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Ricci curvature, geodesics and some geometric properties of Riemannian manifolds with boundary

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Introduction.

Let M be a connected, complete Riemannian manifold with (possibly empty) boundary ∂M. Cheeger and Gromoll proved in [**4**] that if ∂M is empty and the Ricci curvature of M is nonnegative, then the Busemann function with respect to any ray is superharmonic on M. From this result, they showed that M as above is the isometric product $N\times \boldsymbol{R}^k$ ($k\geqq 0$), where N contains no lines and $\boldsymbol{R}^k$ has its standard flat metric. They also proved in [**5**] that if M is a convex subset with boundary ∂M in a positively curved manifold, then the distance function to ∂M is concave on M. Later, making use of this result, Burago and Zalgaller obtained in [**3**] a theorem on such a manifold M saying that

(1) the number of components of ∂M is not greater than 2,

(2) if there are two components Γ_1 and Γ_2 of ∂M, then M is isometric to the direct product $[0, a]\times\Gamma_1$,

(3) if ∂M is connected and compact, but M is noncompact, then M is isometric to the direct product $[0, \infty)\times\partial M$.

Recently we have obtained in [**9**] a sharp and general Laplacian comparison theorem, which tells us the behavior of the Laplacian of a distance function or a Busemann function on M in terms of the Ricci curvature of M. In this paper, using our comparison theorem, we shall study Riemannian manifolds with boundary and obtain, roughly speaking, a generalization of the above result by Burago and Zalgaller from the viewpoint of Ricci curvature.

We shall now describe our main theorems. Let M be a connected, complete Riemannian manifold of dimension m with smooth boundary ∂M. We call M *complete* if it is complete as a metric space with the distance induced by the Riemannian metric of M. Let R and Λ be two real numbers. We say M is *of class* (R, Λ) if the Ricci curvature of $M\geqq(m-1)R$ and (the trace of $S_\xi)\leqq(m-1)\Lambda$ for any unit inner normal vector field ξ of ∂M, where S_ξ is the second fundamental form of ∂M with respect to ξ (i.e., $\langle S_\xi X, Y\rangle=\langle\nabla_X\xi, Y\rangle$). We write $i(M)$ for the inradius of M (i.e., $i(M)=\sup\{\mathrm{dis}_M(x, \partial M): x\in M\}\leqq+\infty$). Let f

be the solutions of the equation:

$$f''+Rf=0 \quad \text{with} \quad f(0)=0 \quad \text{and} \quad f'(0)=1.$$

Let h be the solution of the equation:

$$h''+Rh=0 \quad \text{with} \quad h(0)=1 \quad \text{and} \quad h'(0)=\Lambda.$$

Set $C_1(R, \Lambda)=\inf\{t: t>0 \text{ and } h(t)=0\}$ and $C_2(R, \Lambda)=\inf\{t: t>0 \text{ and } h'(t)=0\}$. If $h>0$ (resp. $h'>0$) on $[0, \infty)$, we understand $C_1(R, \Lambda)=+\infty$ (resp. $C_2(R, \Lambda)=+\infty$).

THEOREM A. *Let M be a connected, complete Riemannian manifold of class (R, Λ). Then:*

(1) $i(M)\leqq C_1(R, \Lambda)$.

(2) *If $C_1(R, \Lambda)<+\infty$ and $\mathrm{dis}_M(p, \partial M)=C_1(R, \Lambda)$ for some $p\in M$, then M is isometric to the closed ball $B(C_1(R, \Lambda): R)$ in the simply connected space form $M^m(R)$ of constant curvature R.*

(3) *$C_1(R, \Lambda)<+\infty$ if and only if $R>0$, $R=0$ and $\Lambda<0$, or $R<0$ and $\Lambda<-\sqrt{-R}$.*

THEOREM B. *Let M be a connected, complete Riemannian manifold of class (R, Λ). Suppose ∂M is disconnected and it has a compact connected component Γ_1. Then:*

(1) *If $R=0$ and $\Lambda=0$, M is the isometric product $[0, a]\times\Gamma_1$.*

(2) *If $R>0$, then $\Lambda>0$ and $\min_{2\leq j}\mathrm{dis}_M(\Gamma_1, \Gamma_j)\leqq 2C_2(R, \Lambda)$, where $\{\Gamma_j\}_{j=1,2,\cdots}$ are the connected components of ∂M. Moreover if $\min_{2\leq j}\mathrm{dis}_M(\Gamma_1, \Gamma_j)=2C_2(R, \Lambda)$, then M is isometric to the warped product $[0, 2C_2(R, \Lambda)]\times_h\Gamma_1$.*

For the definition of warped products, see [**1**].

THEOREM C. *Let M be a connected, complete Riemannian manifold of class (R, Λ). Suppose ∂M is compact but M is noncompact. Then:*

(1) $R\leqq 0$.

(2) *If $R=0$ and $\Lambda=0$, then ∂M is connected and M is the isometric product $[0, \infty)\times\partial M$.*

(3) *If $\Lambda<0$, $R<0$ and $\Lambda\geqq-\sqrt{-R}$. Moreover if $\Lambda=-\sqrt{-R}$, then M is isometric to the warped product $[0, \infty)\times_h\partial M$.*

In the first assertion of Theorem B, we cannot delete the assumption that ∂M has a compact component, in contrast to the theorem of Burago and Zalgaller cited above. In fact, it is well known that there is a nonparametric minimal hypersurface in $\boldsymbol{R}^m$ ($m\geqq 9$) with the form: $x^m=k(x^1, \cdots, x^{m-1})$ defined for all $(x^1, \cdots, x^{m-1})$, where k is not linear (cf. [**2**]). Set $M=\{(x^1, \cdots, x^m)\in\boldsymbol{R}^m: k(x^1, \cdots, x^{m-1})\leqq x^m\leqq k(x^1, \cdots, x^{m-1})+1\}$. Then M satisfies all the conditions of (1) in Theorem B except that ∂M has a compact component, but M is not the isometric product $[0, a]\times\Gamma$.

We shall illustrate the method to prove our main theorems briefly. For example, let M be a Riemannian manifold of class (R, Λ). We assume there is a point p in the interior M_0 of M such that $\rho_1(p)=i(M)$ $(\rho_1=\mathrm{dis}_M(\partial M, *))$. Then the Laplacian comparison theorem of [**9**] tells us that the Laplacians $\Delta\rho_1$ and $\Delta\rho_2$ of ρ_1 and ρ_2 $(\rho_2=\mathrm{dis}_M(p, *))$ in the sense of distributions satisfy

$$\Delta\rho_1 \leqq (m-1)(h'/h)(\rho_1)$$

and

$$\Delta\rho_2 \leqq (m-1)(f'/f)(\rho_2)$$

on $\Omega=\{x\in M_0 : 0<\rho_1(x)<i(M) \text{ and } 0<\rho_2(x)<i(M)\}$. Therefore if $i(M)\geqq C_1(R, \Lambda)$, then

$$\Delta(\rho_1+\rho_2) \leqq (m-1)\{(h'/h)(\rho_1)+(f'/f)(\rho_2)\} \leqq 0$$

as a distribution on Ω, that is, $\rho_1+\rho_2$ is superharmonic on Ω. Set $\omega=\{x\in M_0\setminus\{p\} : \rho_1(x)+\rho_2(x)=i(M)\}$. Then ω is a nonempty closed subset contained in Ω. Since $\rho_1+\rho_2$ takes the minimum $i(M)$ on ω $(\subset\Omega)$, it is equal to $i(M)$ everywhere on M. Therefore we see that the exponential map $\exp_p$ at p restricted to the closed ball V with radius $i(M)$ in the tangent space M_p at p induces a diffeomorphism between V and M. The equality discussion in the Laplacian comparison theorem shows that for every $v\in V$, the sectional curvature of any plane tangent to $\dot\sigma$ $(\sigma(t)=\exp_p tv)$ is equal to R. This shows that $i(M)=C_1(R, \Lambda)$ and M is isometric to $B(C_1(R, \Lambda); R)$.

