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On the degree of symmetry of a certain manifold

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Introduction.

In their paper [10], R. Schoen and S. T. Yau have studied compact Lie group actions on the manifold which admits a map of degree one into a Riemannian manifold with non-positive sectional curvature. One of our purpose of this paper is to prove the topological part of results of Theorem 7 in [10] without differential geometrical methods. Since a Riemannian manifold with non-positive sectional curvature is *aspherical*, i.e. a manifold whose universal covering is contractible, we restrict ourselves to manifolds which admit a map of degree one into an aspherical manifold. In this note, we shall first prove a result which is analogous to [5] and then apply it to the study of a compact connected Lie group action on the manifold which admits a map of degree one into an aspherical manifold. We shall also consider the degree of symmetry of a connected sum $M\sharp N$, where M is a closed manifold and N is an aspherical manifold.

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In this note, we shall only consider continuous action and the term "*manifold*" will mean compact connected topological manifold without boundary. Note that manifolds have the homotopy type of a finite CW complex [12].

1. Statement of results.

Unless the contrary is stated, the manifold is assumed to be oriented from now on.

Let M be an m-dimensional manifold. Assume there is a map $f: M\rightarrow N$, where N is an aspherical manifold such that $f^{*}: H^{k}(N: Z)\rightarrow H^{k}(M: Z)$ is nontrivial for some integer $k\,(1\leqq k\leqq \dim M)$, where Z denotes the group of integers. We shall prove the following

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THEOREM A. *Let M, N and k be as above. Assume M admits almost effective action of a compact connected Lie group G. Then the following statements are all valid.*

(1) $\dim G' \leqq (m-k)(m-k+1)/2$, *where G' is the semi-simple part of G.*

(2) *If $k=m$, then G is a torus whose rank is at most the rank of the center of the fundamental group of M, G_x is finite for all $x \in M$ and the Euler characteristic $\chi(M)$ is zero.*

(3) *If $k=m$, $f_*: \pi_1(M) \to \pi_1(N)$ is surjective and $\pi_1(N)$ is centerless, then G is trivial.*

REMARKS. (1) If M is aspherical, then we can take the identity map as f. Hence the statement (2) implies Theorem 5.6 in [**5**].

(2) According to [**9**], we call M a *hypertoral manifold* if there are 1-dimensional cohomology classes $w_1, w_2, \cdots, w_m$ such that $(w_1 \cup w_2 \cup \cdots \cup w_m)[M]=1$. It is clear that M is a hypertoral manifold if and only if there is a map $f: M \to T^m$ of degree one. Thus we obtain Theorem A in [**2**].

(3) The following Proposition was pointed to us by Professor R. Schultz.

PROPOSITION. *Let M be an m-dimensional manifold with the fundamental group $\pi_1(M)=Z \oplus Z \oplus \cdots \oplus Z$ (m-times). Then M is hypertoral.*

(4) Let M and N be manifolds of the same dimension m. It is easy to see that there is a map $f: M \sharp N \to N$ such that $f^*: H^m(N; Z) \to H^m(M \sharp N: Z)$ is an isomorphism. It follows from this observation and Theorem A above that we obtain the following

PROPOSITION (cf. [**11**]. Corollary 2 to Theorem 3). *Let M, N and N′ be m-dimensional, n-dimensional manifolds and (m−n)-dimensional aspherical manifold, respectively. Then we have $S_t^s(M \sharp (N \times N')) \leqq n(n+1)/2$, where $S_t^s(X)$, the topological semi-simple degree of symmetry of X, is the maximal dimension of compact connected semisimple Lie group which acts on X almost effectively.*

Finally we shall consider the degree of symmetry of a connected sum $M \sharp N$, where M is an m-dimensional manifold and N is an m-dimensional aspherical manifold. We shall prove the following

THEOREM B. *Let M and N be as above. Assume M is not a homotopy sphere. Then we have $S_t(M \sharp N)=0$, where $S_t(X)$, the topological degree of symmetry, is the maximal dimension of compact connected Lie group which acts on X almost effectively.*

2. Proof of Theorems.

In this section, we shall prove Theorems A and B stated in Section 1. To prove Theorem A, we consider a slightly more general situation. Unless the contrary is stated, the manifold M is assumed to admit a map f into a finite dimensional Eilenberg-Maclane space N such that $f^*: H^k(N: Z) \to H^k(M: Z)$ is non-trivial for some integer $k\,(1\leqq k\leqq \dim M)$. Let a compact connected Lie group G act on M effectively. We define the evaluation map $\mathrm{ev}^x: G\to M$ by $\mathrm{ev}^x(g)=gx$ for $x\in M$. Now we obtain the following

LEMMA 1. *If the image of* $\mathrm{ev}^x_*: \pi_1(G, E)\to\pi_1(M, x)$ *is contained in the kernel of* $f_*: \pi_1(M, x)\to\pi_1(N, f(x))$, *then the composition* $f\cdot i$ *is homotopic to the constant map, where* $i: G(x)\to M$ *is the inclusion.*

