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An axiomatization theorem

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Let L be a first order finitary predicate logic with equality $L_{\omega,\omega}$, or a first order infinitary predicate logic with equality $L_{\omega_1,\omega}$, and I, J, K three sets of formulas in L. Then, I and J are said to be equivalent in L over K if A is provable from I in L iff A is provable from J in L, for any formula A in K. Also, I is said to be an axiomatization of J in L, if I is a subset of J and, I and J are equivalent in L over J. An axiomatization theorem of J in L is a statement to give us a "concrete" method to construct a "simple" axiomatization of J in L. Of course, "concrete" and "simple" have no precise mathematical meanings and we use them rather informally. But, two remarks on them will be given in the following. First, if $L=L_{\omega,\omega}$ and J is a recursively enumerable set of formulas in L under a nice Gödel-numbering, then there is a method to construct a primitive recursive axiomatization of J in L by the well-known theorem due to W. Craig (cf. Craig [2]). But the axiomatization of J obtained by Craig's method seems to be so complicated that, in practice, one can not easily tell whether or not a given formula belongs to it, generally (cf. p. 141 in Keisler [4]). This shows us that it is meaningful to give a concrete method to construct a simple axiomatization of J, even if J is recursively enumerable. Secondly, the set J is usually defined using some parameters $p_1, p_2, \cdots, p_n$. So, in order to give an axiomatization theorem of J in L, we should clearly state how to construct an axiomatization of J from each values of parameters $p_1, p_2, \cdots, p_n$, concretely. To define sets of formulas which we are going to deal with, and to state their axiomatization theorem, we require some definitions. Suppose that W is a set of predicate symbols. Then, W-free (W-positive, W-negative) formulas are formulas which have no (no negative, no positive) occurrences of predicate symbols in W. W-atomic formulas are formulas of the form $P(\bar{t})$ for some $P\in W$ and some sequence $\bar{t}$ of terms. A formula A is said to belong to a formula B syntactically, if every predicate symbol occurring in A positively (negatively) occurs in B positively (negatively). For each sentence A in L, and each two sets S, Q of predicate symbols in L, let

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$\mathrm{Pr}_L(A; S, Q)$ be the set of S-positive and Q-negative formulas in L which are provable from A in L. In this paper, we construct a subset $\mathrm{Ax}_L(A; S, Q)$ of $\mathrm{Pr}_L(A; S, Q)$ from A, S, Q concretely, and prove the following

THEOREM I (Axiomatization Theorem). *$\mathrm{Ax}_L(A; S, Q)$ is an axiomatization of the set $\mathrm{Pr}_L(A; S, Q)$ in L, which consists of sentences belonging to A syntactically. In fact, for any formula C in $\mathrm{Pr}_L(A; S, Q)$, we are able to obtain a sentence B in $\mathrm{Ax}_L(A; S, Q)$ such that $B \supset C$ is provable in L, concretely.*

Also, we give a purely syntactical proof of this theorem, which gives us a concrete procedure to obtain a sentence B in $\mathrm{Ax}_L(A; S, Q)$ such that $B \supset C$ is provable in L, from a proof-figure of a given S-positive and Q-negative formula C in L from A, (cf. Remark 1.7 below). This is an advantage of syntactical methods.

To explain the construction of $\mathrm{Ax}_L(A; S, Q)$ from A, S, Q, we have to introduce some notions. Primitive formulas are atomic formulas, negations of atomic formulas, and open formulas which have no occurrences of predicate symbols. Simple existence conditions of W are sentences of the form $\forall \bar{x} \exists y \bigvee_{i \in I} P_i(\bar{x}, y)$, where $P_i \in W$ $(i \in I)$. $\forall x \exists y P(x, y)$ is a typical example of simple existence conditions of $\{P\}$, where P is a binary predicate symbol. Uniqueness conditions of W are sentences of the form $\forall \bar{x}(B(\bar{x}) \supset A(\bar{x}))$, where $B(\bar{x})$ is a finite (possibly empty) conjunction of W-atomic formulas, and $A(\bar{x})$ is a W-free formula, called the principal formula of this uniqueness condition. $\forall x \forall y \forall z(P(x, y) \wedge P(x, z) .\supset y = z)$ is a typical example of uniqueness conditions of $\{P\}$, where P is a binary predicate symbol. Primitive uniqueness conditions of W are uniqueness conditions of W whose principal formulas are primitive. A formula F is said to be in negation normal form (abbre. by n. n. f.) if F is obtained from primitive formulas by applying $\wedge$ (conjunctions), $\vee$ (disjunctions), $\forall$ (universal quantifications), and $\exists$ (existential quantifications), (cf. p. 90 in [8]). It is well-known that any formula F in L can be transformed into a formula F' in n. n. f., canonically, which belongs to F syntactically and is equivalent to F in L (cf. p. 11 in [3]). This F' is called the negation normal form (n. n. f.) of F, and denoted by $F(\neg)$.