After the preparation of this paper, the author was informed that Ichida [8] has also proved the assertion (1) of Theorem B, independently.

Finally the author would like to express sincere thanks to Professor T. Ochiai for his helpful advice and encouragement.

1. Preliminary.

Let M be a connected, complete Riemannian manifold of dimmension m. We assume the boundary ∂M of M is nonempty and smooth, unless otherwise stated. We write M_0 for the interior of M. For a connected open subset A of M, we denote by $\mathrm{dis}_A(x, y)$ the distance between two points x and y in A induced by the Riemannian metric of M restricted to A. Then, in general, $\mathrm{dis}_A(x, y)\geqq \mathrm{dis}_M(x, y)$. Let N be a closed subset of M. We write Ω_N for the subset of all the points x in $M_0\setminus N$ which can be joined to N by a geodesic $\sigma: [0, l]\to M$ such that $\sigma((0, l))\subset M_0$, $\mathrm{dis}_M(N, \sigma(t))=t$ for $t\in[0, l]$, and $\sigma(l_x)=x$ for some $l_x\in(0, l)$. Then by the definition of Ω_N, we see that for $x\in\Omega_N$, a geodesic σ as above is uniquely determined up to l_x, so that we write σ_x for the above geodesic σ restricted to $[0, l_x]$.

Suppose N is a point p of M_0. Let x be a point of Ω_p. Then it follows from the definition of Ω_p that, in the tangent space M_p at p, there is a connected, open neighborhood $\hat{C}_{p,x}$ of a segment $\{t\dot{\sigma}_x(0): 0 \leqq t \leqq l_x (l_x = \mathrm{dis}_M(x, p))\}$ which has the following properties:

(1) For any $t \in [0, 1]$, $tv \in \hat{C}_{p,x}$ if $v \in \hat{C}_{p,x}$.

(2) The exponential map $\exp_p$ at p restricted to $\hat{C}_{p,x}$ induces a diffeomorphism between $\hat{C}_{p,x}$ and its image. We write $C_{p,x}$ for the image $\exp_p(\hat{C}_{p,x})$ of such a $\hat{C}_{p,x}$. Then for any $y = \exp_p v$ $(v \in \hat{C}_{p,x})$, $\mathrm{dis}_{C_{p,x}}(p, y) = \|v\|$, so that the distance function $\rho = \mathrm{dis}_{C_{p,x}}(p, *)$ is a smooth function on $C_{p,x} \setminus \{p\}$. Furthermore we have the following

(1.1) LEMMA (cf. [**9**: Lemma 2.5]). *Suppose M is of class (R, Λ). Then for any $x \in \Omega_p$, the distance function $\rho = \mathrm{dis}_{C_{p,x}}(p, *)$ satisfies*

$$\Delta\rho \leqq (m-1)(f'/f)(\rho) \tag{1.2}$$

on $C_{p,x} \setminus \{p\}$, where f is the solution of the equation: $f'' + Rf = 0$ with $f(0) = 0$ and $f'(0) = 1$. Moreover the equality in (1.2) *holds at $y = \exp_p v$ $(v \in \hat{C}_{p,x})$ if and only if the sectional curvature of any plane containing the tangent vector of $\exp_p(tv)$ $(0 \leqq t \leqq 1)$ is equal to R.*

Next we consider the case N is a connected component of ∂M or ∂M itself. Let $\xi: \partial M \to (\partial M)^\perp$ be the unit inner normal vector field on ∂M. Set $V_N = \{t\xi(y): y \in N, 0 \leqq t\}$. Let x be a point of Ω_N. Then it follows from the definition of Ω_N that, in V_N, there is an open neighborhood $\hat{C}_{N,x}$ of a segment $\{t\dot{\sigma}_x(0): 0 \leqq t \leqq l_x (l_x = \mathrm{dis}_M(x, N))\}$ which has the following properties:

(1) For any $t \in [0, 1]$, $tv \in \hat{C}_{N,x}$ if $v \in \hat{C}_{N,x}$.

(2) The exponential map $\exp_N$ of N restricted to $\hat{C}_{N,x}$ induces a diffeomorphism between $\hat{C}_{N,x}$ and its image. We write $C_{N,x}$ for the image $\exp_N(\hat{C}_{N,x})$ of such a $\hat{C}_{N,x}$. Then for any $y = \exp_N v$ $(v \in \hat{C}_{N,x})$, $\mathrm{dis}_{C_{N,x}}(N \cap C_{N,x}, y) = \|v\|$, so that the distance function $\rho = \mathrm{dis}_{C_{N,x}}(N \cap C_{N,x}, *)$ is a smooth function on $C_{N,x} \setminus N$. Furthermore we have the following

(1.3) LEMMA (cf. [**9**: Lemma 2.8]). *Suppose M is of class (R, Λ). Then for any $x \in \Omega_N$, the distance function $\rho = \mathrm{dis}_{C_{N,x}}(N \cap C_{N,x}, *)$ satisfies*

$$\Delta\rho \leqq (m-1)(h'/h)(\rho) \tag{1.4}$$

on $C_{N,x} \setminus N$, where h is the solution of the equation: $h'' + Rh = 0$ with $h(0) = 1$ and $h'(0) = \Lambda$. Moreover the equality in (1.4) *holds at $y = \exp_N v$ $(v \in \hat{C}_{N,x})$ if and only if the sectional curvature of any plane containing the tangent vector of $\exp_N(tv)$ $(0 \leqq t \leqq 1)$ is equal to R and N is umbilic at $z = \exp_N 0 \cdot v$.*

(1.5) LEMMA. *Let M be a connected, complete Riemannian manifold with boundary ∂M. Then for any $x \in M_0$, there is a geodesic $\sigma: [0, l] \to M$ such that $\mathrm{dis}_M(\partial M, \sigma(t)) = t$ for $t \in [0, l]$ and $\sigma(l) = x$. In particular, if ∂M is compact but M is noncompact, then there is a geodesic $\sigma: [0, \infty) \to M$ such that $\mathrm{dis}_M(\partial M, \sigma(t))$*

$=t$ *for* $t\geqq 0$.

