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On solutions to the initial-boundary value problem for $\frac{\partial}{\partial t}u-\Delta\beta(u)=f$

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0. Introduction.

This paper is concerned with the solutions to the problem

$$(1)\qquad \begin{cases} \dfrac{\partial}{\partial t}u-\Delta\beta(u)=f(x, t) & \text{on} \quad \Omega\times(0, \infty) \\ u|_{\partial\Omega}=0, \quad u(x, 0)=u_0(x) \end{cases}$$

where β is a (smooth) monotone increasing function admitting $\beta'(0)=0$, Ω is a bounded domain in n-dimensional Euclidean space R^n with smooth boundary $\partial\Omega$ and Δ is the Laplacian. A typical example of $\beta(u)$ is $|u|^{\alpha}u$ $(\alpha>0)$.

This equation is a typical model of nonlinear degenerate parabolic equations and has been studied by many authors from various points of view (Oleinik, Kalashnikov and Yui-Lin [**16**], Dubinskii [**7**], Raviart [**18**], Brezis [**4**, **5**], Lions [**12**], Crandall [**6**], Konishi [**9**, **10**] etc.; see also Aronson [**1**], Peletier [**17**], Atkinson and Peletier [**2**], Boillet, Saravia and Villa [**3**] etc. for related topics). Thus interesting existence-uniqueness theorems have been established.

However, the asymptotic behavior of the solutions to the problem (1) seems not to be so much investigated. And in the present paper we shall study some decay property of the solutions as $t\rightarrow\infty$. More precisely, we are interested in the decay of $\|\beta(u(\cdot, t))\|_{\mathring{H}_1(\Omega)}$ $\left(\text{and also } \int_t^{t+1}\int_{\Omega}\beta'(u)|u_t|^2 dxds\right)$ which seems to be the most natural quantity as the potential energy to the equation.

Concerning decay of solutions, Brezis [**5**] has already showed that

$$\|\beta(u(\cdot, t))\|_{\mathring{H}_1(\Omega)}\leqq ct^{-1} \quad \text{as} \quad t\rightarrow\infty$$

under appropriate conditions on f, while here we shall give a preciser estimate on the rate of decay. Our assumption on the smoothness of $\beta(u)$ is somewhat stronger than Brezis' one, but the interesting example $\beta(u)=|u|^{\alpha}u$ $(\alpha>0)$ is contained in our case.

Most of recent works approach to the problem (1) by the theory of monotone operators or nonlinear semi-groups, while here we shall employ a rather classical method, i.e., an elliptic regularization method and some energy inequalities are used. By this we can show the existence and the decay property of solutions at the same time. Existence part of our result is related to Lions [**12**].

As another typical example of nonlinear degenerate parabolic equations we know the problem:

$$\frac{\partial}{\partial t}u-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p-2}\frac{\partial}{\partial x_i}u\right)=f(x,t) \quad \text{in } \Omega\times(0,\infty),\quad p>2, \tag{2}$$

$$u|_{\partial\Omega}=0,\quad u(x,0)=u_0(x).$$

The decay property of the solutions to (2) was discussed in our previous paper [**14**] (see also [**15**]) and our method of the present paper is related to the one used there.

1. Assumptions and result.

First we state our precise assumptions on β, u_0 and f.

A. 1. $\beta(u)$ is in $C^2(R-\{0\})\cap C^1(R)$ with $\beta''(u)$ (locally) Hölder continuous on $R-\{0\}$, $\beta(0)=0$, $\beta'(u)>0$ if $u\neq 0$ and

$$k_0|u|^{\alpha+1}\leqq|\beta(u)|,\qquad k_0|\beta(u)|\leqq\beta'(u)|u|\leqq k_1(|\beta(u)|^{\alpha/(\alpha+1)}+1)|u|$$

for some positive constants k_0, k_1 and α.

For $0<\varepsilon<1$ we take a smooth function $\theta_\varepsilon(u)$ such that

$$\theta_\varepsilon(u)=\begin{cases}1 & \text{if } |u|\geqq\varepsilon\\ 0 & \text{if } |u|\leqq\varepsilon/2\end{cases}\qquad 0\leqq\theta_\varepsilon(u)\leqq 1 \quad\text{and}\quad \theta'_\varepsilon(u)u\geqq 0.$$

Setting

$$\beta_\varepsilon(u)=\int_0^u\beta'(s)\theta_\varepsilon(s)ds,$$

$\beta_\varepsilon(u)$ belongs to $C^2(R)$ and $\beta''_\varepsilon(u)$ is (locally) Hölder continuous on R. Moreover it is easily verified from A. 1 that

A. 1′. $$k'_0|u|^{\alpha+1}-k'_1\beta(\varepsilon)\leqq|\beta_\varepsilon(u)|,$$

$$0\leqq\beta'_\varepsilon(u)\leqq k_2(|\beta_\varepsilon(u)|^{\alpha/(\alpha+1)}+1),$$

and

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(u)=\beta(u)\quad\text{and}\quad\lim_{\varepsilon\to 0}\beta'_\varepsilon(u)=\beta'(u)\quad\text{uniformly on } R.$$

The functions $\beta_\varepsilon(u)$ will be used for construction of approximate solutions.