The construction of $\mathrm{Ax}_L(A; S, Q)$ from A, S, Q consists of the following four steps:

1-ST STEP. Construction of the n. n. f. $A(\neg)$ of A.

2-ND STEP. Constructions of a countable set $V(A)$ of new predicate symbols which do not occur in A, a countable set $E(A)$ of simple existence conditions of $V(A)$, and a countable set $U(A)$ of primitive uniqueness conditions of $V(A)$, whose principal formulas belong to A syntactically, such that $\{A\}$ and $E(A) \cup U(A)$ are equivalent in L over the set of $V(A)$-free formulas. For the sake of simplicity, we show, here, how to construct $V(A)$, $E(A)$, $U(A)$ from A

in case that A is a finitary sentence (i. e. sentence in $L_{\omega,\omega}$). The exact method to construct them in a general case will be given in § 2 below (cf. Definitions 2.1, 2.3, 2.5 below). First, we construct a sequence of three finite sets $V_k(A)$, $E_k(A)$, $U_k(A)$, $k=0, 1, 2, \cdots$, such that;

$$(*)_k \quad \left\{\begin{array}{l} V_k(A) \text{ is a finite set of predicate symbols which do not occur in } A, \\ E_k(A) \text{ is a finite set of simple existence conditions of } V_k(A),\ U_k(A) \text{ is} \\ \text{a finite set of uniqueness conditions of } V_k(A), \text{ whose principal formulas} \\ \text{belong to } A \text{ syntactically, and, } \{A\} \text{ and } E_k(A)\cup U_k(A) \text{ are equivalent in} \\ L \text{ over the set of } V_k(A)\text{-free formulas.} \end{array}\right.$$

Let $V_0(A)=E_0(A)=\emptyset$ (the empty set) and $U_0(A)=\{\wedge\emptyset\supset A(\neg)\}$. Assume that $V_k(A)$, $E_k(A)$, $U_k(A)$ are defined and satisfy $(*)_k$. If every uniqueness condition of $V_k(A)$ in $U_k(A)$ is primitive, the construction comes to an end. If there is at least one uniqueness condition of $V_k(A)$ in $U_k(A)$ which is not primitive, let F be such one. Then, F has the form $\forall\bar{x}(B(\bar{x})\supset C(\bar{x}))$, where B is a finite (possibly empty) conjunction of $V_k(A)$-atomic formulas, and C is a formula in n. n. f. which belongs to A syntactically.

Case 1. If C is $C_1\wedge C_2$, let $V_{k+1}(A)=V_k(A)$, $E_{k+1}(A)=E_k(A)$, and $U_{k+1}(A)=U_k(A)\cup\{\forall\bar{x}(B(\bar{x})\supset C_1(\bar{x})), \forall\bar{x}(B(\bar{x})\supset C_2(\bar{x}))\}-\{F\}$.

Case 2. If C is $\forall yD(y)$, let $V_{k+1}(A)=V_k(A)$, $E_{k+1}(A)=E_k(A)$, and $U_{k+1}(A)=U_k(A)\cup\{\forall\bar{x}\forall y(B(\bar{x})\supset D(\bar{x}, y))\}-\{F\}$.

Case 3. If C is $C_1\vee C_2$. Let P_1 and P_2 be two distinct $(n+1)$-ary predicate symbols which occur neither in A nor in $V_k(A)$, where n is the length of $\bar{x}$. Let $V_{k+1}(A)=V_k(A)\cup\{P_1, P_2\}$, $E_{k+1}(A)=E_k(A)\cup\{\forall\bar{x}\exists y(P_1(\bar{x}, y)\vee P_2(\bar{x}, y))\}$, and $U_{k+1}(A)=U_k(A)\cup\{\forall\bar{x}\forall y(B(\bar{x})\wedge P_1(\bar{x}, y).\supset C_1(\bar{x})), \forall\bar{x}\forall y(B(\bar{x})\wedge P_2(\bar{x}, y).\supset C_2(\bar{x}))\}-\{F\}$.