PROOF. For the proof of the first assertion, see, e.g., the proof of Corollary (2.44) in [**9**]. Now suppose ∂M is compact but M is noncompact. Then by the first assertion, we see that $\{x\in M: \mathrm{dis}_M(\partial M, x)\leqq a\}$ is compact for each $a\geqq 0$. Therefore there is a sequence $\{x_n\}$ of M_0 such that $\mathrm{dis}_M(\partial M, x_n)$ is divergent as $n\to\infty$. Let $\sigma_n: [0, l_n]\to M$ be a geodesic such that $\mathrm{dis}_M(\partial M, \sigma_n(t))=t$ for $t\in[0, l_n]$ and $\sigma_n(l_n)=x_n$. Since ∂M is compact, we can find a subfamily of $\{\sigma_n\}$ which converges to a geodesic $\sigma: [0, \infty)\to M$. Such a geodesic σ is a required one. This completes the proof of Lemma (1.5).

(1.6) LEMMA. *Let M be a connected, complete Riemannian manifold with boundary ∂M. Suppose ∂M is disconnected and has a compact component, say Γ_1, and suppose* $\mathrm{dis}_M(\Gamma_1, \Gamma_2)=\min_{2\leqq j\leqq k}\mathrm{dis}_M(\Gamma_1, \Gamma_j)$, *where* $\{\Gamma_j\}_{j=1,2,\dots,k}$ $(k\in\{2, 3, \cdots, \infty\})$ *are the connected components of ∂M. Then the set* $\omega=\{x\in M_0: \mathrm{dis}_M(\Gamma_1, x)+\mathrm{dis}_M(\Gamma_2, x)=\mathrm{dis}_M(\Gamma_1, \Gamma_2)\}$ *is a nonempty closed subset of M_0 and for each $x\in\omega$, there is a unique geodesic $\sigma: [0, l]\to M$ $(l=\mathrm{dis}_M(\Gamma_1, \Gamma_2))$ through x such that* $\mathrm{dis}_M(\Gamma_1, \sigma(t))=t$ *and* $\mathrm{dis}_M(\Gamma_2, \sigma(t))=l-t$ *for* $t\in[0, l]$.

PROOF. Since Γ_1 is compact, there is a point x of Γ_1 such that $\mathrm{dis}_M(x, \Gamma_2)=\mathrm{dis}_M(\Gamma_1, \Gamma_2)$. Let $\{y_n\}_{n=1,2,\dots}$ be a sequence of Γ_2 such that $\lim_{n\to\infty}\mathrm{dis}_M(x, y_n)=\mathrm{dis}_M(\Gamma_1, \Gamma_2)$. Then for each y_n, (i) there is a geodesic $\sigma_n: [0, l_n]\to M$ such that $\sigma_n(0)=x$, $\sigma_n(l_n)=y_n$, and $\mathrm{dis}_M(\Gamma_1, \sigma_n(t))=t$ for $t\in[0, l_n]$ or (ii) there is a geodesic $\gamma_n: [0, \delta_n]\to M$ such that $\gamma_n(0)=x$, $\gamma_n(\delta_n)\in\partial M\setminus\{y_n\}$, $\mathrm{dis}_M(\Gamma_1, \gamma_n(t))=t$ for $t\in[0, \delta_n]$ and further $\mathrm{dis}_M(x, \gamma_n(t))+\mathrm{dis}_M(\gamma_n(t), y_n)=\mathrm{dis}(x, y_n)$ for $t\in[0, \delta_n]$ (cf. [**10**: Theorem 7.1]). Suppose the case (i) takes place for infinitely many y_n, say $\{y_{n_j}\}_{j=1,2,\dots}$. Then we find a subfamily of $\{\sigma_{n_j}\}$ which converges to a geodesic $\sigma: [0, l]\to M$. Then $\sigma(t)$ is clearly an element of ω for $t\in(0, l)$. Now suppose the case (ii) takes place for infinitely many y_n, say $\{y_{n_k}\}_{k=1,2,\dots}$. Then $\lim_{k\to\infty}\mathrm{dis}_M(z_{n_k}, y_{n_k})=0$ $(z_{n_k}=\gamma_{n_k}(\delta_{n_k}))$, since $\mathrm{dis}_M(x, z_{n_k})\geqq l$ and $\lim_{k\to\infty}\mathrm{dis}_M(x, y_{n_k})=l$. Therefore $\lim_{k\to\infty}\mathrm{dis}_M(x, z_{n_k})=l$. Hence we can find a subfamily of $\{\gamma_{n_k}\}$ which converges to a geodesic $\gamma: [0, l]\to M$ such that $\gamma(t)$ is contained in ω for each $t\in(0, l)$. Thus we see that ω is not empty. Now let x be any point of ω. Then by Lemma (1.5) and the above argument, we see that there are two geodesics $\sigma_1: [0, l_1]\to M$ and $\sigma_2: [0, l_2]\to M$ such that $\mathrm{dis}_M(\Gamma_1, \sigma_1(t))=t$ for $t\in[0, l_1]$, $\sigma_1(l_1)=x$, $\sigma_2(0)=x$, and $\mathrm{dis}_M(\Gamma_2, \sigma_2(t))=l_2-t$ for $t\in[0, l_2]$. Define a curve $\sigma: [0, l]\to M$ by $\sigma(t)=\sigma_1(t)$ for $t\in[0, l_1]$ and $\sigma(t)=\sigma_2(t-l_1)$ for $t\in[l_1, l]$. Then by the definition of ω, we see that the above σ is the required geodesic. This completes the proof of Lemma (1.6).

(1.7) LEMMA. *Let M be a connected, complete Riemannian manifold with boundary ∂M. Let p be a point of M_0. Then for any x such that* $\mathrm{dis}_M(x, p)<\mathrm{dis}_M(\partial M, p)$, *there is a geodesic $\sigma: [0, l]\to M_0$ such that* $\mathrm{dis}_M(p, \sigma(t))=t$ *for*

$t\in[0, l]$ *and* $\sigma(l)=x$.

For the proof, see, e.g., that of Corollary (2.35) in [**9**].

Now we consider the case ∂M is empty. Let $\tilde{N}$ be a Riemannian manifold of dimension n without boundary. We assume there is an isometric immersion $\iota: \tilde{N}\to M$ such that the image $N=\iota(\tilde{N})$ is closed. Then we have the following two lemmas.

(1.8) LEMMA (cf. [**9**: Corollary 2.35 and Lemma 2.8]). *Suppose the Ricci curvature of* $M\geqq(m-1)R$ $(R\in\boldsymbol{R})$ *and suppose* $n=m-1$ *and* $\iota: \tilde{N}\to M$ *is a minimal immersion. Then the distance function* $\rho=\mathrm{dis}_M(N, *)$ *satisfies* $\rho\leqq C_1(R, 0)$ *and*

$$\Delta\rho\leqq(m-1)(h'/h)(\rho) \tag{1.9}$$

as a distribution on $\{x\in M: 0<\rho(x)<C_1(R, 0)\}$, *where* h *is the solution as in Lemma* (1.4) $(\Lambda=0)$ *and* $C_1(R, 0)=\inf\{t: t\geqq 0, h(t)\leqq 0\}(\leqq+\infty)$. *Moreover when* N *is a (imbedded) submanifold, the equality in* (1.9) *holds at* $x\in\Omega_N$ *if and only if the sectional curvature of any plane tangent to* σ_x *is equal to* R *and the second fundamental form* $S_{\dot{\sigma}_x(0)}$ *of* N *with respect to* $\dot{\sigma}_x(0)$ *vanishes.*