PROOF. It follows from the homotopy exact sequence of the fibering $G_x\to G\to G(x)$ that the index of $\mathrm{Im\,ev}^x_*$ in $\pi_1(G(x))$ is finite. Moreover it follows from the assumption that the correspondence between $\pi_1(G(x))/\mathrm{ev}^x_*(\pi_1(G))$ (=the set of cosets) and $\pi_1(G(x))/\mathrm{Ker}(f\,|\,G(x))_*$ is surjective, which implies $\pi_1(G(x))/\mathrm{Ker}(f\,|\,G(x))_*$ and hence $\mathrm{Im}(f\,|\,G(x))_*$ is finite. It is well known that $\pi_1(N)$ has no element of finite order (see [4] Chapter 9 for example). It follows that $\mathrm{Im}(f\,|\,G(x))_*$ is trivial. Since N is an Eilenberg-Maclane space $K(\pi_1(N), 1)$, the composition $f\cdot i$ is homotopic to the constant map. This completes the proof of Lemma 1.

Let $\tilde{N}$ be the universal covering space of N and $\tilde{M}$ the pullback of $\tilde{N}$ by f. If $f_*: \pi_1(M)\to\pi_1(N)$ is surjective, then $\tilde{M}$ is also a covering space. If f_* is not surjective, $\tilde{M}$ is not arcwise connected. In this case, consider the covering space N' of N corresponding to the subgroup $\mathrm{Im}\,f_*$. It is well known that the map $f: M\to N$ can be lifted to a map $f': M\to N'$ such that f is homotopic to the composition $p'\cdot f'$, where $p': N'\to N$ is the projection. Since $f'^*: H^k(N': Z)\to H^k(M: Z)$ is non-trivial, we may assume $f_*: \pi_1(M)\to\pi_1(N)$ is surjective.

We have the following

LEMMA 2. *Assume the hypothesis in Lemma* 1. *Then the action of* G *on* M *can be lifted to the action of* G *on* $\tilde{M}$ *and the natural mapping* $\tilde{M}/G\to M/G$ *is a covering projection such that the following diagram is commutative, where* $\tilde{q}$, q *are the orbit maps.*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\ \tilde{q}\ } & \tilde{M}/G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ M & \xrightarrow{\ q\ } & M/G\,. \end{array}$$

PROOF. It follows from a result in [5] (Theorem 4.3 in [5]) that the action of G on M can be lifted to an action of G on M. It follows from Lemma 1 that for every point x in M, $p^{-1}(G(x))=G(x)\times\pi_1(N)$ as fiber bundle over $G(x)$. This implies that the action of $\pi_1(N)$ on $\tilde{M}/G$ is free and hence $\tilde{M}/G\to M/G$ is a covering projection. It is easy to see that the above diagram is commutative. This completes the proof of Lemma 2.

Note that the covering $\tilde{N}\to N$ can be considered as the universal $\pi_1(N)$-bundle. Let $g: M/G\to N$ be the classifying map of the fiber bundle $\tilde{M}/G\to M/G$. It follows from Lemma 2 that f is homotopic to the composition $g \cdot q$. Recall that $f^*: H^k(N: Z) \to H^k(M: Z)$ is not zero. Therefore $g^*: H^k(N: Z) \to H^k(M/G: Z)$ is not zero, which implies that $\dim M/G$ is at least k. In particular, the dimension of a principal orbit is at most $m-k$. It follows from a well known result that $\dim G$ is at most $(m-k)(m-k+1)/2$. Thus we have proved the following

PROPOSITION 3. *Assume the hypothesis in Lemma* 1. *Then we have* $\dim G \leqq (m-k)(m-k+1)/2$.

REMARK. In the proof of Proposition 3 we have used the fact that M and M/G have the homotopy type of a finite CW complex. This fact is guaranteed by the works of P. E. Conner, R. Oliver and J. E. West ([**3**], [**8**] and [**12**]).

Now we assume that G is semisimple. Then $\pi_1(G)$ is finite and hence the hypothesis in Lemma 1 is satisfied since $\pi_1(N)$ has no element of finite order. Thus Proposition 3 implies the part (1) of Theorem A.

Consider the case in which k is equal to m. It follows from the above arguments that G is a torus. Now we can show the following

PROPOSITION 4. *The action of G on M is injective; in other words,* $ev_*^{\sharp}$: $\pi_1(G, e) \to \pi_1(M, x)$ *is injective for every point x in M.*

PROOF. Assume $ev_*^{\sharp}(a)=1$. Let $h: S^1 \to G$ be a homomorphism representing the class a. Assume $a \neq 1$. Then the action of S^1 on M induced from h is non-trivial, because G acts on M effectively. It is clear that the action (S^1, M) satisfies the hypothesis in Lemma 1. It follows from Proposition 3 that we have $\dim S^1=0$, which is absurd. This completes the proof of Proposition.