Case 4. If C is $\exists yD(y)$. Let P be an $(n+1)$-ary predicate symbol which occurs neither in A nor in $V_k(A)$. Let $V_{k+1}(A)=V_k(A)\cup\{P\}$, $E_{k+1}(A)=E_k(A)\cup\{\forall\bar{x}\exists yP(\bar{x}, y)\}$, and $U_{k+1}(A)=U_k(A)\cup\{\forall\bar{x}\forall y(B(\bar{x})\wedge P(\bar{x}, y).\supset D(\bar{x}, y))\}-\{F\}$.

(Using the notions $V^*(F)$, $E^*(F)$, $U^*(F)$ in Definition 2.1 below, $V_{k+1}(A)$ is $V_k(A)\cup V^*(F)$, $E_{k+1}(A)$ is $E_k(A)\cup E^*(F)$, and $U_{k+1}(A)$ is $(U_k(A)\cup U^*(F))-\{F\}$.)

Since $V_k(A)$, $E_k(A)$, $U_k(A)$ satisfy $(*)_k$, and $V_k(A)\subseteqq V_{k+1}(A)$, we can easily see that $V_{k+1}(A)$, $E_{k+1}(A)$, $U_{k+1}(A)$ satisfy $(*)_{k+1}$, by using the following two obvious facts (1) and (2) (cf. the proof of Theorem 7.2 in [**6**], or (i), (ii) in Corollary 2.2 below);

(1) $\{\forall\bar{x}(B(\bar{x})\supset.C_1(\bar{x})\vee C_2(\bar{x}))\}$ and $\{\forall\bar{x}\exists y(P_1(\bar{x}, y)\vee P_2(\bar{x}, y)), \forall\bar{x}\forall y(B(\bar{x})\wedge P_1(\bar{x}, y).\supset C_1(\bar{x})), \forall\bar{x}\forall y(B(\bar{x})\wedge P_2(\bar{x}, y).\supset C_2(\bar{x}))\}$ are equivalent in L over the set of $\{P_1, P_2\}$-free formulas.

(2) $\{\forall\bar{x}(B(\bar{x})\supset\exists yD(\bar{x}, y))\}$ and $\{\forall\bar{x}\exists yP(\bar{x}, y), \forall\bar{x}\forall y(B(\bar{x})\wedge P(\bar{x}, y).\supset D(\bar{x}, y))\}$ are equivalent in L over the set of $\{P\}$-free formulas.

Since A is finitary, this construction comes to an end within a finite number of steps and we obtain three finite sets $V_N(A)$, $E_N(A)$, $U_N(A)$ which satisfy $(*)_N$ such that every uniqueness condition of $V_N(A)$ in $U_N(A)$ is primitive. In fact, N is no more than the number of occurrences of logical symbols in $A(\neg)$. Let $V(A)=V_N(A)$, $E(A)=E_N(A)$, and $U(A)=U_N(A)$.

3-RD STEP. Construction of a countable set $\tilde{U}(A)$ of primitive uniqueness conditions of $V(A)$ from $U(A)$, S and Q such that $E(A)\cup\tilde{U}(A)$ and $E(A)\cup U(A)$ are equivalent in L over the set of S-positive and Q-negative formulas, and every principal formula of any uniqueness condition in $\tilde{U}(A)$ is not only a formula which belongs to A syntactically, but also S-positive and Q-negative. $\tilde{U}(A)$ is the set obtained from $U(A)$ by deleting every uniqueness condition of $V(A)$ whose principal formula is not S-positive or not Q-negative, and by adding every primitive uniqueness condition of $V(A)$ of the form; $\forall\bar{x}\forall\bar{y}(B(\bar{x})\wedge C(\bar{y}).\supset \neg\bar{t}(\bar{x})=\bar{s}(\bar{y}))$, where $\forall\bar{x}(B(\bar{x})\supset P(\bar{t}(\bar{x})))\in U(A)$, $\forall\bar{y}(C(\bar{y})\supset\neg P(\bar{s}(\bar{y})))\in U(A)$ for some P in $S\cup Q$, $\bar{t}(\bar{x})$ and $\bar{s}(y)$ are two sequences $\langle t_1(\bar{x}), t_2(\bar{x}), \cdots, t_n(\bar{x})\rangle$, $\langle s_1(\bar{y}), s_2(\bar{y}), \cdots, s_n(\bar{y})\rangle$ of terms, and $\bar{t}(\bar{x})=\bar{s}(\bar{y})$ is the formula $t_1(\bar{x})=s_1(\bar{y})\wedge t_2(\bar{x})=s_2(\bar{y})\wedge\cdots\wedge t_n(\bar{x})=s_n(\bar{y})$. Then, $\tilde{U}(A)$ has the required properties (cf. Lemma 1.3 below).