(1.10) LEMMA (cf. [**9**: Corollary 2.40 and Lemma 2.11]). *Suppose the sectional curvature of* $M\geqq R$ $(R\in\boldsymbol{R})$ *and suppose* $\iota: \tilde{N}\to M$ *is a minimal immersion. Then the distance function* $\rho=\mathrm{dis}_M(N, *)$ *satisfies* $\rho\leqq C_1(R, 0)$ *and*

$$\Delta\rho\leqq(m-n-1)(f'/f)(\rho)+n(h'/h)(\rho) \tag{1.11}$$

as a distribution on $\{x\in M: 0<\rho(x)<C_1(R, 0)\}$, *where* f *and* h *are respectively the solutions as in Lemma* (1.1) *and as in Lemma* (1.3) $(\Lambda=0)$. *Moreover when* N *is a (imbedded) submanifold, the equality in* (1.11) *holds at* $x\in\Omega_N$ *if and only if the sectional curvature of any plane tangent to* σ_x *is equal to* R *and* $S_{\dot{\sigma}_x(0)}=0$.

For the rest of this section, we assume M is noncompact. (The boundary ∂M may be nonempty.) A *ray* $\gamma: [0, \infty)\to M$ $(\gamma((0, \infty))\subset M_0$ if $\partial M\neq\emptyset)$ is by definition a geodesic such that $\mathrm{dis}_M(\gamma(t), \gamma(s))=|t-s|$ for any $t\geqq 0$ and $s\geqq 0$. Let $\gamma: [0, \infty)\to M$ be a ray. Define the open *half-space* B_γ with respect to γ by

$$B_\gamma=\bigcup_{t>0} B_t(\gamma(t))$$

where $B_t(\gamma(t))=\{x\in M: \mathrm{dis}_M(\gamma(t), x)<t\}$. A *Busemann function* $\eta_\gamma: M\to\boldsymbol{R}$ with respect to γ is defined as follows: for every point $x\in M$, let

$$\eta_\gamma(x)=\lim_{t\to\infty}\{\mathrm{dis}_M(\gamma(t), x)-t\}.$$

The right-hand side is bounded from below by $-\mathrm{dis}_M(\gamma(0), x)$ and is monotone decreasing with t. Hence it converges uniformly on each compact set of M as $t\to\infty$. Then we have the following two lemmas.

(1.12) LEMMA. *Suppose* ∂M *is empty and the Ricci curvature of* $M\geqq(m-1)R$ $(R\leqq 0)$. *Then for every ray* $\gamma: [0, \infty)\to M$, *the Busemann function* η_γ

satisfies

$$\Delta\eta_{\gamma}\leqq(m-1)\sqrt{-R}$$

as a distribution on M.

PROOF. Set $\rho_{t}=\mathrm{dis}_{M}(\gamma(t), *)$. Then Corollary (2.42) in [**9**] tells us that ρ_{t} satisfies

$$\Delta(\rho_{t}-t)\leqq(m-1)(f'/f)(\rho_{t})$$

as a distribution on $M\setminus\{\gamma(t)\}$, where f is the solution as in Lemma (1, 1). Since $\rho_{t}-t$ converges uniformly on each compact set of M as $t\rightarrow\infty$, we see that

$$\Delta\eta_{\gamma}=\lim_{t\rightarrow\infty}\Delta\rho_{t}\leqq(m-1)\lim_{t\rightarrow\infty}(f'/f)(\rho_{t})=(m-1)\sqrt{-R}$$

as a distribution on M. This completes the proof of Lemma (1.12).

(1.13) LEMMA. *Suppose* ∂M *is nonempty and the Ricci curvature of* $M\geqq(m-1)R$ $(R\leqq 0)$ *and suppose there is a ray* $\gamma:[0, \infty)\rightarrow M$ *such that* $\mathrm{dis}_{M}(\partial M, \gamma(t))=t$ *for* $t\geqq 0$. *Then the Busemann function* η_{γ} *with respect to* γ *satisfies*

$$\Delta\eta_{\gamma}\leqq(m-1)\sqrt{-R}$$

as a distribution on B_{γ}.

PROOF. Set $\rho_{t}=\mathrm{dis}_{M}(\gamma(t), *)$. Then Corollary (2.45) in [**9**] tells us that ρ_{t} satisfies

$$\Delta(\rho_{t}-t)\leqq(m-1)(f'/f)(\rho_{t})$$

as a distribution on $B_{t}(\gamma(t))\setminus\{\gamma(t)\}$, where f is the solution as in Lemma (1.1). Since $B_{t}(\gamma(t))\subset B_{s}(\gamma(s))\ (s>t)$ and $\rho_{t}-t$ converges uniformly on each compact set of M as $t\rightarrow\infty$, we see that

$$\Delta\eta_{\gamma}=\lim_{t\rightarrow\infty}\Delta\rho_{t}\leqq(m-1)\lim_{t\rightarrow\infty}(f'/f)(\rho_{t})=(m-1)\sqrt{-R}$$

as a distribution on $B_{\gamma}=\bigcup_{t>0}B_{t}(\gamma(t))$. This completes the proof of Lemma (1.13).

Finally we shall give the following

(1.14) LEMMA. *Let* f *and* h *be, respectively, the solutions as in Lemma* (1.1) *and as in Lemma* (1.3). *Set* $C_{1}(R, \Lambda)=\inf\{t: t\geqq 0, h(t)\leqq 0\}\ (\leqq+\infty)$, $C_{2}(R, \Lambda)=\inf\{t: t\geqq 0, h'(t)\leqq 0\}\ (\leqq+\infty)$, $C_{3}(R, \Lambda)=\inf\{t: t>0, f(t)\leqq 0\}\ (\leqq+\infty)$. *Suppose* $C_{1}(R, \Lambda)<+\infty$. *Then we have*

$$(h'/h)(t)+(f'/f)(s)\leqq 0 \tag{1.15}$$

for $t\in[0, C_{1}(R, \Lambda))$ *and* $s\in[C_{1}(R, \Lambda)-t, C_{3}(R, \Lambda))$. *Equality in* (1.15) *holds if and only if* $s=C_{1}(R, \Lambda)-t$. *Moreover suppose* $R>0$ *and* $\Lambda>0$, *then we have*

$$(h'/h)(t)+(h'/h)(s)\leqq 0 \tag{1.16}$$

for $t\in[0, 2C_{2}(R, \Lambda))$ *and* $s\in[2C_{2}(R, \Lambda)-t, C_{1}(R, \Lambda))$. *Equality in* (1.16) *holds if and only if* $s=2C_{1}(R, \Lambda)-t$.

PROOF. Both (1.15) and (1.16) follow from simple computations.

REMARKS. (1) In Lemma (1.8), inequality (1.9) implies that ρ is superharmonic on $M \setminus N$ when $R=0$. This fact was proved by Wu [**11**]. (2) Lemma (1.12) tells us that if M is a complete Riemannian manifold of nonnegative Ricci curvature without boundary, then the Busemann function with respect to any geodesic ray γ is superharmonic on M. This fact was proved by Cheeger and Gromoll [**4**].

