|(A^kf)(x)|+\sup_{x\in M}|f(x)|)\\&\leqq C_3(\|A^kf\|_{2mk_0}+\|f\|_{2mk_0})\leqq C_4\|f\|_{2m(k+k_0)}.\end{aligned}$$

Here the constants C_i depend neither on $f\in H_{2m(k+k_0)}(M)$ nor on $\varphi\in G(A)$, and this shows the lemma.

As the manifold M is compact, the compact-open topology for the group $G(P)$ is metrizable. Therefore it is sufficient to show that $G(P)$ is sequencially compact. To prove this we use the following

LEMMA 3. *Let $\{T_n\}$ be a sequence of bounded linear operators defined on a normed space H into a normed space H'. Suppose that $\{T_n\}$ is uniformly bounded and $\{T_n\}$ converges pointwisely on a dense subspace, then $\{T_n\}$ converges pointwisely on all of H to a bounded operator $T: H\to H'$.*

This is a standard fact in functional analysis, so the proof is omitted.

Let $\{\varphi_i\}_{i\geqq 1}$ be a sequence in $G(P)$. For each fixed integer $k\geqq 0$ we regard $\{\varphi_i^*\}$ as a sequence of bounded operators from $H_{2m(k+k_0)}(M)$ to $H_{2mk}(M)$. Then we have

LEMMA 4. *There exists a subsequence $\{\psi_i\}$ of the sequence $\{\varphi_i\}$ such that $\{\psi_i^*\}$ converges pointwisely to a bounded operator ψ: $H_{2m(k+k_0)}(M)\to H_{2mk}(M)$.*

PROOF. By Lemma 2 we see that $\{\varphi_i^*\}$ is uniformly bounded as operators from $H_{2m(k+k_0)}(M)$ to $H_{2mk}(M)$. Also we see that each space $E_\lambda(A)$ ($A=\bar{P}\circ P$) is invariant under the operators φ_i^*. As each space $E_\lambda(A)$ is finite dimensional, the sequence $\{\varphi_i^*\}$ is a bounded set in the finite dimensional space $\mathrm{Hom}(E_\lambda(A), E_\lambda(A))$. Therefore, by applying the diagonal process of choice to the sequence $\{\varphi_i^*\}$, we can obtain a subsequence $\{\psi_i\}$ of $\{\varphi_i\}$ such that $\{\psi_i^*\}$ converges pointwisely on the subspace $\sum_{\lambda\in\mathrm{Sp}(A)}E_\lambda(A)$. Here we use the fact that the set $\mathrm{Sp}(A)$ consists of countably many elements. Therefore, by Lemma 3 the subsequence $\{\psi_i^*\}$ actually converges pointwisely to a bounded operator ψ: $H_{2m(k+k_0)}(M)\to H_{2mk}(M)$.

By using Lemma 4 one after another for $k=0, 1, 2, \cdots$ we again apply the diagonal process of choice to get a subsequence $\{\sigma_i\}$ of $\{\varphi_i\}$ such that $\{\sigma_i^*\}$ converges on the space $\bigcap_{k\geq 0} H_k(M)=C^\infty(M)$. For $f\in C^\infty(M)$ put $\sigma(f)=\lim \sigma_i^*(f)$. Then we can easily show the following

LEMMA 5. (i) *For any* $f\in C^\infty(M)$ *the function* $\sigma(f)$ *is also smooth, and the sequence* $\{\sigma_i^*(f)\}$ *converges to* $\sigma(f)$ *with respect to* C^∞*-topology,*

(ii) *for any* $f, g\in C^\infty(M)$ $\sigma(fg)=\sigma(f)\sigma(g)$,

(iii) $\sigma\circ P=P\circ\sigma$.

By the same argument for $\{\sigma_i^{-1}\}$ as for $\{\varphi_i\}$, we see that the map σ is an isomorphism of the ring $C^\infty(M)$. Therefore, there exists a diffeomorphism $\varphi\in G(P)$ such that $\sigma=\varphi^*$ (see the remark below), and σ_i converges to φ with respect to the compact-open topology. This shows, together with Proposition 6, the compactness of $G(P)$.

REMARK. The diffeomorphism φ in the above proof is obtained as follows: let $I_x=\{f\in C^\infty(M): f(x)=0\}$, then I_x is a maximal ideal of the ring $C^\infty(M)$. Conversely, any maximal ideal of $C^\infty(M)$ coincides with an I_x for some $x\in M$. Also $\sigma(I_x)$ is a maximal ideal of $C^\infty(M)$, so that there exists a unique point $y\in M$ such that $\sigma(I_x)=I_y$. The desired map φ is defined by $\varphi(x)=y$. (For details see [**1**, 11-14].)

