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# Weak $L$-spaces are free $L$-spaces

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## 1. Introduction.

In order to discuss the dimension theory, K. Nagami [4], [5] introduced the concepts of free $L$-spaces and weak $L$-spaces. He posed the following two problems in [5] and [4] respectively.

1. Does the class of weak $L$-spaces coincide with the class of free $L$-spaces?
2. Is the perfect image of a free $L$-space again a free $L$-space? (Problem 2.11.)

The main purpose of this paper gives a positive answer to the first problem. In Section 4 we give a partial answer to the second problem as follows.

*The closed continuous image of a free L-space need not be a free L-space.*

In this paper all spaces are assumed to be Hausdorff topological spaces. The letter $N$ denotes the positive integers. For undefined terminology refer to [2].

The author thanks Professor K. Nagami for his guidance.

## 2. Definition.

DEFINITION 2-1. Let $X$ be a space and $F$ a closed subset of $X$. A family $\mathcal{U}$ of open sets is said to be an *anti-cover* of $F$ if $\mathcal{U}^{\#}(=\bigcup\{U: U\in\mathcal{U}\})=X-F$.

Let $\mathcal{U}$ be an anti-cover of $F$. For a subset $S$ of $X$ $\mathrm{St}^i_{\mathcal{U}}(S)$ is defined inductively by the formulae

$$\mathrm{St}^1_{\mathcal{U}}(S)=\mathrm{St}_{\mathcal{U}}(S)=\{U\in\mathcal{U}: U\cap S\neq\emptyset\}^{\#},$$

$$\mathrm{St}^i_{\mathcal{U}}(S)=\mathrm{St}_{\mathcal{U}}(\mathrm{St}^{i-1}_{\mathcal{U}}(S)).$$

An open neighborhood $W$ of $F$ is said to be a *canonical* (*semi-canonical*) *neighborhood* of $F$ with respect to $\mathcal{U}$ if $F\cap \mathrm{Cl}\,\mathrm{St}^i_{\mathcal{U}}(X-W)=\emptyset$ for each $i\in N$ ($F\cap \mathrm{Cl}\,\mathrm{St}_{\mathcal{U}}(X-W)=\emptyset$) respectively.

Let $\mathcal{W}=\{W_a: a\in A\}$ be a family of neighborhoods of $F$. $\mathcal{W}$ is said to be an *anti-closure-preserving family* if $\{(X-W_a)\cup F: a\in A\}$ is closure-preserving.

DEFINITION 2-2. For a space $X$ consider a pair $\mathcal{P}=(\mathcal{F}, \{\mathcal{U}_F: F\in\mathcal{F}\})$ such that $\mathcal{F}$ is a family of closed sets of $X$ and each $\mathcal{U}_F$ is an anti-cover of $F$. $\mathcal{P}$

is said to be a *free* (*weak*) *L-structure* of $X$ if $\mathcal{F}$ is $\sigma$-discrete ($\sigma$-locally finite) and for each $x\in X$ and each neighborhood $U$ of $x$ there exist a finite subfamily $\{F_1, \cdots, F_n\}$ of $\mathcal{F}$ and a canonical (semi-canonical) neighborhood $U_i$ of $F_i$ $(1\leqq i\leqq n)$ such that $x\in\bigcap_{i=1}^n F_i\subset\bigcap_{i=1}^n U_i\subset U$ respectively.

A paracompact space $X$ is said to be a *free* (*weak*) *L-space* if $X$ has a free (weak) $L$-structure respectively.