4-TH STEP. Construction of $\mathrm{Ax}_L(A; S, Q)$. Let $\mathrm{Ax}_L(A; S, Q)$ be the set of simple approximations of $\tilde{U}(A)$ by $E(A)$ (cf. Definition 1.4 below). The simple approximation theorem of uniqueness conditions by simple existence conditions (cf. Theorem 1.6 below) and the definition of simple approximations of uniqueness conditions by simple existence conditions (cf. Definition 1.4 below) show us that $\mathrm{Ax}_L(A; S, Q)$ has the desired properties. Therefore, we complete our construction of $\mathrm{Ax}_L(A; S, Q)$. The construction of $\mathrm{Ax}_L(A; S, Q)$ from A, S, Q, explained above roughly, is so simple that we can easily tell whether or not a given formula belongs to it. As immediate consequences of Theorem I, we have the following two corollaries.

COROLLARY II (Axiomatization of $\sum_1^1$-sentences, cf. Harnik [**5**]). *For each sentence A, and each predicate symbol S, the set $\mathrm{Ax}_L(A; \{S\}, \{S\})$ is essentially an axiomatization of the set of all the first order formulas in L which are provable from the second order sentence $\exists S(A)$.*

COROLLARY III (Lyndon-Lopezescobar Interpolation Theorem, cf. [**3**]). *If a sentence $A\supset B$ is provable in L, then there is a sentence C which belongs to A and B, syntactically, such that $A\supset C$ and $C\supset B$ are provable in L. In fact, we can choose such a C in $\mathrm{Ax}_L(A; S, Q)$, where $S(Q)$ is the set of predicate symbols in A which do not occur in B negatively (positively).*

Note that, in [**5**], V. Harnik gave a similar axiomatization of $\sum_1^1$-sentences by using Vaught sentences. But, Vaught sentences are very big sentences which do not belong to $L_{\omega_1,\omega}$. On the other hand, we do not use any such big sentences in this paper. Moreover, if A is finitary and $L=L_{\omega,\omega}$, then the construction of $\mathrm{Ax}_L(A;\{S\},\{S\})$ from A and S is also finitary in Hilbert's sense (cf. p. 81 in Takeuti [8]). Therefore, our result here is more constructive than Harnik's.

In § 1 of this paper, we shall explain our notions and notations which are not introduced above, and state some necessary results about them, including the simple approximation theorem in [6] (Theorem 1.6 below), with slight modifications. Note that Lemma 1.3 below is an essentially new result which can be considered as one of elimination theorems treated in [7]. In § 2, we shall give a procedure to construct the set $\mathrm{Ax}_L(A;S,Q)$ from A, S, Q and prove Theorem I.

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§ 1. Preliminaries.

Suppose that L is a fixed infinitary first order predicate logic $L_{\omega_1,\omega}$. Every result in this paper holds for any finitary first order predicate logic with a few natural modifications. So, we treat only the case that $L=L_{\omega_1,\omega}$. Also, we shall consider the extended logic $L(W)$ for each set W of new predicate symbols. As logical symbols in L, we use: $\neg$ (negation), $\bigwedge$ (countable conjunction), $\bigvee$ (countable disjunction), $\forall$ (universal quantifier), $\exists$ (existential quantifier) and $=$ (equality). $\supset$ (implication) will be used as an abbreviation as usual. If $\bar{k}$ and $\bar{l}$ are two sequences of symbols, then $\bar{k}^\frown\bar{l}$ is the concatenation of $\bar{k}$ and $\bar{l}$. If $\bar{t}$ and $\bar{s}$ are two sequences $\langle t_1, t_2, \cdots, t_n\rangle$, $\langle s_1, s_2, \cdots, s_n\rangle$ of terms of the same length, then $\bar{t}=\bar{s}$ is the formula;

$$t_1=s_1\wedge t_2=s_2\wedge\cdots\wedge t_n=s_n\,.$$

By a k-ary formula in L, we mean a pair $(A,\bar{a})$ of a formula A in L and a sequence $\bar{a}$ of distinct free variables of length k such that every free variable in A occurs in $\bar{a}$. But, $A(\bar{a}),\cdots$, will be used to denote both the formula $A,\cdots$, and the k-ary formula $(A,\bar{a}),\cdots$, if no confusion is likely to occur. A replacing function f is a mapping such that the domain of f is a set of predicate symbols and $f(P)$ is a k-ary formula for each k-ary predicate symbol P in the domain of f. If f is a replacing function and A is a formula, then $A[f]$ is the formula obtained from A by replacing every P in the domain of f by $f(P)$, simultaneously. For any k-ary formula $(A,\bar{a})$, let $(A,\bar{a})[f]$ be the k-ary formula $(A[f],\bar{a})$. The next lemma is an obvious, but important fact;

