J. Math. Soc. Japan
Vol. 34, No. 3, 1982

The role of boundary Harnack principle in the study of Picard principle

By Toshimasa TADA

(Received Nov. 26, 1980)

A nonnegative locally Hölder continuous function P on $0<|z|\leqq 1$ will be referred to as a *density* on $\Omega: 0<|z|<1$. Here we consider Ω as an end of the punctured sphere $0<|z|\leqq+\infty$ so that the point $z=0$ is viewed as the ideal boundary $\delta\Omega$ of Ω, the unit circle $|z|=1$ as the relative boundary $\partial\Omega$ of Ω, and the punctured closed unit disk $0<|z|\leqq 1$ as the relative closure $\bar{\Omega}$ of Ω. Similar notations are used for subregions of Ω. For example we denote by $\partial\Omega_a$ and $\bar{\Omega}_a$ the relative boundary $|z|=a$ and the relative closure $0<|z|\leqq a$ of the subregion $\Omega_a: 0<|z|<a$ $(a\in(0, 1])$ of Ω, respectively. A density P on Ω gives rise to an elliptic operator $L=L_P$ on Ω defined by

$$Lu=L_Pu=\Delta u-Pu\,, \qquad \Delta=\partial^2/\partial x^2+\partial^2/\partial y^2\,.$$

Since $\delta\Omega$ is of parabolic character, there exists a unique bounded solution $e(z, a)=e_P(z, a)$, referred to as the *P-unit* on Ω_a, of $Lu=0$ on Ω_a with continuous boundary values 1 on $\partial\Omega_a$. We simply denote by $e(z)=e_P(z)$ the P-unit $e(z, 1)=e_P(z, 1)$ on Ω. With the operator $L=L_P$ we associate an elliptic operator $\hat{L}=\hat{L}_P$ on Ω, referred to as the *associate operator* to $L=L_P$, given by

$$\hat{L}v=\hat{L}_Pv=\Delta v+2\nabla\log e_P\cdot\nabla v\,, \qquad \nabla=(\partial/\partial x,\ \partial/\partial y)\,.$$

After Bouligand we say that the *Picard principle* is valid for P at $\delta\Omega$ if the dimension of the half module of nonnegative solutions of $Lu=0$ on Ω with continuous boundary values 0 on $\partial\Omega$ is 1. We also say that the *Riemann theorem* is valid for $\hat{L}$ at $\delta\Omega$ if the limit $\lim_{z\to\delta\Omega}v(z)$ exists for every bounded solution v of $\hat{L}v=0$ on Ω. Then we have the following *duality theorem* (cf. Heins [3], Hayashi [2], Nakai [8]): The Picard principle is valid for P at $\delta\Omega$ if and only if the Riemann theorem is valid for $\hat{L}$ at $\delta\Omega$. As a sufficient condition of the Riemann theorem for $\hat{L}$ at $\delta\Omega$ we have, what we call, the following *boundary Harnack principle* for L at $\delta\Omega$:

(1) *For every a in $(0, 1]$ there exists a Jordan curve K_a in Ω_a which separates $\delta\Omega$ from $\partial\Omega_a$ and satisfies $C(K_a; \Omega_a, L)=O(1)$ $(a\to 0)$, where for every*

This work was done partly supported by Grant-in-Aid for Scientific Research (No. 374055), Ministry of Education.

subregion S of Ω and compact subset K of S $C(K; S, L)$ is the Harnack constant given by

$$C(K; S, L)=\sup\left\{\frac{u(\zeta)}{u(\xi)}; \zeta, \xi\in K \text{ and } u \text{ is any positive solution of } Lu=0 \text{ on } S \text{ with } C^2 \text{ boundary values on } \partial S\right\}.$$

The boundary Harnack principle (1) for L at $\delta\Omega$ implies the following *boundary Harnack principle* for $\hat{L}$ at $\delta\Omega$ in the same fashion as (1) originally considered by Kawamura [6]:

(2) *For every a in* (0, 1] *there exists a Jordan curve K_a in Ω_a which separates $\delta\Omega$ from $\partial\Omega_a$ and satisfies $C(K_a; \Omega_a, \hat{L})=O(1)$ $(a\to 0)$, where $C(K_a; \Omega_a, \hat{L})$ is the Harnack constant given by the similar way in* (1).