## 3. Main theorem.

LEMMA 3-1. *Let $X$ be a monotonically normal space, $F$ a closed subset of $X$ and $\{W_a: a\in A\}$ an anti-closure-preserving family of open neighborhoods of $F$. Then there exists an anti-cover $\mathcal{U}$ of $F$ such that each $W_a$ is semi-canonical with respect to $\mathcal{U}$.*

PROOF. Let D be a monotonic operator in $X$. Set

$$U_x=\mathrm{D}[\{x\}, \{(X-W_a): a\in A, W_a\ni x\}^{\#}\cup F], \quad x\in X-F,$$

$$\mathcal{U}=\{U_x: x\in X-F\}.$$

Now we show that for every $a\in A$, $W_a$ is semi-canonical with respect to $\mathcal{U}$. Set $G=\mathrm{D}[(X-W_a), F]$. From the definition of $\mathcal{U}$ if $x\in W_a$ then $U_x\cap(X-W_a)=\emptyset$. Therefore if $U_x\cap(X-W_a)\neq\emptyset$ then $x\in X-W_a$ and $U_x\subset G$. This implies that $W_a$ is semi-canonical with respect to $\mathcal{U}$. That completes the proof.

Now we note that every weak $L$-space is hereditarily paracompact and monotonically normal.

REMARK 3-2. Let $X$ be a weak $L$-space and $(\mathcal{F}, \{\mathcal{U}_F: F\in\mathcal{F}\})$ a weak $L$-structure of $X$ such that each $\mathcal{U}_F$ is locally finite mod $F$ (i. e. locally finite in $X-F$). Set

$$\mathcal{U}(F)=\{F\cup\mathcal{W}^{\#}: \mathcal{W}\subset\mathcal{U}_F, F\cup\mathcal{W}^{\#} \text{ is open}\}, \quad F\in\mathcal{F}.$$

Then $(\mathcal{F}, \{\mathcal{U}(F): F\in\mathcal{F}\})$ satisfies the following conditions.

1. *Each $\mathcal{U}(F)$ is an anti-closure-preserving family of open neighborhoods of $F$.*
2. *For each $x\in X$ and each neighborhood $U$ of $x$ there exist a finite subfamily $\{F_1, \cdots, F_n\}$ of $\mathcal{F}$ and a member $U_i$ of $\mathcal{U}(F_i)$ $(1\leqq i\leqq n)$ such that $x\in\bigcap_{i=1}^n F_i\subset\bigcap_{i=1}^n U_i\subset U$.*

Conversely, let $\mathcal{H}$ be a $\sigma$-locally finite family of closed sets and $(\mathcal{H}, \{\mathcal{V}(H): H\in\mathcal{H}\})$ satisfies the above two conditions. Let $\mathcal{V}_H$ be an anti-cover of $H$ constructed from $\mathcal{V}(H)$ as in Lemma 3-1, $H\in\mathcal{H}$. Then $(\mathcal{H}, \{\mathcal{V}_H: H\in\mathcal{H}\})$ is a weak $L$-structure of $X$.

THEOREM 3-3. *Let $X$ be a weak $L$-space. Then $X$ is a free $L$-space.*

PROOF. Part 1. Let us prove that $X$ has a weak $L$-structure $(\mathcal{H}, \{\mathcal{V}_H:$

$H\in\mathcal{H}\})$ such that $\mathcal{H}$ is $\sigma$-discrete.

Let $(\mathcal{F}, \{\mathcal{U}_F: F\in\mathcal{F}\})$ be a weak $L$-structure of $X$ such that each $\mathcal{U}_F$ is locally finite mod $F$. Let $\mathcal{F}=\bigcup_{i\in N}\mathcal{F}_i$ where each $\mathcal{F}_i$ is locally finite, $\mathcal{F}_i\subset\mathcal{F}_{i+1}$ and $\mathcal{F}_i=\{F_a: a\in A_i\}$. For $F\in\mathcal{F}$ set

$$\mathcal{U}(F)=\{F\cup\mathcal{W}^{\#}: \mathcal{W}\subset\mathcal{U}_F,\ F\cup\mathcal{W}^{\#} \text{ is open}\}.$$