A. 2. $$u_0\in V\equiv\{u\in L^2(\Omega)\,|\,\beta(u)\in\mathring{H}_1(\Omega)\}.$$

A. 3. $$f \in L^2_{\mathrm{loc}}(R^+;\ L^{4(\alpha+1)/(\alpha+2)}(\Omega)) \qquad (R^+ \equiv [0, \infty)).$$

We set

$$\delta(t) \equiv \left(\int_t^{t+1} \|f(s)\|^2_{L^{4(\alpha+1)/(\alpha+2)}(\Omega)} ds\right)^{1/2}, \qquad t \geqq 0.$$

Now we give our definition of solutions to the problem (1).

DEFINITION. A function $u \in L^2_{\mathrm{loc}}(R^+;\ L^2(\Omega))$ is said to be a solution of the problem (1) if $\beta(u) \in L^2_{\mathrm{loc}}(R^+;\ \mathring{H}_1(\Omega))$ and it holds that

$$\int_0^\infty \int_\Omega \left\{-u(x, t)\phi_t(x, t) + \sum_{i=1}^n \frac{\partial}{\partial x_i}\beta(u)\frac{\partial}{\partial x_i}\phi(x, t) - f(x, t)\phi(x, t)\right\} dx dt - \int_\Omega u_0(x)\phi(x, 0) dx = 0 \tag{3}$$

for $\forall \phi \in C_0^1(\Omega \times R^+)$.

REMARK. By a standard way (Lions [**12**]) we see that if u is a solution defined above, $u' \in L^2_{\mathrm{loc}}(R^+;\ H^{-1}(\Omega))$ ($H^{-1}(\Omega)$ is the dual space of $\mathring{H}_1(\Omega)$ with respect to L^2-inner product) and

$$\begin{cases} u'(t) - \Delta\beta u(t) = f(\cdot, t) & \text{in} \quad H^{-1}(\Omega) \quad \text{a. e. } t \\ u(0) = u_0. \end{cases} \tag{1'}$$

Then, our results read as follows.

THEOREM 1. *Under the assumptions* A. 1–A. 3 *the problem* (1) *admits a unique solution u such that*

$$u \in L^\infty_{\mathrm{loc}}(R^+;\ L^2(\Omega)), \quad \beta(u) \in L^\infty_{\mathrm{loc}}(R^+;\ \mathring{H}_1(\Omega)), \quad \int_0^u \sqrt{\beta'(\eta)}\, d\eta \in L^2_{\mathrm{loc}}(R^+;\ \mathring{H}_1(\Omega))$$

and

$$\frac{\partial}{\partial t}\int_0^u \sqrt{\beta'(\eta)}\, d\eta \in L^2_{\mathrm{loc}}(R^+;\ L^2(\Omega)),$$

and moreover, if $0 \leqq \delta(t) \leqq \text{const.}\,(1+t)^{-1-\theta}$ $(\theta > 1/2 + 1/\alpha)$, *we have*

$$\left(\int_t^{t+1}\int_\Omega \left|\frac{\partial}{\partial t}\left(\int_0^u \sqrt{\beta'(\eta)}\, d\eta\right)\right|^2 dx ds\right)^{1/2} + \|\beta(u(\cdot, t))\|_{\mathring{H}_1(\Omega)} \leqq C(\|\beta(u_0)\|_{\mathring{H}_1(\Omega)})(1+t)^{-1-1/\alpha}. \tag{4}$$

THEOREM 2. *In addition to* A. 1–A. 3 *suppose that* $u_0 \geqq 0$ *and* $f \geqq 0$. *Then the problem* (1) *has a unique nonnegative solution u satisfying the same properties as Theorem* 1.

Finally in this section we state a lemma which is used to prove the decay rate of solutions.

LEMMA 1. *Let* $\phi(t)$ *be a nonnegative function on* R^+ *satisfying*

$$\sup_{t\leq s\leq t+1}\phi(s)^{1+r}\leqq C_0\{\phi(t)-\phi(t+1)\}+g(t)+\varepsilon\,,\qquad 0<\varepsilon\leqq 1$$

for some $r, C_0>0$. *Then, if* $0\leqq g(t)\leqq \text{const.}(1+t)^{-1-\theta}$ $(\theta>1/r)$ *we have*

$$\phi(t)\leqq C_1(\phi(0))(1+t)^{-1/r}\qquad for\quad 0<t<C_2\varepsilon^{-1/\theta(1+r)}\qquad (C_2>0)\,.$$

For a proof see Appendix.