4.5. Proof of Theorem 2, (ii) and (iii). For a vector field X on M we denote by $\{\varphi_{X,t}\}_{t\in\boldsymbol{R}}$ the one-parameter group of transformations of M generated by X.

LEMMA 6. *Let* $X\in X(A)$, *then* $(\varphi_{X,t})^*\circ A=A\circ(\varphi_{X,t})^*$ *for any* $t\in\boldsymbol{R}$.

PROOF. Let $\{u_i\}$ $(1\leqq i\leqq \dim E_\lambda(A))$ be a base of $E_\lambda(A)$ and $X(u_i)=\sum_j c_{ij}u_j$. Put

$$h_i(t, x)=(A\circ(\varphi_{X,t})^*-(\varphi_{X,t})^*\circ A)u_i(x)=[A, (\varphi_{X,t})^*]u_i(x),$$

then

$$\left(\frac{d}{dt}h_i\right)(t, x)=[A, (\varphi_{X,t})^*]X(u_i)(x)=\sum_j c_{ij}h_j(t, x).$$

Since $h_i(0, x)=0$ $(1\leqq i\leqq \dim E_\lambda(A))$, all h_i must be identically zero. Therefore, A commutes with $(\varphi_{X,t})^*$ for any $t\in\boldsymbol{R}$ on all eigenspaces $E_\lambda(A)$. Hence by Proposition 2 the operator A commutes with $(\varphi_{X,t})^*$ for any $t\in\boldsymbol{R}$ on the space $C^\infty(M)$.

Lemma 6 and the first part of Theorem 2 imply the second part of Theorem 2 at once. The third part of Theorem 2 follows also from Lemma 6, because the same argument as for $A=\bar{P}\circ P$ holds for P itself in the proof of Lemma 6.

4.6. Proof of Theorem 3. The principal symbol of the differential operator $D=\sum_{1\leq i\leq j} Y_i^2$ is $\sigma(D)_x(\xi)=\sum_i\langle\xi, (Y_i)_x\rangle^2$, $\xi\in T_x^*(M)$. Hence, by the assumption, D is elliptic. So the group is a subgroup of the compact Lie group $G(D)$, and in fact we can show that the group is a closed subgroup of $G(D)$ by the same argument as the proof of the compactness of $G(D)$. Finally, it is immediate from Theorem B to determine the Lie algebra of this group.

4.7. Finally we give a proposition which implies the finite dimensionality of the Lie algebra $X(A)$.

PROPOSITION 7. *The representation of $X(A)$ in the finite dimensional space $\sum_{|\lambda|\leq s} E_\lambda(A)$ is faithful for a sufficiently large $s>0$.*

PROOF. Let $\{s_i\}_{i=1}^n$ be a smooth local frame of the cotangent bundle $T^*(M)$ defined on an open set $U\subset M$. Also let $\{f_i\}_{i=1}^n$ be a family of smooth functions on M such that $(df_i)_{x_0}=s_i(x_0)$ at a point $x_0\in U$. Then $\{df_i\}_{i=1}^n$ is also a local frame of $T^*(M)$ on an open set $V\subset U$. If we take a number $s>0$ sufficiently large, then by Proposition 2 there exist generalized eigenfunctions $f_{\lambda,i}\in E_\lambda(A)$ ($i=1, \cdots, n$, $|\lambda|\leqq s$) such that $\|df_i-\sum_{|\lambda|\leq s} df_{\lambda,i}\|<\delta$ for any $\delta>0$, where $\|\cdot\|$ is an arbitrarily taken norm on $T^*(M)$. Therefore, if δ is sufficiently small, then $\{\sum_{|\lambda|\leq s} df_{\lambda,i}\}_{i=1}^n$ is also a local frame of $T^*(M)$ on an open set $W\subset V$. Hence any $X\in X(A)$ satisfying $X(\sum_{|\lambda|\leq s} f_{\lambda,i})=0$ on M ($i=1, \cdots, n$) vanishes on W. Therefore, owing to the compactness of M we can take a positive s as desired.

§ 5. Some special cases.