LEMMA 1.1 (Positive Lemma). *Suppose that S and Q are countable sets of* *predicate symbols. If A is an S-positive and Q-negative formula, then the formula*

$$\bigwedge_{P\in S-Q}\forall\bar{x}(f(P)(\bar{x})\supset P(\bar{x}))\wedge\bigwedge_{P\in Q-S}\forall\bar{x}(P(\bar{x})\supset f(P)(\bar{x}))\wedge A[f].\supset A$$

is provable in L.

DEFINITION 1.2. Suppose that U is a set of primitive uniqueness conditions of a set W of predicate symbols, and P is a predicate symbol. Then, $\mathrm{Eq}(U; P)$ is the set of all the primitive uniqueness conditions of W of the form;

$\forall\bar{x}\forall\bar{y}(B(\bar{x})\wedge C(\bar{y}).\supset\neg(\bar{t}(\bar{x})=\bar{s}(\bar{y})))$, where $\forall\bar{x}(B(\bar{x})\supset P(\bar{t}(\bar{x})))\in U$, and $\forall\bar{y}(C(\bar{y})\supset\neg P(\bar{s}(\bar{y})))\in U$. For each two sets S, Q of predicate symbols, let $\mathrm{De}(U; S, Q)$ be the set $(U\cup\bigcup_{P\in S\cup Q}\mathrm{Eq}(U; P))-(U_1\cup U_2)$, where U_1 is the set of primitive uniqueness conditions in U whose principal formulas are negations of S-atomic formulas and U_2 is the set of primitive uniqueness conditions in U whose principal formulas are Q-atomic formulas.

Then, the following result is a new elimination theorem (cf. [7]).

LEMMA 1.3. *Suppose that U is a countable set of primitive uniqueness conditions of W, and S, Q are two sets of predicate symbols such that $W\cap(S\cup Q)=\emptyset$ (the empty set). Then, U and* $\mathrm{De}(U; S, Q)$ *are equivalent in L over the set of* S-positive and Q-negative formulas.

PROOF. Since $\wedge\mathrm{Eq}(U; P)$ is provable from U for any predicate symbol P, it is sufficient to prove that if A is provable from U in L, then A is provable from $\mathrm{De}(U; S, Q)$ in L, for any S-positive, Q-negative formula A. Assume that A is an S-positive, Q-negative formula which is provable from U in L. Without loss of generality, we can assume that S and Q are countable. Let f be the replacing function defined by: the domain of f is $S\cup Q$, and $f(P)(\bar{a})$ is the k-ary formula $\bigvee\{\exists\bar{y}(\bar{a}=\bar{t}(\bar{y})\wedge B(\bar{y}))\mid\forall\bar{y}(B(\bar{y})\supset P(\bar{t}(\bar{y})))\in U\}$, if $P\in S$, and $\neg\bigvee\{\exists\bar{y}(\bar{a}=\bar{t}(\bar{y})\wedge B(\bar{y}))\mid\forall\bar{y}(B(\bar{y})\supset\neg P(\bar{t}(\bar{y})))\in U\}$, if $P\in Q-S$. Suppose that F is a sentence in U. If the principal formula of F is a $(Q-S)$-atomic formula or the negation of an S-atomic formula, then $F[f]$ is provable from $\bigcup_{P\in S\cup Q}\mathrm{Eq}(U; P)$ in L. If the principal formula of F is an S-atomic formula or the negation of a $(Q-S)$-atomic formula, then $F[f]$ is provable in L. Hence, we conclude that every sentence in $U[f]=\{F[f]\mid F\in U\}$ is provable from $\mathrm{De}(U; S, Q)$ in L. On the other hand, $A[f]$ is provable from $U[f]$ in L, because A is provable from U in L. Therefore, $A[f]$ is provable from $\mathrm{De}(U; S, Q)$ in L. Also, the sentences $\forall\bar{x}(f(P)(\bar{x})\supset P(\bar{x}))$, $P\in S-Q$, and the sentences $\forall\bar{x}(P(\bar{x})\supset f(P)(\bar{x}))$, $P\in Q-S$, are all provable from $\mathrm{De}(U; S, Q)$ in L. Since A is S-positive and Q-negative, the formula $\bigwedge_{P\in S-Q}\forall\bar{x}(f(P)(\bar{x})\supset P(\bar{x}))\wedge\bigwedge_{P\in Q-S}\forall\bar{x}(P(\bar{x})\supset f(P)(\bar{x}))\wedge A[f].\supset A$ is provable in L by Lemma 1.1. This means that A is provable from $\mathrm{De}(U; S, Q)$ in L. (Q. E. D.)