2. Proof of Theorem 1.

First we assume $u_0\in C_0^3(\Omega)$ and $f\in C_0^1(\Omega\times R^+)$. Let $0<\varepsilon<1$ and let us consider the regularized problem

$$(\mathrm{P}_\varepsilon)\qquad \begin{cases} \dfrac{\partial}{\partial t}u-\Delta(\varepsilon u+\beta_\varepsilon(u))=f \qquad \text{on}\quad \Omega\times R^+ \\ u\,|_{\partial\Omega}=0\,,\qquad u(x,0)=u_0(x)\,. \end{cases}$$

Then, by Theorem 1 of Ladyzhenskaya, Solonnikov and Ural'tseva [**11**, Chap. V], the problem (P_ε) admits a unique classical solution u_ε with

$$\frac{\partial^2}{\partial t\partial x_i}u_\varepsilon\in L^2_{\mathrm{loc}}(R^+;\,L^2(\Omega))\,,\qquad i=1,2,\cdots,n\,.$$

In the sequel we shall estimate u_ε by the desired norms.

Multiplying (P_ε) by $\dfrac{\partial}{\partial t}(\beta_\varepsilon(u_\varepsilon)+\varepsilon u)$ $(\in L^2_{\mathrm{loc}}(R^+;H_1(\Omega)))$ and integrating over $\Omega\times[t,t+1]$ we have

$$(5)\qquad \begin{aligned}&\int_t^{t+1}\int_\Omega(\beta'_\varepsilon(u_\varepsilon)+\varepsilon)\left|\frac{\partial}{\partial t}u_\varepsilon\right|^2dxds\\ &\quad=E_\varepsilon(u_\varepsilon(t))-E_\varepsilon(u_\varepsilon(t+1))+\int_t^{t+1}\int_\Omega f\Big(\varepsilon\frac{\partial}{\partial t}u_\varepsilon+\frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon)\Big)dxds\end{aligned}$$

where we set $u(t)\equiv u(\cdot,t)$ for simplicity and

$$(6)\qquad E_\varepsilon(u(t))=\frac{1}{2}\|\beta_\varepsilon(u(t))\|^2_{\dot H_1}+\varepsilon\int_\Omega\beta'_\varepsilon(u(t))\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}u(t)\right|^2dx+\frac{\varepsilon^2}{2}\|u(t)\|^2_{\dot H_1}\,,$$

$$\Big(\|u\|_{\dot H_1}\equiv\Big(\int_\Omega\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}u\right|^2dx\Big)^{1/2}\Big).$$

Now,

$$(7)\qquad \begin{aligned}&\left|\int_t^{t+1}\int_\Omega f\frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon)\,dxds\right|\\ &\leqq\int_t^{t+1}\int_\Omega|f|\sqrt{\beta'_\varepsilon(u_\varepsilon)}\cdot\sqrt{\beta'_\varepsilon(u_\varepsilon)}\left|\frac{\partial}{\partial t}u_\varepsilon\right|dxds\\ &\leqq\frac{1}{2}\int_t^{t+1}\int_\Omega|f|^2\beta'_\varepsilon(u_\varepsilon)dxds+\frac{1}{2}\int_t^{t+1}\int_\Omega\beta'_\varepsilon(u_\varepsilon)\left|\frac{\partial}{\partial t}u_\varepsilon\right|^2dxds\,,\end{aligned}$$

and

$$\int_t^{t+1}\int_\Omega |f|^2\beta_\varepsilon'(u_\varepsilon)\,dxds$$

(8)
$$\begin{aligned}&\leqq C\int_t^{t+1}\int_\Omega |f|^2(|\beta_\varepsilon(u_\varepsilon)|^{\alpha/(\alpha+1)}+1)\,dxds\\ &\leqq C\int_t^{t+1}\|f(s)\|^2_{L^{4(\alpha+1)/(\alpha+2)}}(\|\beta_\varepsilon(u_\varepsilon)\|^{\alpha/(\alpha+1)}_{\mathring{H}_1}+1)\,ds\\ &\leqq C\delta(t)^2\{\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))^{\alpha/2(\alpha+1)}+1\}\end{aligned}$$

where C denotes positive constants.

From (6)-(8) we have

(9)
$$\begin{aligned}&\int_t^{t+1}\int_\Omega(\beta_\varepsilon'(u_\varepsilon)+\varepsilon)\left|\frac{\partial}{\partial t}u_\varepsilon\right|^2dxds\\ &\leqq 2\{E_\varepsilon(u_\varepsilon(t))-E_\varepsilon(u_\varepsilon(t+1))\}+C\delta(t)^2\{1+\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))^{\alpha/2(\alpha+1)}\}\\ &\equiv D(t)^2\,.\end{aligned}$$

On the other hand, multiplying (P_ε) by $\varepsilon u_\varepsilon+\beta(u_\varepsilon)$ and integrating, we get

(10)
$$\begin{aligned}&\varepsilon^2\int_t^{t+1}\|u_\varepsilon(s)\|^2_{\mathring{H}_1}ds+\int_t^{t+1}\int_\Omega\left\{2\varepsilon\sum_{i=1}^n\beta_\varepsilon'(u_\varepsilon)\left|\frac{\partial}{\partial x_i}u_\varepsilon\right|^2+\sum_{i=1}^n\left|\frac{\partial}{\partial x_i}\beta_\varepsilon(u_\varepsilon)\right|^2\right\}dxds\\ &=\int_t^{t+1}\int_\Omega\left\{-\beta_\varepsilon(u_\varepsilon)\frac{\partial}{\partial t}u_\varepsilon-\varepsilon\frac{\partial}{\partial t}u_\varepsilon\cdot u_\varepsilon+f(\varepsilon u_\varepsilon+\beta_\varepsilon(u_\varepsilon))\right\}dxds\,.\end{aligned}$$