2. Proofs of Theorems A, B and C.

Throughout this section, we use the same notations as in the previous sections.

PROOF OF THEOREM A. The first assertion is well known. But for the completeness, we shall show it from our version. Suppose $i(M)>C_1(R, \Lambda)$. Then by Lemma (1.5), we can find a point p in $\Omega_{\partial M}$ such that $\mathrm{dis}_M(p, \partial M)=C_1(R, \Lambda)$. On some neighborhood $C_{\partial M, p} \setminus \partial M$ of p, the distance function $\rho=\mathrm{dis}_{C_{\partial M, p}}(C_{\partial M, p} \cap \partial M, *)$ is smooth and satisfies: $\Delta\rho(p) \leqq (m-1)(h'/h)(\rho(p))=-\infty$ by (1.4). This is a contradiction. Therefore we see that $i(M) \leqq C_1(R, \Lambda)$. Now suppose $\mathrm{dis}_M(p, \partial M)=C_1(R, \Lambda)$ for some $p \in M_0$. Put $\omega=\{x \in M_0 \setminus \{p\} : \mathrm{dis}_M(\partial M, x)+\mathrm{dis}_M(p, x)=C_1(R, \Lambda)\}$. Then by Lemma (1.5) and Lemma (1.7), we see that ω consists of all the points in $M_0 \setminus \{p\}$ which lie on geodesics $\sigma: [0, C_1(R, \Lambda)] \to M$ such that $\mathrm{dis}_M(\partial M, \sigma(t))=t$ for $t \in [0, C_1(R, \Lambda)]$ and $\mathrm{dis}_M(p, \sigma(t))=C_1(R, \Lambda)-t$, so that ω is a nonempty closed subset of $M_0 \setminus \{p\}$ contained in $\Omega_{\partial M} \cap \Omega_p$. Now we claim that ω is an open subset, so that $\omega=M_0 \setminus \{p\}$. In fact, for any $x \in \omega$ $(\subset \Omega_{\partial M} \cap \Omega_p)$, we see by (1.2) that, on some neighborhood $C_{p, x} \setminus \{p\}$ of x, the distance function $\rho_1=\mathrm{dis}_{C_{p, x}}(p, *)$ is smooth and satisfies

$$\Delta\rho_1 \leqq (m-1)(f'/f)(\rho_1), \tag{2.2}$$

and by (1.4) that, on some neighborhood $C_{\partial M, x} \setminus \partial M$ of x, the distance function $\rho_2=\mathrm{dis}_{C_{\partial M, x}}(C_{\partial M, x} \cap \partial M, *)$ is smooth and satisfies

$$\Delta\rho_2 \leqq (m-1)(h'/h)(\rho_2). \tag{2.3}$$

By (2.2) and (2.3), we have

$$\Delta(\rho_1+\rho_2) \leqq (m-1)\{(f'/f)(\rho_1)+(h'/h)(\rho_2)\} \tag{2.4}$$

on some neighborhood U of x $(U=(C_{p, x} \setminus \{p\}) \cap (C_{\partial M, x} \setminus \partial M))$. On the other hand, it follows from the definitions of ρ_1 and ρ_2 that $\rho_1+\rho_2 \geqq C_1(R, \Lambda)$ and $\rho_1(x)+\rho_2(x)=C_1(R, \Lambda)$. Moreover (1.15) implies that the right-hand side of (2.4) is nonpositive on U, that is, $\rho_1+\rho_2$ is superharmonic on U. Hence by the minimum principle for superharmonic functions, we see that, on a neighborhood U' of x $(U' \subset U)$,

$$\rho_1+\rho_2=C_1(R, \Lambda) \tag{2.5}$$

and hence

$$\Delta\rho_1=(m-1)(f'/f)(\rho_1). \tag{2.6}$$

Then (2.5) shows that x is an interior point of ω, since $\rho_1+\rho_2 \geqq \mathrm{dis}_M(p, *)+\mathrm{dis}_M(\partial M, *) \geqq C_1(R, \Lambda)$. Thus ω is an open and closed subset of $M_0 \setminus \{p\}$, so that $\omega = M_0 \setminus \{p\} = \{x \in M: 0 < \mathrm{dis}_M(p, x) < C_1(R, \Lambda)\}$. This implies that the exponential map $\exp_p$ at p restricted to the closed ball $V=\{v \in M_p: \|v\| \leqq C_1(R, \Lambda)\}$ induces a diffeomorphism between V and M. Moreover (2.6) and the equality discussion in Lemma (1.1) tell us that for any $x \in M_0 \setminus \{p\}$, the sectional curvature of any plane tangent to $\dot{\sigma}_x$ is equal to R, where $\sigma_x: [0, l] \to M$ is the unique geodesic joining p to x. Therefore we can conclude that M is isometric to $\bar{B}(C_1(R, \Lambda); R)$. The assertion (3) of the theorem follows from the definition of h. This completes the proof of Theorem A.

PROOF OF THEOREM B. Let $\{\Gamma_j\}_{j=1,\dots,k}$ be the connected components of ∂M $(2 \leqq k \leqq +\infty)$. We assume Γ_1 is compact and $\mathrm{dis}_M(\Gamma_1, \Gamma_2)=\min_{2 \leq j \leq k} \mathrm{dis}_M(\Gamma_1, \Gamma_j)$. Set $\omega=\{x \in M_0: \mathrm{dis}_M(\Gamma_1, x)+\mathrm{dis}_M(\Gamma_2, x)=\mathrm{dis}_M(\Gamma_1, \Gamma_2)\}$. Then by Lemma (1.6), we see that ω is a nonempty closed subset of M_0 contained in $\Omega_{\Gamma_1} \cap \Omega_{\Gamma_2}$. Now we shall prove the assertion (1) of the theorem. Let x be any point of ω. Then we see by (1.4) that, on some neighborhoods $C_{\Gamma_i, x} \setminus \Gamma_i$ of x $(i=1, 2)$, the distance functions $\rho_i=\mathrm{dis}_{C_{\Gamma_i, x}}(C_{\Gamma_i, x} \cap \Gamma_i, *)$ are smooth and satisfy

$$\Delta\rho_i \leqq (m-1)(h'/h)(\rho_i). \tag{2.7}$$

Since $R=0$ and $\Lambda=0$, the right-hand side of (2.7) is equal to 0, so that both ρ_1 and ρ_2 are superharmonic on some neighborhood U of x $(U=(C_{\Gamma_1, x} \setminus \Gamma_1) \cap (C_{\Gamma_2, x} \setminus \Gamma_2))$. In particular, $\rho_1+\rho_2$ is superharmonic on U. Since $\rho_1+\rho_2$ takes the minimum $\mathrm{dis}_M(\Gamma_1, \Gamma_2)$ at x, we see that, on a neighborhood U' of x $(U' \subset U)$,

$$\rho_1+\rho_2=\mathrm{dis}_M(\Gamma_1, \Gamma_2) \tag{2.8}$$

and hence

$$\Delta\rho_1=0. \tag{2.9}$$

Then (2.8) implies that x is an interior point of ω, since $\rho_1+\rho_2 \geqq \mathrm{dis}_M(\Gamma_1, *)+\mathrm{dis}_M(\Gamma_2, *) \geqq \mathrm{dis}_M(\Gamma_1, \Gamma_2)$. Thus ω is an open and closed subset of M_0, so that $\omega=M_0=\{x \in M: 0<\mathrm{dis}_M(\Gamma_1, x)<\mathrm{dis}_M(\Gamma_1, \Gamma_2)\}$. Moreover (2.9) and the equality discussion in Lemma (1.3) tell us that for any $x \in M_0$, the sectional curvature of any plane tangent to $\dot{\sigma}_x$ vanishes, where $\sigma_x: [0, l] \to M$ is the unique geodesic joining Γ_1 to x such that $\mathrm{dis}_M(\Gamma_1, \sigma_x(t))=t$ for $t \in [0, l]$, and Γ_1 is totally geodesic. Define a map $\Psi: [0, \mathrm{dis}_M(\Gamma_1, \Gamma_2)] \times \Gamma_1 \to M$ by $\Psi(t, y)=\exp_{\Gamma_1}(t\xi(y))$, where ξ:

$\Gamma_1 \to \Gamma_1^{\perp}$ is the unit inner normal vector field on Γ_1. Then Ψ induces an isometry from the Riemannian product manifold $[0, \mathrm{dis}_M(\Gamma_1, \Gamma_2)] \times \Gamma_1$ onto M. Next we shall show the assertion (2) of Theorem B. It follows from (1) of Theorem B that $\Lambda > 0$. Suppose $\mathrm{dis}_M(\Gamma_1, \Gamma_2) \geqq 2C_2(R, \Lambda)$. Let x be a point of ω; let ρ_1, ρ_2 and U be as in the preceding proof. Then by (1.4), we see that

$$\Delta(\rho_1+\rho_2) \leqq (m-1)\{(h'/h)(\rho_1)+(h'/h)(\rho_2)\} \tag{2.10}$$

on U. Since $\rho_1+\rho_2 \geqq \mathrm{dis}_M(\Gamma_1, \Gamma_2) \geqq 2C_2(R, \Lambda)$, we see by (1.16) that the right-hand side of (2.10) is nonpositive, that is, $\rho_1+\rho_2$ is superharmonic on U, and further it takes the minimum $\mathrm{dis}_M(\Gamma_1, \Gamma_2)$ at x. Therefore taking account of the equality discussion after (1.14), we see that, on a neighborhood U' of x $(U' \subset U)$,

$$\rho_1+\rho_2 = \mathrm{dis}_M(\Gamma_1, \Gamma_2) = 2C_2(R, \Lambda) \tag{2.11}$$

and hence

$$\Delta\rho_1 = (m-1)(h'/h)(\rho_1)\,. \tag{2.12}$$

Then (2.11) implies that x is an interior point of ω, since $\rho_1+\rho_2 \geqq \mathrm{dis}_M(\Gamma_1, *) + \mathrm{dis}_M(\Gamma_2, *) \geqq \mathrm{dis}_M(\Gamma_1, \Gamma_2)$. Thus ω is an open and closed subset of M_0, so that $\omega = M_0 = \{x \in M:\ 0 < \mathrm{dis}_M(\Gamma_1, x) < 2C_2(R, \Lambda)\}$. Moreover (2.12) and the equality discussion in Lemma (1.3) tell us that for any $x \in M_0$, the sectional curvature of any plane tangent to $\dot{\sigma}_x$ is equal to R, where $\sigma_x : [0, l] \to M$ is the unique geodesic joining Γ_1 to x such that $\mathrm{dis}_M(\Gamma_1, \sigma_x(t)) = t$ for $t \in [0, l]$ and Γ_1 is totally umbilic (i.e., $\langle S_\xi X, Y\rangle = h'(0) \times \langle X, Y\rangle$). Let Ψ: $[0, 2C_2(R, \Lambda)] \times \Gamma_1 \to M$ be the map defined as above. Then Ψ induces an isometry from the warped product $[0, 2C_2(R, \Lambda)] \times_h \Gamma_1$ onto M. This completes the proof of Theorem B.

Before the proof of Theorem C, we remark that the proof of Theorem B and Lemma (1.8) will yield the following

THEOREM B′. *Let M be a Riemannian manifold of class (R, Λ). Suppose ∂M is connected and suppose there is a minimal immersion ι: $\tilde{N} \to M_0$ from a Riemannian manifold N without boundary into M such that* $\dim N = \dim M - 1$ *and the immage $N = \iota(\tilde{N})$ is compact. Then:*

(1) *If $R=0$, $\Lambda=0$ and $M \setminus N$ is connected, there exists an involutive isometry σ of ∂M without fixed points and M is isometric to the quotient space $[0, 2l] \times \partial M/G_\sigma$ where G_σ is the isometric group of $[0, 2l] \times \partial M$ which consists of the identity element and the involutive isometry $\hat{\sigma}$ defined by $\hat{\sigma}((t, x)) = (2l-t, \sigma(x))$.*

(2) *If $R > 0$, then $\Lambda > 0$ and $\mathrm{dis}_M(\partial M, N) \leqq C_2(R, \Lambda)$. Moreover if $\mathrm{dis}_M(\partial M, N) = C_2(R, \Lambda)$, there is an involutive isometry σ of ∂M without fixed points and M is isometric to the quotient space $[0, 2C_2(R, \Lambda)] \times_h \partial M/G_\sigma$, where h is as in Theorem B and G_σ is as above.*

PROOF OF THEOREM C. It follows from the assumptions that there is a geodesic ray $\gamma: [0, \infty) \to M$ such that $\mathrm{dis}_M(\partial M, \gamma(t))=t$ for $t \geqq 0$ (cf. Lemma (1.5)). Since γ has no conjugate point, we see that $R \leqq 0$. Let η_γ be the Busemann function with respect to γ. Then $\eta_\gamma+\rho_{\partial M} \geqq 0$ ($\rho_{\partial M}=\mathrm{dis}_M(\partial M, *)$) on the half-space B_γ by the triangle inequality and $(\eta_\gamma+\rho_{\partial M})(\gamma(t))=0$ for any $t \geqq 0$ by the definition of η_γ. We remark here the following properties of η_γ: $B_\gamma=\{x \in M: \eta_\gamma(x)<0\}$ and for any $a<0$,

$$\eta_\gamma(x)=a+\mathrm{dis}_M(x, \eta_\gamma^{-1}(a)) \tag{2.13}$$

on $\{x \in B_\gamma: \eta_\gamma(x) \geqq a\}$ (cf. [**11**: Lemma 7] and Lemma (1.7)). Set $\omega=\{x \in M: \eta_\gamma(x)+\rho_{\partial M}(x)=0\}$. Then by (2.13) and Lemma (1.5), we see that ω is a closed subset of M_0 contained in $\Omega_{\partial M} \cap B_\gamma$. Let x be a point of ω. Then we see by (1.4) that, on some neighborhood $C_{\partial M, x} \setminus \partial M$ of x, the distance function $\rho_1= \mathrm{dis}_{C_{\partial M, x}}(C_{\partial M, x} \cap \partial M, *)$ is smooth and satisfies