Suppose that X is a countable set of W-atomic formulas and F is a uniqueness condition of W. Then, $F[X]$ is the formula

$$\bigwedge_{P_1(\bar{s}_1)\in X} \cdots \bigwedge_{P_n(\bar{s}_n)\in X} \forall \bar{x}(\bar{t}_1(\bar{x})=\bar{s}_1 \wedge \cdots \wedge \bar{t}_n(\bar{x})=\bar{s}_n . \supset A(\bar{x})),$$

where F is $\forall \bar{x}(P_1(\bar{t}_1(\bar{x})) \wedge \cdots \wedge P_n(\bar{t}_n(\bar{x})). \supset A(\bar{x}))$.

DEFINITION 1.4. Suppose that U is a countable set of uniqueness conditions of W, E is a set of simple existence conditions of W, X is a countable set of W-atomic formulas, and α is a countable ordinal number. Then, the set of α-th approximations of U by E over X, denoted by $\mathrm{Ap}^{\alpha}(U, E, X)$, is defined by the following (i), (ii), and (iii):

(i) $\mathrm{Ap}^{0}(U, E, X)=\{\bigwedge_{F\in U} F[X]\}$.

(ii) $\mathrm{Ap}^{\alpha}(U, E, X)=\{\bigwedge_{\beta<\alpha} A_{\beta} \mid A_{\beta}\in \mathrm{Ap}^{\beta}(U, E, X)\}$, if α is a limit ordinal number.

(iii) $\mathrm{Ap}^{\alpha+1}(U, E, X)=\{\forall \bar{x} \exists y \bigvee_{i\in I} A_i(\bar{x}, y) \mid \forall \bar{x} \exists y \bigvee_{i\in I} P_i(\bar{x}, y)\in E,\ A_i(\bar{a}, b)\in \mathrm{Ap}^{\alpha}(U, E, X\cup\{P_i(\bar{a}, b)\})$ for some sequence $\bar{a}$, b of variables none of which occurs in any formula in X, $i\in I\}$.

Let $\mathrm{Ap}(U, E, X)=\bigcup_{\alpha<\omega_1} \mathrm{Ap}^{\alpha}(U, E, X)$ and $\mathrm{Ap}(U, E)=\mathrm{Ap}(U, E, \emptyset)$. Formulas in $\mathrm{Ap}(U, E)$ are called "simple approximations" of U by E. Note that the definition of $\mathrm{Ap}(U, E)$ given above is slight different from that in [**6**]. The following lemma is an immediate consequence of Definition 1.4.

LEMMA 1.5. (i) *Every formula in* $\mathrm{Ap}(U, E)$ *is a W-free sentence which is provable from $U\cup E$.*

(ii) *Suppose that U is a set of primitive uniqueness conditions of W, P is a predicate symbol, and A is a simple approximation of U by E. If P occurs in A positively (negatively), then there is a primitive uniqueness condition of W in U, whose principal formula is (the negation of) a $\{P\}$-atomic formula.*

By the simple approximation theorem in [**6**], we have:

THEOREM 1.6. *Suppose that W is a countable set of predicate symbols, U is a countable set of uniqueness conditions of W, and E is a set of simple existence conditions of W. Then, for any W-free formula A, A is provable from $U\cup E$ in L if and only if A is provable from some simple approximation of U by E, in L.*

REMARK 1.7. The simple approximation theorem in [**6**] is proved syntactically, and we can obtain a simple approximation B of U by E such that $B\supset A$ is provable in L, from a given proof-figure of a W-free formula A from $U\cup E$ in L, concretely.

§2. A construction of $\mathrm{Ax}_L(A; S, Q)$.

Let V be a set of new predicate symbols which do not belong to L. We assume that V has sufficiently many predicate symbols.