Take $i\in N$. For $A'\subset A_i$ set

$$(\bigcap_{a\in A'} F_a)-(\bigcup_{a\in A_i\setminus A'} F_a)=\bigcup_{n\in N} K_n \text{ where each } K_n \text{ is closed.}$$

Take $j\in N$. Set $L(A')=K_j$. Then

$$\{L(A'): A'\in B_i^j\}=\{L(A'): A'\subset A_i,\ L(A')\neq\emptyset\}$$

is a discrete family of closed sets. So we can take a discrete family $\{D(A'): A'\in B_i^j\}$ of open sets such that $L(A')\subset D(A')$ for each $A'\in B_i^j$. Since $\mathcal{F}_i$ is locally finite, for $A'\in B_i^j$ we can put

$$A'=\{a(A', 1), \cdots, a(A', n_{A'})\}.$$

Take open sets $G_1$, $G_2$ such that

$$L(A')\subset G_1\subset \mathrm{Cl}\, G_1\subset G_2\subset \mathrm{Cl}\, G_2\subset D(A').$$

For $1\leqq k\leqq n_{A'}$ set

$$H(A')_{k1}=F_{a(A',k)}\cap \mathrm{Cl}\, G_1,\quad H(A')_{k2}=L(A').$$

Set

$$U(A')_{k1}=(U\cap G_2)\cup(G_2-\mathrm{Cl}\, G_1),\quad U\in\mathcal{U}(F_{a(A',k)}),$$

$$\mathcal{V}(H(A')_{k1})=\{U(A')_{k1}: U\in\mathcal{U}(F_{a(A',k)})\}.$$

Then the family $\mathcal{V}(H(A')_{k1})$ is an anti-closure-preserving family of open neighborhoods of $H(A')_{k1}$. Set

$$U(A')_{k2}=G_1,\quad U\in\mathcal{U}(F_{a(A',k)}),$$

$$\mathcal{V}(H(A')_{k2})=\{U(A')_{k2}: U\in\mathcal{U}(F_{a(A',k)})\}.$$

Obviously $\mathcal{V}(H(A')_{k2})$ is an anti-closure-preserving family of open neighborhoods of $H(A')_{k2}$. Now we note that

$$U(A')_{k1}\cap U(A')_{k2}\subset U\cap G_1\subset U.$$

For convenience' sake we put $H(A')_{kh}=\emptyset$ and $\mathcal{V}(H(A')_{kh})=\{\emptyset\}$ for $k>n_{A'}$ and $h=1, 2$. Since $\mathcal{H}_{ikh}^j=\{H(A')_{kh}: A'\in B_i^j\}$ is discrete for $(i, j, k, h)\in N\times N\times N\times \{1, 2\}$, then

$$\mathcal{H}=\{H: H\in\mathcal{H}_{ikh}^j,\ (i, j, k, h)\in N\times N\times N\times\{1, 2\}\}$$

is $\sigma$-discrete. By Remark 3-2, we have only to check that $(\mathcal{H}, \{\mathcal{V}(H): H\in\mathcal{H}\})$ satisfies the two conditions of Remark 3-2.

Let $x\in X$ and $W$ a neighborhood of $x$. Then there exist $F_1, \cdots, F_m\in\mathcal{F}$ and $U_i\in\mathcal{U}(F_i)$ $(1\leqq i\leqq m)$ such that $x\in\bigcap_{i=1}^m F_i\subset\bigcap_{i=1}^m U_i\subset W$. We can assume that $F_1, \cdots, F_m\in\mathcal{F}_n$ for some $n\in N$. Let $A'=\{a\in A_n: x\in F_a\}$. Then there exists $j\in N$ such that $A'\in B_n^j$ and $x\in L(A')$. Also we can assume that $F_i=F_{a(A',i)}$ $(1\leqq i\leqq m)$. Then $\{H(A')_{ih}: 1\leqq i\leqq m,\ h=1, 2\}\subset\mathcal{H}$, $U_i(A')_{ih}\in\mathcal{V}(H(A')_{ih})$ $(1\leqq i\leqq m,$ $h=1, 2)$ and

$$x\in\bigcap_{\substack{i=1\\h=1,2}}^m H(A')_{ih}\subset\bigcap_{\substack{i=1\\h=1,2}}^m U_i(A')_{ih}\subset\bigcap_{i=1}^m U_i\subset W.$$

Actually $\mathcal{H}$ constitutes a network of $X$.