5.1. Let M and N be compact Riemannian manifolds, and we denote by Δ_M and Δ_N the Laplace operators of M and N, respectively. Also we denote by Δ the Laplace operator of $M\times N$ with the product metric: $\Delta=\Delta_M+\Delta_N$. Let us consider an operator P on $M\times N$ such that

$$P=\Delta_M^2+\Delta_N^2 : C^\infty(M\times N) \longrightarrow C^\infty(M\times N)\,.$$

We can see that the operator P is elliptic and positive definite in $H_0(M\times N)$, here the inner product is taken with respect to the volume element of the product metric on $M\times N$.

PROPOSITION 8. *Assume that for any λ_i, $\lambda_j\in \mathrm{Sp}(\Delta_M)$ and μ_k, $\mu_l\in \mathrm{Sp}(\Delta_N)$,*

$$\lambda_i^2+\mu_k^2=\lambda_j^2+\mu_l^2 \quad \textit{implies} \quad \lambda_i=\lambda_j \quad \textit{and} \quad \mu_k=\mu_l\,.$$

Then,

$$G(P)\subset G(\Delta)\,.$$

This follows from the fact that $E_\nu(P)=E_\lambda(\Delta_M)\otimes E_\mu(\Delta_N)$ ($\lambda^2+\mu^2=\nu$) and the following lemma.

LEMMA 7. *Let an elliptic differential operator Q on a compact manifold M be selfadjoint with respect to some inner product in $H_0(M)$. If a diffeomorphism*

ϕ of M leaves each eigenspace $E_\lambda(Q)$ of the operator Q invariant, then $\phi\in G(Q)$.

This is proved by using Proposition 2. Notice that in this case all the generalized eigenfunctions are eigenfunctions.

EXAMPLE. We give an example satisfying the assumption of Proposition 8. Let ω_1 and ω_2 be real numbers such that 1, ω_1^4, ω_2^4 and $\omega_1^2\omega_2^2$ are linearly independent over $\boldsymbol{Z}$. Let $\Gamma=\left\{\frac{1}{2\pi}(n_1\omega_1,\ n_2\omega_2)\in\boldsymbol{R}^2:\ n_i\in\boldsymbol{Z}\right\}$ be a lattice in $\boldsymbol{R}^2$, and denote by Γ^* the dual lattice of Γ, i.e., $\Gamma^*=\left\{(x_1,\ x_2):\ \frac{1}{2\pi}\sum_i n_i x_i\omega_i\in\boldsymbol{Z},\ \text{for any}\ n_i\in\boldsymbol{Z}\right\}$.

We take $M=S^n=\{(x_1,\cdots,x_{n+1})\in\boldsymbol{R}^{n+1}:\ \sum x_i^2=1\}$ with the standard metric, and $N=\boldsymbol{R}^2/\Gamma^*$ with the metric induced from the Euclidean metric. Then it is well known that

$$\mathrm{Sp}(\Delta_{S^n})=\{k(k+n-1):\ k=0,\ 1,\ 2,\ \cdots\}$$

and

$$\mathrm{Sp}(\Delta_{\boldsymbol{R}^2/\Gamma^*})=\{n_1^2\omega_1^2+n_2^2\omega_2^2:\ n_i\in\boldsymbol{Z}\}\,.$$

Therefore, if

$$k^2(k+n-1)^2+(n_1^2\omega_1^2+n_2^2\omega_2^2)^2=l^2(l+n-1)^2+(m_1^2\omega_1^2+m_2^2\omega_2^2)^2\,,$$

then $k=l$, $n_1^2=m_1^2$ and $n_2^2=m_2^2$. This implies that the assumption of Proposition 8 is satisfied in this case.

5.2. It seems difficult to see what properties of an elliptic operator P imply the non-triviality of $G(P)$ or $X(P)$. But in a certain sense, for generic P's in the space of all elliptic operators on compact manifolds, $X(P)=\{0\}$. That is, we have

PROPOSITION 9. *Assume that an elliptic operator P satisfies the following three conditions, then $X(P)=\{0\}$, and $G(P)$ is at most a finite group:*

(i) *the principal symbol of P is real,*

(ii) *for any $\lambda\in\mathrm{Sp}(P)$, $\dim E_\lambda(P)=1$,*

(iii) *each eigenspace $E_\lambda(P)$ $(\lambda\in\mathrm{Sp}(P))$ contains a non-zero real-valued function.*

PROOF. Let $X\in X(P)$ and $u\in E_\lambda(P)$, real-valued, then there exists a constant c such that $u(\varphi_{X,t}(x))=e^{ct}u(x)$, which means that the constant c must be real and pure imaginary, hence $c=0$. Therefore, $X(u)=cu=0$ on every eigenspace. So $X=0$.

REMARK. On the meaning of 'generic' see [3] or [9].

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