DEFINITION 2.1. For each uniqueness condition F of V of the form $\forall\bar{x}(B(\bar{x})\supset C(\bar{x}))$, where $C(\bar{x})$ is in n. n. f., we associate a countable set $V^*(F)$ of predicate symbols in V, which do not occur in F, a countable set $E^*(F)$ of simple existence conditions of $V^*(F)$, a countable set $U^*(F)$ of uniqueness conditions of $V^*(F)$, and a replacing function f_F^*, whose domain is $V^*(F)$, by the following (i)-(v):

(i) If $C(\bar{x})$ is primitive, let $V^*(F)=E^*(F)=f_F^*=\emptyset$ and $U^*(F)=\{F\}$.

(ii) If $C(\bar{x})$ is $\bigwedge_{i\in I}C_i(\bar{x})$, let $V^*(F)=E^*(F)=f_F^*=\emptyset$ and $U^*(F)=\{\forall\bar{x}(B(\bar{x})\supset C_i(\bar{x}))\,|\,i\in I\}$.

(iii) If $C(\bar{x})$ is $\forall yD(\bar{x}, y)$, let $V^*(F)=E^*(F)=f_F^*=\emptyset$ and $U^*(F)=\{\forall\bar{x}\forall y(B(\bar{x})\supset D(\bar{x}, y))\}$.

(iv) If $C(\bar{x})$ is $\bigvee_{i\in I}C_i(\bar{x})$, let $\{P_i\}_{i\in I}$ be a set of distinct $(n+1)$-ary predicate symbols in V, which do not occur in F, where n is the length of the sequence $\bar{x}$. Let $V^*(F)=\{P_i\,|\,i\in I\}$, $E^*(F)=\{\forall\bar{x}\exists y\bigvee_{i\in I}P_i(\bar{x}, y)\}$, $U^*(F)=\{\forall\bar{x}\forall y(B(\bar{x})\wedge P_i(\bar{x}, y).\supset C_i(\bar{x})\,|\,i\in I\}$, and $f_F^*(P_i)=(B(\bar{x})\supset C_i(\bar{x}), \bar{x}^\frown y)$, $i\in I$.

(v) If $C(\bar{x})$ is $\exists yD(\bar{x}, y)$, let P be an $(n+1)$-ary predicate symbol in V, which does not occur in F, where n is the length of the sequence $\bar{x}$. Let $V^*(F)=\{P\}$, $E^*(F)=\{\forall\bar{x}\exists yP(\bar{x}, y)\}$, $U^*(F)=\{\forall\bar{x}\forall y(B(\bar{x})\wedge P(\bar{x}, y).\supset D(\bar{x}, y))\}$, and $f_F^*(P)=(B(\bar{x})\supset D(\bar{x}, y), \bar{x}^\frown y)$.

Note that $V^*(F)$, $E^*(F)$, and $U^*(F)$ are finite sets if F is a finitary sentence. Then, the following facts are immediate consequences of Definition 2.1.

COROLLARY 2.2. (i) *F is provable from $E^*(F)\cup U^*(F)$ in $L(V)$.*

(ii) *$C[f_F^*]$ is provable from F in $L(V)$ for any C in $E^*(F)\cup U^*(F)$.*

(iii) *If F is not primitive, then the principal formula of any uniqueness condition in $U^*(F)$ is a proper subformula of the principal formula of F* (cf. p. 30 in [8] for the notion "subformula").

DEFINITION 2.3. For each sentence A in L, and each natural number k, we define a countable set $V_k(A)$ of predicate symbols in V, a countable set $E_k(A)$ of simple existence conditions of $V_k(A)$, a countable set $U_k(A)$ of uniqueness conditions of $V_k(A)$, whose principal formula belongs to A syntactically, and a replacing function f_A^k, by the following: Let $V_0(A)=E_0(A)=f_A^0=\emptyset$, and $U_0(A)=\{\bigwedge\emptyset\supset A(\neg)\}$. Assume that $V_k(A)$, $E_k(A)$, $U_k(A)$, and f_A^k are defined. Without loss of generality, we can assume that $V^*(F)$ $(F\in U_k(A))$, $V_k(A)$ are all disjoint. Let $V_{k+1}(A)=V_k(A)\cup\bigcup_{F\in U_k(A)}V^*(F)$, $E_{k+1}(A)=E_k(A)\cup\bigcup_{F\in U_k(A)}E^*(F)$, $U_{k+1}(A)=\bigcup_{F\in U_k(A)}U^*(F)$, and $f_A^{k+1}(P)=f_A^k(P)$ if $P\in V_k(A)$, and $f_A^{k+1}(P)=f_F^*(P)[f_A^k]$ if $P\in V^*(F)$, $F\in U_k(A)$.