Each term of the right hand side of (10) is estimated as follows;

$$\begin{aligned}&\int_t^{t+1}\int_\Omega\left|\beta_\varepsilon(u_\varepsilon)\frac{\partial}{\partial t}u_\varepsilon\right|dxds\\ &=\int_t^{t+1}\int_{\{x\,|\,|u_\varepsilon(x,s)|>\varepsilon/2\}}\sqrt{\beta_\varepsilon'(u_\varepsilon)}\left|\frac{\partial}{\partial t}u_\varepsilon\right|\cdot\frac{|\beta_\varepsilon(u_\varepsilon)|}{\sqrt{\beta_\varepsilon'(u_\varepsilon)}}\,dxds\\ &\leqq CD(t)\left(\int_t^{t+1}\int_\Omega\{|\beta_\varepsilon(u_\varepsilon)|^{(\alpha+2)/(\alpha+1)}+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\}\,dxds\right)^{1/2}\quad\text{(by (9) and A.1')}\\ &\qquad\left(\text{note that }\frac{|\beta_\varepsilon(u_\varepsilon)|^2}{\beta_\varepsilon'(u_\varepsilon)}\leqq u_\varepsilon\cdot\beta_\varepsilon(u_\varepsilon)/k_0\quad\text{if }|u_\varepsilon|>\frac{\varepsilon}{2}\right)\\ &\leqq CD(t)\left(\int_t^{t+1}\|\beta_\varepsilon(u_\varepsilon)\|^{(\alpha+2)/(\alpha+1)}_{\mathring{H}_1}+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}ds\right)^{1/2}\\ &\leqq CD(t)\{\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))^{(\alpha+2)/4(\alpha+1)}+\beta(\varepsilon)^{(\alpha+2)/2(\alpha+1)}\}\,,\end{aligned}$$

$$\varepsilon\int_t^{t+1}\int_\Omega\left|\frac{\partial}{\partial t}u_\varepsilon\cdot u_\varepsilon\right|dxds$$

$$\leqq\Big(\int_t^{t+1}\varepsilon\Big\|\frac{\partial}{\partial t}u_\varepsilon\Big\|_{L^2}^2 ds\Big)^{1/2}\Big(\varepsilon\int_t^{t+1}\int_\Omega u_\varepsilon^2 dxds\Big)^{1/2}$$

$$\leqq D(t)\Big(\varepsilon^2\int_t^{t+1}\|u_\varepsilon(s)\|_{L^2}^2 ds\Big)^{1/4}\Big(\int_t^{t+1}\int_\Omega \beta_\varepsilon(u_\varepsilon)^{2/(\alpha+1)}+\beta(\varepsilon)^{2/(\alpha+1)}dxds\Big)^{1/4}$$

$$\leqq CD(t)(\sqrt{\varepsilon}\sup_{t\leqq s\leqq t+1}\|u_\varepsilon(s)\|_{H_1}^{1/2})\{\sup_{t\leqq s\leqq t+1}\|\beta_\varepsilon(u_\varepsilon)\|_{H_1}^{1/2(\alpha+1)}+\beta(\varepsilon)^{1/2(\alpha+1)}\}$$

$$\leqq CD(t)\{\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))^{(\alpha+2)/4(\alpha+1)}+\beta(\varepsilon)^{(\alpha+2)/2(\alpha+1)}\}$$

and

$$\int_t^{t+1}\int_\Omega f(\varepsilon u_\varepsilon+\beta_\varepsilon(u_\varepsilon))\,dxds\leqq C\delta(t)\sup_{t\leqq s\leqq t+1}\sqrt{E_\varepsilon(u_\varepsilon(s))}\,.$$

Thus we have

$$\int_t^{t+1}E_\varepsilon(u_\varepsilon(s))\,ds\leqq C\{D(t)(\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))^{(\alpha+2)/4(\alpha+1)}+\beta(\varepsilon)^{(\alpha+2)/2(\alpha+1)})$$

$$+\delta(t)\sup_{t\leqq s\leqq t+1}\sqrt{E_\varepsilon(u_\varepsilon(s))}\}\tag{11}$$

$$\equiv A(t)$$

which implies that there exists $t^*\in[t, t+1]$ such that

$$E_\varepsilon(u_\varepsilon(t^*))=A(t)\,.$$

Now, using a similar equality as (6) and the inequalities just above we obtain

$$\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))\leqq E_\varepsilon(u_\varepsilon(t^*))+\int_t^{t+1}\int_\Omega\Big\{\Big|f\Big(\varepsilon\frac{\partial}{\partial t}u_\varepsilon+\frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon)\Big)\Big|$$

$$+(\beta_\varepsilon'(u_\varepsilon)+\varepsilon)\Big|\frac{\partial}{\partial t}u_\varepsilon\Big|^2\Big\}dxds$$

$$\leqq C\{D(t)^2+A(t)\}\,,$$

by virtue of (11) and Young's inequality,

$$\leqq C\{D(t)^2+D(t)^{4(\alpha+1)/(3\alpha+2)}+\delta(t)^2+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\}$$

$$+\frac{1}{2}\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))$$

and consequently

$$\sup_{t\leqq s\leqq t+1}E_\varepsilon(u_\varepsilon(s))\leqq C\{D(t)^2+D(t)^{4(\alpha+1)/(3\alpha+2)}+\delta(t)^2+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\}\,.\tag{12}$$