Now we shall prove the rest of Theorem A. Assume that k is equal to m. It follows from Theorem 4.2 in [**5**] that $\operatorname{Im} ev_*^{\sharp}$ is contained in the center of $\pi_1(M)$ which implies the first part of (2). If there is a point x such that $\dim G_x > 0$, then Proposition 4 does not hold for this point x. Next assume $\chi(M) \neq 0$. Then the fixed point set is not empty, which contradicts the fact $\dim G_x=0$ for every point x in M. If f_* is surjective, then $f_*((\operatorname{Im} ev_*^{\sharp})$ is a central subgroup of $\pi_1(N)$. The assumption implies that $f_*((\operatorname{Im} ev_*^{\sharp})=1$. Now Proposition 3 implies the part (3) of Theorem A. This completes the proof of Theorem A.

Finally we shall prove Theorem B. Let M and N be m-dimensional manifolds and N aspherical. Assume the connected sum $M \sharp N$ admits a non-trivial S^1-action. Put $X=M \sharp N$. If $\pi_1(M) \neq 1$, then $\pi_1(X)$ is centerless. Then the part (2) of Theorem A leads to a contradiction. Thus we have $\pi_1(M)=1$. Consider the covering space X_Z of X corresponding to the subgroup $\operatorname{Im} ev_*^{\sharp}=Z$ of $\pi_1(X)=\pi_1(N)$. Assume $\Gamma=\pi_1(N)/Z$ is not trivial. It follows from a result in [**6**] (Theorem 3.1 in [**6**]) that X_Z is equivariantly homeomorphic to $S^1 \times M'$ ($M'=X_Z/S^1$). Note that M' is simply connected. On the other hand, it follows from the argument in [**1**] that X_Z is homeomorphic to the space

$$(\sharp)\qquad (N_Z-\text{int}\, D^m\times\Gamma)\underset{S^{m-1}\times\Gamma}{\cup}(M-\text{int}\, D^m)\times\Gamma,$$

where N_Z is the covering space of N corresponding to the subgroup Z of $\pi_1(N)$. Since N_Z is also aspherical and $\pi_1(N_Z)=Z$, N_Z is homotopy equivalent to S^1. It follows from the excision property that $H_i(N_Z, N_Z-\text{int}\, D^m\times\Gamma: Z)=0$ for $i\neq m$, which implies that $H_i(N_Z-\text{int}\, D^m\times\Gamma: Z)=0$ for $2\leqq i\leqq m-2$. The Mayer-Vietoris exact sequence applied to the space ($\sharp$) implies that $H_i(X_Z: Z)=H_i((M-\text{int}\, D^m)\times\Gamma: Z)$ for $2\leqq i\leqq m-2$. Since M is assumed to be simply connected and not a homotopy sphere, there is an integer i such that $H_i(M': Z)\neq 0$. Let r be the minimal value of i such that $H_i(M': Z)\neq 0$. It is easy to see that $2\leqq r\leqq m-2$. Thus we have that $H_r(M': Z)=H_r((M-\text{int}\, D^m)\times\Gamma; Z)$. Let $\tilde{X}$ be the universal covering space of X. Since $\tilde{X}$ is also the universal covering space of X_Z, we have $\tilde{X}=R^1\times M'$ and $\pi_1(X_Z)=Z$ acts on $H_r(\tilde{X}: Z)=H_r(M'; Z)$ trivially. But this is proved to be impossible. In fact, it follows from the argument in [**1**] that $\tilde{X}$ is homeomorphic to the space

$$(\tilde{N}-\text{int}\, D^m\times\pi_1(N))\underset{S^{m-1}\times\pi_1(N)}{\cup}(M-\text{int}\, D^m)\times\pi_1(N),$$

where $\tilde{N}$ is the universal covering space of N. It is easy to see that $H_r(M'; Z)=H_r(\tilde{X}; Z)=H_r((M-\text{int}\, D^m)\times\pi_1(N); Z)$ on which Z acts non-trivially. Thus we have shown that Γ is trivial, in other words, $\text{ev}_*^{\sharp}: \pi_1(S^1, e)\to\pi_1(X, x)$ is an isomorphism. It follows again from a result in [**6**] (Theorem 3.1 in [**6**]) that X is equivariantly homeomorphic to $S^1\times(X/S^1)$. Note that X/S^1 is simply connected. Consider the universal covering space $\tilde{X}$ of X. Then we have $\tilde{X}=R^1\times(X/S^1)$ and $\tilde{X}$ is proved to be homeomorphic to the space

$$(*)\qquad (N-\text{int}\, D^m\times Z)\underset{S^{m-1}\times Z}{\cup}(M-\text{int}\, D^m)\times Z\,.$$

Let s be the minimal value of i such that $H_i(M-\text{int}\, D^m; Z)\neq 0$. Since M is not a homotopy sphere, s is smaller than $m-1$. Since X/S^1 is compact, $H_s(\tilde{X}: Z)$ is finitely generated. However the space (*) has non-finitely generated s-dimensional homology group. This is a contradiction and completes the proof of Theorem B.

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