$$\Delta \rho_1 \leqq (m-1)(h'/h)(\rho_1). \tag{2.14}$$

On the other hand, we see by Lemma (1.13) that

$$\Delta \eta_\gamma \leqq (m-1)\sqrt{-R} \tag{2.15}$$

as a distribution on B_γ. Therefore by (2.14) and (2.15), we have

$$\Delta(\rho_1+\eta_\gamma) \leqq (m-1)\{(h'/h)(\rho_1)+\sqrt{-R}\} \tag{2.16}$$

as a distribution on a neighborhood U of x. Suppose $R=0$ and $\Lambda=0$. Then the right-hand side of (2.16) is equal to 0, that is, $\rho_1+\eta_\gamma$ is superharmonic on U, and further it takes the minimum 0 at x. Therefore we see that, on a neighborhood U' of x ($U' \subset U$),

$$\rho_1+\eta_\gamma=0 \tag{2.17}$$

and hence

$$\Delta \rho_1=0\,. \tag{2.18}$$

Then (2.17) implies that x is an interior point of ω, since $\rho_1+\eta_\gamma \geqq \mathrm{dis}_M(\partial M, *)+\eta_\gamma \geqq 0$. Thus ω is an open and closed subset of M_0, so that $\omega=\Omega_{\partial M}=B_\gamma=M_0$. Moreover (2.18) and the equality discussion in Lemma (1.3) tell us that for any $x \in M_0$, the sectional curvature of any plane tangent to $\dot{\sigma}_x$ vanishes, where $\sigma_x: [0, l] \to M$ is the unique geodesic joining ∂M to x such that $\mathrm{dis}_M(\partial M, \sigma_x(t))=t$ for $t \in [0, l]$, and ∂M is totally geodesic. Let $\Psi: [0, \infty) \times \partial M \to M$ be the map defined by $\Psi(t, y)=\exp_{\partial M}(t\xi(y))$, where $\xi: \partial M \to (\partial M)^\perp$ is the unit inner normal vector field on ∂M. Then Ψ is an isometry from the Riemannian product manifold $[0, \infty) \times \partial M$ onto M. This completes the proof for the assertion

(2) of Theorem C. As for the last assertion (3), we see by the preceding assertion (2) that $R<0$ and by (3) of Theorem A that $\Lambda \geqq -\sqrt{-R}$. Now suppose $\Lambda = -\sqrt{-R}$. Then the right-hand side of (2.16) is equal to 0. Therefore the same argument as in the preceding assertion (2) shows that the map $\Psi: [0, \infty) \times \partial M \to M$ as above induces an isometry from the warped product $[0, \infty) \times_h \partial M$ onto M. This completes the proof of Theorem C.

Now we shall show the following

THEOREM 1. *Let M be a connected, complete Riemannian manifold of dimension m without boundary. We assume M is noncompact and has nonnegative Ricci curvature. Suppose there exists an isometric minimal immersion $\iota: \tilde{N} \to M$ from a connected, compact Riemannian manifold $\tilde{N}$ of dimension $m-1$ into M. Then a noncompact connected component of $M \setminus N$ ($N = \iota(\tilde{N})$) is the isometric product $(0, \infty) \times L$, where L is compact. In particular, N is a totally geodesic (imbedded) submanifold of M.*

PROOF. Let Ω be a noncompact connected component of $M \setminus N$. Since $\partial\Omega$ is compact, there is a ray $\gamma: [0, \infty) \to \Omega$ such that $\rho_{\partial\Omega}(\gamma(t)) = t$ for any $t \geqq 0$, where $\rho_{\partial\Omega} = \mathrm{dis}_M(\partial\Omega, *)$. Let η_γ be the Busemann function with respect to γ. Then by Lemma (1.12) ($R=0$), η_γ is superharmonic on M. Moreover Lemma (1.8) ($R=0$) tells us that $\rho_N = \mathrm{dis}_M(N, *)$ is superharmonic on $M \setminus N$. In particular, $\eta_\gamma + \rho_{\partial\Omega}$ is superharmonic on B_γ ($\rho_{\partial\Omega} = \rho_{N|\Omega}$) and further it takes the minimum 0 on B_γ, since $(\eta_\gamma + \rho_{\partial\Omega})(\gamma(t)) = 0$ for any $t \geqq 0$. Therefore by the same reason as in the proof of (2) in Theorem C, we see that $\eta_\gamma + \rho_{\partial\Omega} = 0$ on Ω, so that both η_γ and $\rho_{\partial\Omega}$ are harmonic. In particular, $\eta^{-1}(a)$ is a (imbedded) minimal hypersurface for any $a<0$, since the trace of the second fundamental form of $\eta^{-1}(a)$ is equal to $(\Delta\eta)|_{\eta^{-1}(a)}$ by (2.13). Therefore $\{x \in \Omega : \eta(x) < a\}$ for any $a<0$ satisfies all the conditions of (2) in Theorem C. This shows that the map $\Psi_a: (a, \infty) \times \eta^{-1}(a) \to \Omega$ ($a<0$) defined by $\Psi_a(t, x) = \exp_x(t\xi(x))$ ($\xi(x) = -\mathrm{grad}\,\eta(x)$) is an isometry from the Riemannian product $(a, \infty) \times \eta^{-1}(a)$ onto Ω. This completes the proof of the theorem.

Now it would be interesting to give an alternative proof of the following Cheng's theorem [6] as an application of Theorem A.

THEOREM (Cheng). *Let M be a connected compact Riemannian manifold without boundary. Suppose the Ricci curvature of $M \geqq (m-1)R$ ($m = \mathrm{dim} M$ and $R>0$) and the diameter of $M = \pi/\sqrt{R}$. Then M is isometric to the standard sphere $S^m(R)$ of constant curvature R.*

PROOF. Let p and q be two points in M such that $\mathrm{dis}_M(p, q) = \pi/\sqrt{R}$. Fix any small $\varepsilon>0$. We write $B_\varepsilon(q)$ for the metric open ball centered at q with radius ε. Then Lemma (1.1) shows that $\Delta\rho_q \leqq (m-1)\Lambda_\varepsilon$ on $\partial B_\varepsilon(q)$, where $\rho_q = \mathrm{dis}_M(q, *)$ and $\Lambda_\varepsilon = \sqrt{R}\cos\sqrt{R}\varepsilon/\sin\sqrt{R}\varepsilon$. This implies that

$$\text{the trace of } S_\xi \leqq (m-1)\Lambda_\varepsilon, \tag{2.19}$$

where S_ξ is the second fundamental form of $\partial B_\varepsilon(q)$ with respect to a unit normal vector ξ of $\partial B_\varepsilon(q)$ which points to $M \setminus B_\varepsilon(q)$. Moreover by the assumption: $\mathrm{dis}_M(p, q)=\pi/\sqrt{R}$, we see that

$$i(M \setminus B_\varepsilon(q))=C_1(R, \Lambda_\varepsilon). \tag{2.20}$$

Thus by (2.19) and (2.20), we see that $M \setminus B_\varepsilon(q)$ satisfies all the conditions of Theorem A, so that $M \setminus B_\varepsilon(q)$ is isometric to $\bar{B}(C_1(R, \Lambda_\varepsilon); R)$. Since ε is any small positive constant, we can conclude that M is isometric to $S^m(R)$. This completes the proof of the theorem.