From (12) we can show first the boundedness of $E_\varepsilon(u_\varepsilon(t))$ by a constant independent of ε. Indeed, if $E_\varepsilon(u_\varepsilon(t))\leqq E_\varepsilon(u_\varepsilon(t+1))$ for some t, we have

$$\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))\leqq C\{\delta(t)^2(1+\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))^{\alpha/2(\alpha+1)}) + \delta(t)^{4(\alpha+1)/(3\alpha+2)}(1+\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))^{\alpha/(3\alpha+2)}) + \delta(t)^2+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\}$$

and hence

$$\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))\leqq C\{\delta(t)^2+\delta(t)^{4(\alpha+1)/(\alpha+2)}+\delta(t)^{4(\alpha+1)/(3\alpha+2)}+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\} \equiv C_{0,\varepsilon}(t)\,.$$

Therefore we can conclude, if $0\leqq t\leqq T$,

$$E_\varepsilon(u_\varepsilon(t))\leqq \max(\sup_{0\leqq s\leqq 1} E_\varepsilon(u_\varepsilon(s)),\ \sup_{0\leqq t\leqq T} C_{0,\varepsilon}(t)) \leqq \sup_{0\leqq t\leqq T}\{C_{0,\varepsilon}(t)+E_\varepsilon(u_0)\} \leqq C(T,\ \|\beta(u_0)\|_{\dot{H}^*},\ \|\sqrt{\beta'(u_0)}\,u_0\|_{L^2},\ \|u_0\|_{\dot{H}_1}) <\infty\,. \tag{13}$$

In particular, if $\sup_{0\leqq t<\infty} \delta(t)<\infty$, we have

$$E_\varepsilon(u_\varepsilon(t))\leqq C_{1,\varepsilon}+E_\varepsilon(u_0) \qquad \text{for}\quad t\in R^+ \tag{14}$$

where $C_{1,\varepsilon}\equiv \sup_{t\in R^+} C_{0,\varepsilon}(t)$.

Using (14) we shall derive further estimate of u_ε. We denote by C_ε various constants which depend on $E_\varepsilon(u_0)+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}$ continuously.

By (12) and (14) we have

$$\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))\leqq C_\varepsilon\{D(t)^{4(\alpha+1)/(3\alpha+2)}+\delta(t)^2+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\}$$

(note that $4(\alpha+1)/(3\alpha+2)<2$).

Recalling the definition of $D(t)$ we can derive from above

$$\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))^{(3\alpha+2)/2(\alpha+1)} \leqq C_\varepsilon\{E_\varepsilon(u_\varepsilon(t))-E_\varepsilon(u_\varepsilon(t+1))\} + C_\varepsilon\{\delta(t)^2+\delta(t)^2\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))^{\alpha/2(\alpha+1)}+\delta(t)^{(3\alpha+2)/(\alpha+1)}+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\}$$

and

$$\sup_{t\leqq s\leqq t+1} E_\varepsilon(u_\varepsilon(s))^{(3\alpha+2)/2(\alpha+1)} \leqq C_\varepsilon\{E_\varepsilon(u_\varepsilon(t))-E_\varepsilon(u_\varepsilon(t+1))\}+C_\varepsilon\{\delta(t)^2+\delta(t)^{(3\alpha+2)/(\alpha+1)}+\beta(\varepsilon)^{(\alpha+2)/(\alpha+1)}\}\,. \tag{15}$$

We have now arrived at the desired difference inequality concerning the energy $E_\varepsilon(u_\varepsilon(t))$.

Applying Lemma 1 to (15) with $r=\alpha/2(\alpha+1)$ and $g(t)=C\delta(t)^2$ we obtain

$$E_\varepsilon(u_\varepsilon(t))\leqq C_\varepsilon(1+t)^{-2(\alpha+1)/\alpha} \quad \text{if} \quad 0<t\leqq C_\varepsilon\beta(\varepsilon)^{-2(\alpha+2)/\theta(3\alpha+2)} \tag{16}$$

under the assumption $0\leqq\delta(t)\leqq\text{const.}(1+t)^{-(1+\theta)}$ $(\theta>1/2+1/\alpha)$.