Part 2. For $F\in\mathcal{F}$, where $(\mathcal{F}, \{\mathcal{U}_F: F\in\mathcal{F}\})$ is a weak $L$-structure of $X$ such that $\mathcal{F}$ is $\sigma$-discrete, we construct an anti-cover $\mathcal{V}_F$ of $F$ satisfying the following condition.

*If $U$ is a semi-canonical neighborhood of $F$, then $U$ is a canonical neighborhood of $F$ with respect to $\mathcal{V}_F$.*

Let $\mathcal{U}_F$ be locally finite mod $F$. By induction we define a sequence $\mathcal{U}_F^1$, $\mathcal{U}_F^2, \cdots$ of locally finite (mod $F$) anti-covers of $F$. Set $\mathcal{U}_F^1=\mathcal{U}_F$. If we define a locally finite (mod $F$) anti-cover $\mathcal{U}_F^i$, then the family

$$\mathcal{O}=\{\operatorname{Int}(X-\mathcal{W}^{\#}): \mathcal{W}\subset\mathcal{U}_F^i,\ X-\mathcal{W}^{\#} \text{ is a neighborhood of } F\}$$

is anti-closure-preserving. Therefore we have an anti-cover $\mathcal{U}'$ constructed from $\mathcal{O}$ as in Lemma 3-1. Let $\mathcal{U}_F^{i+1}$ be a locally finite refinement of $\mathcal{U}'$ in $X-F$.

Let $\{G_i: i\in N\}$ be a family of open sets such that

$$X=G_1\supset\operatorname{Cl}G_2\supset G_2\cdots,\quad \bigcap_{i\in N}G_i=F.$$

Set $\mathcal{V}_F$ as follows.

$$U_x=\bigcap\{U\in\bigcup_{j=1}^i\mathcal{U}_F^j: x\in U\}\cap(G_i-\operatorname{Cl}G_{i+2}),\quad x\in G_i-G_{i+1},$$

$$\mathcal{V}_F=\{U_x: x\in X-F\}.$$

We show that this $\mathcal{V}_F$ is the required. Take a semi-canonical neighborhood $U$ of $F$ with respect to $\mathcal{U}_F$. By induction on $i$ we claim that

$$\operatorname{St}^i_{\mathcal{V}_F}(X-U)\subset\operatorname{St}_{\mathcal{U}_F^i}(\cdots(\operatorname{St}_{\mathcal{U}_F^1}(X-U))\cdots)\cup(X-G_{i+1}).$$

For $i=1$, $\mathcal{V}_F$ refines $\mathcal{U}_F^1$, so the assertion is trivial. Let $n\in N$. Assume that the assertion is true for $i<n$. Put

$$\mathcal{V}_1=\{U_x: x\in G_n-F\},$$

$$\mathcal{V}_2=\mathcal{V}_F-\mathcal{V}_1.$$

Then $\mathcal{V}_1$ refines $\mathcal{U}_F^n$, $\mathcal{V}_1^\# \cap (X-G_n)=\emptyset$ and $\mathcal{V}_2^\# \subset X-G_{n+1}$. Therefore we have

$$\mathrm{St}^n_{\mathcal{V}_F}(X-U) \subset \mathrm{St}_{\mathcal{U}_F^n}(\cdots(\mathrm{St}_{\mathcal{U}_F^1}(X-U))\cdots) \cup (X-G_{n+1}),$$

and the induction is completed. Hence $U$ is a canonical neighborhood of $F$ with respect to $\mathcal{V}_F$.