Finally we shall show the following

THEOREM 2. *Let M be a connected, complete Riemannian manifold of dimension m without boundary. Suppose the sectional curvature of $M \geqq R>0$ and suppose there are isometric minimal immersions ι_i: $\tilde{N}_i \to M$ $(i=1, 2)$ from connected, compact Riemannian manifolds $\tilde{N}_i$ of dimensions $n_i>0$ into M. Then* $\mathrm{dis}_M(N_1, N_2) \leqq \pi/2\sqrt{R}$, *where $N_i=\iota_i(\tilde{N}_i)$ $(i=1, 2)$. Moreover if the equality holds and $n_1+n_2 \geqq m-1$, then $n_1+n_2=m-1$ and $(M, N_1, N_2)=(S^m(R), S^m(R) \cap L_1, S^m(R) \cap L_2)/\Gamma$, where $S^m(R)=\{v \in \boldsymbol{R}^{m+1}: \|v\|=1/\sqrt{R}\}$, L_i $(i=1, 2)$ are mutually orthogonal linear subspaces of $\boldsymbol{R}^{m+1}$ with* $\dim L_i=n_i+1$, *and Γ acts reducibly on $S^m(R)$ leaving both L_1 and L_2.*

PROOF. The first assertion of our theorem is well known (cf. e.g., [**10**: Chap. IV, Theorem 6.3] and Lemma (1.10)). Now we shall show the second assertion of the theorem. Set $\rho_i=\mathrm{dis}_M(N_i, *)$ $(i=1, 2)$. Then by Lemma (1.10), we see that each ρ_i satisfies $\rho_i \leqq C_1(R, 0)=\pi/2\sqrt{R}$ and

$$\Delta\rho_i \leqq (m-n_i-1)(f'/f)(\rho_i)+n_i(h'/h)(\rho_i)$$

as a distribution on $A_i=\{x \in M: 0<\rho_i(x)<C_1(R, 0)=\pi/2\sqrt{R}\}$. Therefore we see that $\rho_1+\rho_2$ satisfies

$$\begin{aligned}\Delta(\rho_1+\rho_2) \leqq\, & n_1\{(f'/f)(\rho_2)+(h'/h)(\rho_1)\}+n_2\{(f'/f)(\rho_1)+(h'/h)(\rho_2)\} \\ & +(m-n_1-n_2-1)\{(f'/f)(\rho_1)+(f'/f)(\rho_2)\}\end{aligned} \tag{2.21}$$

as a distribution on $A_1 \cap A_2$. Since $\mathrm{dis}_M(N_1, N_2)=\pi/2\sqrt{R}$ and $n_1+n_2 \geqq m-1$, (1.15) implies that the right-hand side of (2.21) is nonpositive, so that $\rho_1+\rho_2$ is superharmonic on $A_1 \cap A_2$. Set $\omega=\{x \in M: 0<\rho_i(x)$ $(i=1, 2)$, $\rho_1(x)+\rho_2(x)=\pi/2\sqrt{R}\}$. Then ω is a closed subset of $M \setminus (N_1 \cup N_2)$ contained in $A_1 \cap A_2$, and further $\rho_1+\rho_2$ takes the minimum $\pi/2\sqrt{R}$ on ω. Therefore by the minimum principle for superharmonic functions, we see that $\rho_1+\rho_2$ is equal to $\pi/2\sqrt{R}$ on a connected component of $A_1 \cap A_2$ which intersects with ω. This shows that ω is open and closed in $M \setminus (N_1 \cup N_2)$, and

$$\rho_1+\rho_2=\mathrm{dis}_M(N_1, N_2)=\frac{\pi}{2\sqrt{R}} \tag{2.22}$$

on ω. If $n_1+n_2>m-1$, the right-hand side of (2.21) is negative. Therefore by (2.22), we see that $n_1+n_2=m-1$, so that $\omega=M\setminus(N_1\cup N_2)$. That is, equality (2.22) holds everywhere on M. Hence we see that, for each i ($i=1, 2$), N_i is a closed (imbedded) submanifold of M and the exponential map $\exp_{N_i}$ of N_i restricted to $\hat{V}_i=\{v\in N_i^{\perp}: \|v\|<\pi/2\sqrt{R}\}$ induces a diffeomorphism between $\hat{V}_i$ and $V_i=\{x\in M: \rho_i(x)<\pi/2\sqrt{R}\}$. Therefore each ρ_i ($i=1, 2$) is smooth on $V_i\setminus N_i$ and further by equality (2.22) on M, we have

$$\Delta\rho_i=(m-n_i-1)(f'/f)(\rho_i)+n_i(h'/h)(\rho_i) \tag{2.23}$$

on $V_i\setminus N_i$. Set $U_\varepsilon=\{x\in M: \varepsilon\leqq\rho_1(x)\leqq(\pi/2\sqrt{R})-\varepsilon\}$ $(0<\varepsilon<\pi/2\sqrt{R})$, $(\partial U_\varepsilon)_+=\{x\in M: \rho_1(x)=\varepsilon\}$, $(\partial U_\varepsilon)_-=\{x\in M: \rho_1(x)=(\pi/2\sqrt{R})-\varepsilon\}$ and

$$\Phi_\varepsilon=-\int_\varepsilon^{\rho_1}\left(\int_t^{\pi/2\sqrt{R}} f^{n_2}h^{n_1}ds\right)\Big/(f^{n_2}h^{n_1})(t)dt\,.$$

Then by (2.23), we see that Φ_ε satisfies

$$\Delta\Phi_\varepsilon=1$$

on $V_1\setminus N_1$. Therefore we have

$$\begin{aligned}
\mathrm{Vol}(M)&=\int_{V_1\setminus N_1}\Delta\Phi_\varepsilon\\
&=\lim_{\varepsilon\to 0}\int_{U_\varepsilon}\Delta\Phi_\varepsilon\\
&=\lim_{\varepsilon\to 0}\int_{\partial U_\varepsilon}*d\Phi_\varepsilon\\
&=\lim_{\varepsilon\to 0}\Big\{\mathrm{Vol}((\partial U_\varepsilon)_+)\cdot\int_\varepsilon^{\pi/2\sqrt{R}}(f^{n_2}h^{n_1})(t)\,dt\Big/(f^{n_2}h^{n_1})(\varepsilon)\\
&\qquad-\mathrm{Vol}((\partial U_\varepsilon)_-)\cdot\int_{\varepsilon'}^{\pi/2\sqrt{R}}(f^{n_2}h^{n_1})(t)\,dt\Big/(f^{n_2}h^{n_1})(\varepsilon')\Big\}\quad\left(\varepsilon'=\frac{\pi}{2\sqrt{R}}-\varepsilon\right)\\
&=\mathrm{Vol}(N_1)\cdot\mathrm{Vol}(S^{n_2}(1))\cdot\int_0^{\pi/2\sqrt{R}}(f^{n_2}h^{n_1})(t)dt\,,
\end{aligned}$$

where $S^{n_2}(1)$ is the unit sphere in Euclidean space of dimension n_2. Hence Theorem 2 follows from Theorem 4.6 in [7]. This completes the proof of Theorem 2.

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