The estimate (16) is equivalent to the following ones;

$$\|\beta_\varepsilon(u_\varepsilon(t))\|_{\dot{H}_1}\leqq C_\varepsilon(1+t)^{-1-1/\alpha}, \tag{17}$$

$$\varepsilon\|u_\varepsilon(t)\|_{\dot{H}_1}\leqq C_\varepsilon(1+t)^{-1-1/\alpha}, \tag{18}$$

$$\int_t^{t+1}\left\|\sqrt{\beta_\varepsilon'(u_\varepsilon)}\left|\frac{\partial}{\partial t}u_\varepsilon\right|\right\|_{L^2}^2 ds\leqq C_\varepsilon(1+t)^{-2-2/\alpha}. \tag{19}$$

Moreover, multiplying (P_ε) by u_ε we see immediately

$$\int_0^t\int_\Omega\left\{\sum_{i=1}^n\beta_\varepsilon'(u_\varepsilon)\left|\frac{\partial}{\partial x_i}u_\varepsilon(s)\right|^2 dxds+\|u_\varepsilon(t)\|_{L^2}^2\leqq\left(\int_0^t\|f(s)\|_{L^2}^2ds+\|u_0\|_{L^2}^2\right)e^t. \tag{20}$$

Also we note that by (17) and (19)

$$\begin{aligned}
&\int_t^{t+1}\int_\Omega\left|\frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon)\right|dxds\\
&\leqq C\int_t^{t+1}\int\sqrt{\beta_\varepsilon'(u_\varepsilon)}\cdot\sqrt{\beta_\varepsilon'(u_\varepsilon)}\left|\frac{\partial}{\partial t}u_\varepsilon\right|dxds\\
&\leqq C\left(\int_t^{t+1}\int_\Omega\beta'(u_\varepsilon)\left|\frac{\partial}{\partial t}u_\varepsilon\right|^2dxds\right)^{1/2}\left(\int_t^{t+1}\|\beta_\varepsilon(u_\varepsilon)\|_{L^2}^{\alpha/(\alpha+1)}ds+1\right)^{1/2} \qquad (21)\\
&\leqq C_\varepsilon(1+t)^{-1-1/\alpha}.
\end{aligned}$$

When the decay assumption on $\delta(t)$ is not made we should replace the right hand sides of (17)-(21) by $C(T)$. Needless to say, such estimates are valid for $0\leqq t\leqq T$.

In any way, on the basis of above estimates we can choose a converging subsequence from $\{u_\varepsilon(t)\}$ as $\varepsilon\to0$.

First, from (17) and (21), with the aid of Aubin's compactness theorem (see Lions [12]) we see

$$\beta_\varepsilon(u_\varepsilon(t))\longrightarrow\chi(t) \quad \text{strongly in } L^p_{\text{loc}}(R^+;L^2(\Omega)),\quad p>0, \text{ as } \varepsilon\to0 \tag{22}$$

(more precisely, along a subsequence). By (22) we may asssume also

$$\beta_\varepsilon(u_\varepsilon(x,t))\longrightarrow\chi(x,t) \qquad \text{a.e. on} \quad \Omega\times R^+$$

and hence we have

$$u_\varepsilon(x,t)\longrightarrow u(x,t)\equiv\beta^{-1}(\chi(x,t)) \qquad \text{a.e. on} \quad \Omega\times R^+. \tag{23}$$

By this fact and (17), (20) we have

(24) $$\beta_\varepsilon(u_\varepsilon) \longrightarrow \beta(u) \quad \text{weakly* in} \quad L^\infty_{loc}(R^+; \mathring{H}_1(\Omega))$$

and

(25) $$u_\varepsilon \longrightarrow u \quad \text{weakly* in} \quad L^\infty_{loc}(R^+; L^2(\Omega)).$$

Furthermore, by (19) and (20),

(26) $$\frac{\partial}{\partial t}\left(\int_0^{u_\varepsilon} \sqrt{\beta'_\varepsilon(\eta)}\, d\eta\right) \longrightarrow \frac{\partial}{\partial t}\left(\int_0^u \sqrt{\beta'(\eta)}\, d\eta\right) \quad \text{weakly in} \quad L^2_{loc}(R^+; L^2(\Omega))$$

and

(27) $$\int_0^{u_\varepsilon} \sqrt{\beta'_\varepsilon(\eta)}\, d\eta \longrightarrow \int_0^u \sqrt{\beta'(\eta)}\, d\eta \quad \text{weakly in} \quad L^2_{loc}(R^+; \mathring{H}_1).$$

Thus, taking the limit as $\varepsilon \to 0$ along a subsequence in the variational equation

$$\int_0^\infty \int_\Omega \Big\{ -u_\varepsilon(x, t)\phi_t(x, t) + \sum_{i=1}^n \frac{\partial}{\partial x_i}\beta(u_\varepsilon)\frac{\partial}{\partial x_i}\phi(x, t) - \varepsilon u_\varepsilon \Delta\phi(x, t)$$
$$- f(x, t)\phi(x, t) \Big\} dx dt - \int_\Omega u_0(x)\phi(x, 0) dx = 0$$

which is valid for $\phi \in C_0^2(\Omega \times R^+)$, we see that $u(x, t)$ is a solution of the problem (1). We know from (17)-(20)