It is easy to check that this $(\mathcal{F}, \{\mathcal{V}_F: F \in \mathcal{F}\})$ gives a free $L$-structure for $X$, and the proof is completed.

## 4. Example.

DEFINITION 4-1. Let $X$ be a free $L$-space and $x \in X$. The *free L-character* of $x$, denoted by $\chi_L(x)$, is larger than $n$ if the following condition is satisfied.

*For every free L-structure* $(\mathcal{F}, \{\mathcal{U}_F: F \in \mathcal{F}\})$ *of X there exists a neighborhood W of x (depending on* $(\mathcal{F}, \{\mathcal{U}_F: F \in \mathcal{F}\})$*) such that if* $x \in F_i \in \mathcal{F}$ *and* $U_i$ *is a canonical neighborhood of* $F_i$ $(1 \leqq i \leqq n)$ *then* $(\bigcap_{i=1}^n U_i) - W \neq \emptyset$.

We say $\chi_L(x)=n$ if $\chi_L(x)>n-1$ and $\chi_L(x) \not> n$.

EXAMPLE 4-2. Part 1. For each $i \in N$ we construct a free $L$-space $Y_i$ containing a point $o_i$ such that $\chi_L(o_i)>i-1$.

Let

$$C_n = \{(n, b): b=0 \text{ or } 1/m,\ m \in N\} \subset R^2, \quad n \in N,$$

$$A = \{(n, 0): n \in N\},$$

$$X = (\bigcup_{n \in N} C_n)/A,$$

$$g: \bigcup_{n \in N} C_n \longrightarrow X \quad \text{the natural quotient mapping.}$$

Then $g$ is a closed mapping so $X$ is a Lašnev space. Therefore $X$ is a free $L$-space (see [3], Theorem 1.6 and [4]). Let $Y_i = X^i$. Then $Y_i$ is a free $L$-space (see [4], Theorem 1.3). Let

$$o = g(A),$$

$$B_n = g(C_n), \quad n \in N,$$

$$o_i = (o, \cdots, o) \in Y_i.$$

We prove that $\chi_L(o_i)>i-1$.

Let $(\mathcal{F}, \{\mathcal{U}_F: F \in \mathcal{F}\})$ be a free $L$-structure of $Y_i$ and

$$\{(F_1, \cdots, F_{i-1})_n: n \in N\} = \{(F_{a_1}, \cdots, F_{a_{i-1}}): o_i \in F_{a_j} \in \mathcal{F},\ 1 \leqq j \leqq i-1\}.$$

Let $e: N \times N \to N$ be a one-to-one and onto mapping and

$$B(m, n) = B_{e(m,n)}, \quad (m, n) \in N \times N.$$

Letters $G(m, n)$, $(m, n)\in N\times N$ denote sets satisfying the following conditions.

1. *$G(m, n)\subset B(m, n)$, $(m, n)\in N\times N$.*
2. *Each $G(m, n)$ is a neighborhood of $o$ in $B(m, n)$.*

We determine a neighborhood $W$ of $o_i$ which is required in Definition 4-1 as follows.

For $(F_1, \cdots, F_{i-1})_k\in\{(F_1, \cdots, F_{i-1})_n: n\in N\}$ we determine $\{G(k, n): n\in N\}$ and set

$$G=\{G(k, n): (k, n)\in N\times N\}^{\#},$$

$$W=G^i.$$

We determine $\{G(k, n): n\in N\}$.

Let

$$I=(N\cup\{0\})^i-(N^i\cup\{(0, \cdots, 0)\})$$

be an index set. Set

$$B(k, 0)=\{o\},$$

$$S(a)=B(k, n_1)\times\cdots\times B(k, n_i),\quad a=(n_1, \cdots, n_i)\in I.$$

Define $P_j: X^i\to X$ by $P_j(x_1, \cdots, x_i)=x_j$ and $T_j: X^i\to X^{i-1}$ by $T_j(x_1, \cdots, x_i)=(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_i)$ for $1\leqq j\leqq i$.