Then, the following fact is an obvious consequence of Definition 2.3 and (ii) in Corollary 2.2.

COROLLARY 2.4. *For any sentence C in $E_k(A)\cup U_k(A)$, $C[f_A^k]$ is provable from A in L.*

DEFINITION 2.5. For any sentence A in L, let $V(A)=\bigcup_{k<\omega} V_k(A)$, $E(A)=\bigcup_{k<\omega} E_k(A)$, $f_A=\bigcup_{k<\omega} f_A^k$, and $U(A)=\{F\in\bigcup_{k<\omega} U_k(A)\mid F \text{ is primitive}\}$.

Then, from Corollary 2.4, we have;

COROLLARY 2.6. *For any sentence C in $E(A)\cup U(A)$, $C[f_A]$ is provable from A in L.*

Also, we have the following lemma whose proof is essentially due to Professor T. Uesu.

LEMMA 2.7. *Every sentence F in $\bigcup_{k<\omega} U_k(A)$ is provable from $E(A)\cup U(A)$ in $L(V)$. Hence, A is provable from $E(A)\cup U(A)$ in $L(V)$.*

PROOF. Assume that there is a sentence F_0 in $\bigcup_{k<\omega} U_k(A)$ such that F_0 is not provable from $E(A)\cup U(A)$ in $L(V)$. If F_0 is primitive, then F_0 is provable from $E(A)\cup U(A)$ because F_0 belongs to $U(A)$. So, F_0 is not primitive. Since F_0 is provable from $E^*(F_0)\cup U^*(F_0)$, and $E^*(F_0)\subseteqq E(A)$, there is a sentence F_1 in $U^*(F_0)$ such that F_1 is not provable from $E(A)\cup U(A)$ in $L(V)$. By (iii) in Corollary 2.2, the principal formula of F_1 is a proper subformula of the principal formula of F_0. By continuing this process, we have an infinite sequence $\{F_k\}_{k<\omega}$ of sentences in $\bigcup_{k<\omega} U_k(A)$ such that each F_k is not provable from $E(A)\cup U(A)$ in $L(V)$, and the principal formula of F_{k+1} is a proper subformula of the principal formula of F_k for each $k<\omega$. But, this is impossible because there is no infinite sequence $\{C_k\}_{k<\omega}$ of formulas such that C_{k+1} is a proper subformula of C_k for each $k<\omega$. Therefore, every sentence in $\bigcup_{k<\omega} U_k(A)$ is provable from $E(A)\cup U(A)$ in $L(V)$. (Q. E. D.)

From Corollary 2.6 and Lemma 2.7, we have:

LEMMA 2.8. *$\{A\}$ and $E(A)\cup U(A)$ are equivalent in $L(V)$ over the set of $V(A)$-free formulas.*

Now we can define $\mathrm{Ax}_L(A; S, Q)$ explicitly as follows: Suppose that a sentence A and two sets S, Q of predicate symbols in L are given. Take the negation normal form $A(\neg)$ of A, and construct $V(A)$, $E(A)$, $U(A)$ by Definition 2.5. Let $\tilde{U}(A)=\mathrm{De}(U(A); S, Q)$ and $\mathrm{Ax}_L(A; S, Q)=\mathrm{Ap}(\tilde{U}(A), E(A))$. Then, we can easily see that every sentence in $\mathrm{Ax}_L(A; S, Q)$ is S-positive, Q-negative and belongs to A syntactically, by Definition 1.2, (ii) in Lemma 1.5, and Definition 2.5.

Suppose that C is an S-positive, Q-negative formula in L. Then,

C is provable from A in L

$\Longleftrightarrow$ C is provable from $E(A)\cup U(A)$ in $L(V)$, (by Lemma 2.8)

$\Longleftrightarrow$ C is provable from $E(A)\cup\tilde{U}(A)$ in $L(V)$, (by Lemma 1.3)

$\Longleftrightarrow$ C is provable from some B in $\mathrm{Ap}(\tilde{U}(A), E(A))$, in L, (by Theorem 1.6)

$\Longleftrightarrow$ C is provable from some B in $\mathrm{Ax}_L(A; S, Q)$, in L.

This completes our proof of Theorem I.

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