$$\|\beta(u(t))\|_{\mathring{H}_1} \leqq C(\|\beta(u_0)\|_{\mathring{H}_1})(1+t)^{-1-1/\alpha},$$
$$\left(\int_t^{t+1} \left\| \frac{\partial}{\partial t}\left(\int_0^u \sqrt{\beta'(\eta)}\, d\eta\right)\right\|^2_{L^2} ds\right)^{1/2} \leqq C(\|\beta(u_0)\|_{\mathring{H}_1})(1+t)^{-1-1/\alpha},$$

and

$$\int_0^t \left\| \int_0^u \sqrt{\beta'(\eta)}\, d\eta \right\|^2_{\mathring{H}_1} ds + \|u(t)\|^2_{L^2} \leqq \left(\|u_0\|^2_{L^2} + \int_0^t \|f(s)\|^2_2 ds\right) e^t$$

which are the required estimates.

Next we consider the case $u_0 \in V$. In this case we can choose a sequence $\{u_{0m}\} \subset C^3(\bar{\Omega}) \cap C_0^1(\Omega)$ such that $\beta(u_{0m}) \to \beta(u_0)$ in $\mathring{H}_1(\Omega)$ (cf. [8]). We may assume also $u_{0m} \to u_0$ in $L^2(\Omega)$. Let $\{f_m\} \subset C_0^1(\Omega \times R^+)$ be a sequence such that $\lim_{n\to\infty} f_m = f$ in $L^2_{loc}(R^+; L^{4(\alpha+1)/(\alpha+2)}(\Omega))$. Then we can define $u_m(x, t)$ as a solution of (1) with $u_m(x, 0) = u_{0m}$ and $f = f_m$. By almost the same arguments used just above we can obtain the desired solution $u(x, t)$ as a limit of $\{u_m\}$.

Finally we must prove the uniqueness. For this we note that if $u(t)$ is a solution of (1) the equation (1)′ in H^{-1} holds. Let us introduce a new inner-product of H^{-1} as follows:

$$\langle u, v\rangle_{H^{-1}} \equiv \langle \tilde{u}, v\rangle_{\mathring{H}_1 \times H^{-1}} \quad \text{for} \quad u, v \in H^{-1}$$

where $\tilde{u} \in \mathring{H}_1$ is determined by $-\Delta\tilde{u} = u$, and $\langle u, v\rangle_{\mathring{H}_1 \times H^{-1}}$ is the pairing of $u \in \mathring{H}_1$ and $v \in H^{-1}$ such that

$$\langle u, v\rangle_{\mathring{H}_1\times H^{-1}}=\int_\Omega u(x)v(x)dx \qquad \text{if} \quad v\in L^2 .$$

Then it is easy to see $-\Delta\beta(u)$ is a monotone operator from $D\equiv\{u\in L^2 | \beta(u)\in\mathring{H}_1\}$ to H^{-1} (cf. Brezis [**5**], Lions [**12**]). The uniqueness follows immediately from this fact.

REMARK 2. By the proof of Theorem 1 we know that the left hand side of (4) is bounded (tends to 0 as $t\to\infty$) if $\sup_{t\in R^+}\delta(t)<\infty$ ($\delta(t)\to 0$ as $t\to\infty$) (cf. [**13**]).

3. Proof of Theorem 2.

The proof of Theorem 2 is essentially included in the previous section. However, we consider here somewhat different approximate equations (cf. [**16**]), i. e.,

$$(\mathrm{P}_\varepsilon)' \qquad \begin{cases} \dfrac{\partial}{\partial t}u-\Delta(\varepsilon u+\beta(u))=f & \text{on} \quad \Omega\times R^+ \\ u|_{\partial\Omega}=\varepsilon , \quad u(x, 0)=u_0(x)+\varepsilon & (0<\varepsilon<1) . \end{cases}$$

First we assume $u_0\in C_0^3(\Omega)$ and $f\in C_0^1(\Omega\times R^+)$ (this assumption is finally removed as in the preceding section). Then, the problem $(\mathrm{P}_\varepsilon)'$ is equivalent to

$$(\mathrm{P}_\varepsilon)'' \qquad \begin{cases} \dfrac{\partial}{\partial t}u-\Delta(\varepsilon u+\beta(u)\theta_\varepsilon(u))=f & \text{on} \quad \Omega\times R^+ \\ u|_{\partial\Omega}=\varepsilon , \quad u(x, 0)=u_0(x)+\varepsilon \end{cases}$$

as long as $u\geqq\varepsilon$, where $\theta_\varepsilon(u)$ is a smooth function such that

$$\theta_\varepsilon(u)=\begin{cases} 1 & \text{if} \quad u\geqq\varepsilon \\ 0 & \text{if} \quad u\leqq\dfrac{1}{2}\varepsilon \end{cases} \qquad \text{and} \quad \theta_\varepsilon'(u)\geqq 0 .$$