Case 1. $\bigcup_{j=1}^{i-1} F_j|_{S(a)}$ isn't a neighborhood of $o_i$ in $S(a)$ for some $a\in I$.

Without loss of generality, we can assume that $a=(0, n_2, \cdots, n_i)$. Let $\{q_n: n\in N\}=S(a)-\bigcup_{j=1}^{i-1} F_j$. Then there exists $\{G(k, n): n\in N\}$ such that for each $n\in N$,

*if $q_n\in U\in\mathcal{U}_F$ and $F\in\{F_1, \cdots, F_{i-1}\}$ then* $U\cap T_1^{-1}(T_1(q_n))$
$$\supsetneqq P_1^{-1}(G(k, n))\cap T_1^{-1}(T_1(q_n)).$$

Case 2. For each $a\in I$, $\bigcup_{j=1}^{i-1} F_j|_{S(a)}$ is a neighborhood of $o_i$ in $S(a)$.

(2-1) $(\bigcap_{j=1}^{i-1} F_j)\cap\{S(a): a\in I\}^{\#}\ni q\neq o_i$.

In this case there exists $\{G(k, n): n\in N\}$ such that

$$q\notin(\bigcup_{n\in N} G(k, n))^i.$$

(2-2) $(\bigcap_{j=1}^{i-1} F_j)\cap\{S(a): a\in I\}^{\#}=\{o_i\}$.

Without loss of generality, we can assume that

$$\mathrm{Cl}((H_1\cap S(1, 0, \cdots, 0))-\{o_i\})\ni o_i$$

for some $H_1=(\bigcap_{j=1}^{h_1} F_j)-(\bigcup_{j>h_1}^{i-1} F_j)$, $1\leqq h_1\leqq i-2$.

(2-2-1) For each $n\in N$,

$$\mathrm{Cl}\{q:\ q\in(H_1\cap S(1,0,\cdots,0))-\{o_i\},\ |(T_2^{-1}(T_2(q))\cap S(1,n,0,\cdots,0))-H_1|<|N|\}\ni o_i.$$

In this case there exist $\{q_n:\ n\in N\}\subset(H_1\cap S(1,0,\cdots,0))-\{o_i\}$ and $\{G(k,n):\ n\in N\}$ such that for each $n\in N$,

1. $H_1\cap T_2^{-1}(T_2(q_n))\supsetneqq P_2^{-1}(G(k,n))\cap T_2^{-1}(T_2(q_n))$,
2. *if* $q_n\in U\in\mathcal{U}_F$ *and* $F\in\{F_{h_1+1},\cdots,F_{i-1}\}$ *then* $U\cap H_1\cap T_2^{-1}(T_2(q_n))$ $\supsetneqq P_2^{-1}(G(k,n))\cap T_2^{-1}(T_2(q_n))$.

Otherwise, without loss of generality, we can assume that

$$\mathrm{Cl}((H_2\cap S(1,1,0,\cdots,0))-\{o_i\})\ni o_i$$

for some $H_2=(\bigcap_{j=1}^{h_2} F_j)-(\bigcup_{j>h_2}^{i-1} F_2)$, $1\leqq h_2<h_1$. We consider (2-2-1) for the 3rd-axis. If (2-2-1) isn't yet true then we continue analogously. Since $h_1\leqq i-2$ and this program is valid to the $(i-1)$th-axis. Thus we come to (2-2-1) at finite times.

Now we show that this $W$ is the required. Let $(F_1,\cdots,F_{i-1})_k\in\{(F_1,\cdots,F_{i-1})_n:\ n\in N\}$, $F_j\subset U_j$ $(1\leqq j\leqq i-1)$ and $\bigcap_{j=1}^{i-1} U_j\subset W$.