Since $(\mathrm{P}_\varepsilon)''$ has a unique classical solution u_ε with $\dfrac{\partial^2}{\partial t\partial x_i}u_\varepsilon\in L^2_{\mathrm{loc}}(\bar{\Omega}\times R^+)$ ([**11**]) and $u\geqq\varepsilon$ (by the maximum principle), the problem $(\mathrm{P}_\varepsilon)'$ has also the same solution u_ε. To estimate u_ε we rewrite $(\mathrm{P}_\varepsilon)'$ as follows.

$$(\tilde{\mathrm{P}}_\varepsilon) \qquad \begin{cases} \dfrac{\partial}{\partial t}\tilde{u}_\varepsilon-\Delta(\varepsilon\tilde{u}_\varepsilon+\tilde{\beta}_\varepsilon(\tilde{u}_\varepsilon))=f & \text{on} \quad \Omega\times R^+ , \\ \tilde{u}|_{\partial\Omega}=0 , \quad \tilde{u}_\varepsilon(x, 0)=u_0(x) \end{cases}$$

where $\tilde{u}_\varepsilon(x, t)=u_\varepsilon(x, t)-\varepsilon$ ($\geqq 0$) and

$$\tilde{\beta}_\varepsilon(s)=\beta(s+\varepsilon)-\beta(\varepsilon) .$$

Then it is easy to see

$$\tilde{\beta}_\varepsilon(s)\geqq k_0'|s|^{\alpha+1}-\beta(\varepsilon) \quad\text{and}\quad 0\leqq\tilde{\beta}_\varepsilon'(s)\leqq k_1'\{|\tilde{\beta}_\varepsilon(s)|^{\alpha/(\alpha+1)}+1\}\,,\quad 0\leqq s\,.$$

Thus we can obtain almost the same estimates (17)-(21) for $\tilde{u}_\varepsilon$ with β_ε replaced by $\tilde{\beta}_\varepsilon$. Consequently, using almost the same compactness argument, we obtain Theorem 2.

REMARK 3. Using Theorem 1 we can give a direct proof of Theorem 2. Indeed, if we set $u^-=\min\{u, 0\}$ and $B(u^-)=\int_0^{u^-}\beta(\eta)d\eta$ for the solution u we have easily $\int_\Omega B(u^-)(x, t)dx=0$, $t\geqq 0$, i.e., $u^-(x, t)=0$ a.e. x.

Appendix.

Lemma 1 is a variant of Lemma in [**13**] (see also [**15**]). Here we sketch the outline of the proof for completeness.

Setting

$$\psi(t)=\phi(t)+\nu t^{-\theta}$$

we know for large ν and for a certain $T_0>0$

$$\sup_{t\leq s\leq t+1}\psi(s)^{1+r}\leqq C_1(\psi(t)-\psi(t+1))+C_2\varepsilon \qquad\text{if}\quad t\geqq T_0$$

with some C_1, $C_2>0$.

Next, setting $y(t)=\psi(t)^{-r}$, we have

$$\begin{aligned}
y(t+1)-y(t)&=\int_0^1\frac{d}{d\theta}\{\theta\psi(t+1)+(1-\theta)\psi(t)\}^{-r}d\theta\\
&=r\int_0^1\{\theta\psi(t+1)+(1-\theta)\psi(t)\}^{-1-r}d\theta\,\{\psi(t)-\psi(t+1)+C_2/C_1\varepsilon\}\\
&\quad -C_2/C_1\varepsilon r\int_0^1\{\theta\psi(t+1)+(1-\theta)\psi(t)\}^{-1-r}d\theta\\
&\geqq C_3-C_4t^{\theta(1+r)}\varepsilon\\
&\geqq\frac{C_3}{2}>0 \qquad\text{if}\quad T_0<t<C_5\varepsilon^{-r/\theta(1+r)}
\end{aligned}$$

with certain C_3, C_4 and $C_5>0$. From this we obtain

$$y(t)\geqq\min_{s\in[T_0,T_0+1]}y(s)+\frac{C_3}{2}(t-T_0-1)\,,\quad T_0+1\leqq t\leqq C_5\varepsilon^{-r/\theta(1+r)}$$

which implies easily

$$\phi(t)\leqq C(\phi(0))(1+t)^{-1/r} \qquad\text{for}\quad 0\leqq t\leqq C_5\varepsilon^{-r/\theta(1+r)}\,.$$

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Added in proof.

1) Modifying our method we can prove that if $\beta(u)=|u|^{\alpha}u$ the right-hand side of (4) is replaced by $Ct^{-1-1/\alpha}$ with a constant C independent of u_0.

2) After the submittal of the paper we learned some papers which treat the estimates for $\|u(t)\|_{\infty}$; a) N. D. Alikakos, L^p bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations, **4** (1979), 827-868, b) L. Véron, Effets regularisants de semi-groupes non-linéaires dans les espaces de Banach, Ann. Fac. Sci. Toulouse, **1** (1979), 171-200, c) D. G. Aronson and L. A. Peletier, Large time behavior of solutions of the porous medium equation in bounded domain, J. Differential Equations, **39** (1981), 378-412, etc. By a) we see that our estimate (4) is sharp if $f\equiv 0$ and $\beta(u)=|u|^{\alpha}u$.