If Case 1 is true then for each $n\in N$ there exists $p_n\in T_1^{-1}(T_1(q_n))$ such that

1. $p_n\notin W$,
2. *if* $q_n\in U\in\mathcal{U}_F$ *and* $F\in\{F_1,\cdots,F_{i-1}\}$ *then* $p_n\in U$.

Since $(\bigcup_{j=1}^{i-1} F_j)\cap\{q_n:\ n\in N\}=\emptyset$, $\{\mathrm{St}_{\mathcal{U}_{F_j}}(Y_i-U_j):\ 1\leqq j\leqq i-1\}^{\#}\supset\{q_n:\ n\in N\}$. Thus the fact $\mathrm{Cl}\{q_n:\ n\in N\}\ni o_i$ implies that $\mathrm{Cl}\,\mathrm{St}_{\mathcal{U}_{F_n}}(Y_i-U_n)\ni o_i$ for some $n$ $(1\leqq n\leqq i-1)$. Then $U_n$ isn't a canonical neighborhood of $F_n$.

Let Case 2 be true. Since we assume that $\bigcap_{j=1}^{i-1} F_j\subset W$, so case (2-1) does not hold. Thus (2-2) is true. Let (2-2-1) be true. Let us assume that we come to (2-2-1) at one time. (Other case is proved analogously.) For each $n\in N$ there exists $p_n\in T_2^{-1}(T_2(q_n))$ such that

1. $p_n\notin W$,
2. *if* $q_n\in U\in\mathcal{U}_F$ *and* $F\in\{F_{h_1+1},\cdots,F_{i-1}\}$ *then* $p_n\in U\cap H_1$.

By the same reason, there exists $n$ $(h_1<n<i)$ such that $U_n$ isn't a canonical neighborhood of $F_n$.

Therefore $\chi_L(o_i)>i-1$.

Part 2. We define a free $L$-space $Y$, a space $Z$ that isn't a free $L$-space and a closed continuous onto mapping $f: Y \to Z$.

Set $Y=\bigcup_{i\in N} Y_i$, $Z=Y/E$ where $E=\{o_i : i\in N\}$ and $f: Y\to Z$ the natural quotient mapping. It is enough to prove that $Z$ isn't a free $L$-space.

Let $Z$ be a free $L$-space and $(\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\})$ a free $L$-structure of $Z$. Since $f(Y_i)\approx Y_i$, we can assume that $Y_i\subset Z$, $i\in N$. Then $(\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\})|_{Y_i}$ is a free $L$-structure of $Y_i$. Let $W_i$ be a neighborhood of $o_i$ constructed from $(\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\})|_{Y_i}$ as in Part 1. Then $W=\bigcup_{i\in N} W_i$ (in $Z$) is a neighborhood of $\bar{o}=f(E)$. Thus there exist $F_1, \cdots, F_n\in\mathcal{F}$ and a canonical neighborhood $U_i$ of $F_i$ $(1\leq i\leq n)$ such that

$$\bar{o}\in \bigcap_{i=1}^{n} F_i \subset \bigcap_{i=1}^{n} U_i \subset W .$$

Then $F_i|_{Y_{n+1}}\in\mathcal{F}|_{Y_{n+1}}$, $U_i|_{Y_{n+1}}$ is a canonical neighborhood of $F_i|_{Y_{n+1}}$ in $Y_{n+1}$ $(1\leq i\leq n)$ and

$$o_{n+1}\in \bigcap_{i=1}^{n} F_i|_{Y_{n+1}} \subset \bigcap_{i=1}^{n} U_i|_{Y_{n+1}} \subset W_{n+1} .$$

But this contradicts to the construction of $W_{n+1}$. Thus $Z$ isn't a free $L$-space. But $Z$ is an $M_1$-space (see [1], Theorem 3 and [2], Theorem 54.11).

This example shows that an adjunction space of two free $L$-spaces need not be a free $L$-space.

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