The Riemann theorem for $\hat{L}$ at $\delta\Omega$ follows from the boundary Harnack principle (2) for $\hat{L}$ at $\delta\Omega$ (cf. Kawamura [6]). Thus we have the following implications: (1)→(2)→'The Picard principle'.

The purpose of this paper is to give an estimate of the Harnack constant $C(K_a; \Omega_a, L)$ and apply it to the study of the Picard principle. In § 1 we will show that the boundary Harnack principle for L at $\delta\Omega$ follows from the Picard principle for P at $\delta\Omega$. Therefore, with the above implications, we can conclude that the boundary Harnack principle for L and $\hat{L}$, the Riemann theorem for $\hat{L}$, and the Picard principle for P are all equivalent to each other at $\delta\Omega$. In § 2 we will give the Fourier series representation of the normal derivative of the P-Green's function for a *rotation free* density P, i.e. a density satisfying $P(z)=P(|z|)$ on $\bar{\Omega}$. Since we cannot locate the exact reference to the representation which may be well known and also it plays an essential role in § 3, we will include it's proof for the sake of completeness. In § 3 applying results in § 2 to the study of the Picard principle for densities Q on Ω with $P(z)\leqq Q(z)\leqq P(z)+C/|z|^2$ on $\bar{\Omega}$ for a rotation free density P on Ω, we will show that the Picard principle is valid for Q at $\delta\Omega$ if the Picard principle is valid for P at $\delta\Omega$.

The author is very grateful to Professor M. Nakai for his helpful suggestions.

§ 1. The boundary Harnack principle.

1. In nos. 1-3 we consider a density P on Ω for which the Picard principle is valid at $\delta\Omega$ and denote by $G_P^{\Omega_a}(z, \zeta)$ the P-Green's function on Ω_a with pole at ζ. The function $G_P^{\Omega_a}(z, \zeta)/e(\zeta)$ in z uniformly converges on every compact subset of $\bar{\Omega}_a$ as $\zeta\to\delta\Omega$ and hence, the inner normal derivative $\frac{\partial}{\partial n_z}G_P^{\Omega_a}(z, \zeta)/e(\zeta)$ uniformly converges to a positive continuous function $k(z, a)=k_P(z, a)$ on $\partial\Omega_a$ as $\zeta\to\delta\Omega$ (cf. Itô [5]). If we set

$$\varepsilon=\varepsilon(C,\ a)=\frac{\sqrt{C}-1}{\sqrt{C}+1}\min_{z\in\partial\Omega_a}k(z,\ a)$$

for every constant $C>1$, then there exists $\delta=\delta(C,\ a)$ in $(0,\ a)$ such that

$$\left|\frac{\dfrac{\partial}{\partial n_z}G_P^{\Omega_a}(z,\ \zeta)}{e(\zeta)}-k(z,\ a)\right|<\varepsilon \tag{3}$$

for any z in $\partial\Omega_a$ and ζ in Ω_δ.

2. We estimate the Harnack constant $C(K_a\,;\ \Omega_a,\ L)$ for an appropriate Jordan curve K_a in Ω_a. In order to determine K_a we consider two cases separately: $\limsup_{\zeta\to\delta\Omega}e(\zeta)=0$ and >0. First we consider the case $\limsup_{\zeta\to\delta\Omega}e(\zeta)=0$, i.e. $\lim_{\zeta\to\delta\Omega}e(\zeta)=0$. For every λ in $(0,\ 1)$ consider the relatively noncompact connected component A_λ of $\{\zeta\in\Omega\,;\ e(\zeta)<\lambda\}$. Since $\bar{A}_\lambda\downarrow\emptyset$ as $\lambda\to 0$, there exists $\mu=\mu(C,\ a)$ in $(0,\ 1)$ with $\bar{A}_\mu\subset\Omega_\delta$. Then from (3) it follows that for any λ in $(0,\ \mu]$, ζ_1, ζ_2 in ∂A_λ, and z in $\partial\Omega_a$

$$\frac{\dfrac{\partial}{\partial n_z}G_P^{\Omega_a}(z,\ \zeta_1)}{\dfrac{\partial}{\partial n_z}G_P^{\Omega_a}(z,\ \zeta_2)}=\frac{\dfrac{\partial}{\partial n_z}G_P^{\Omega_a}(z,\ \zeta_1)}{e(\zeta_1)}\ \frac{e(\zeta_2)}{\dfrac{\partial}{\partial n_z}G_P^{\Omega_a}(z,\ \zeta_2)}<\frac{k(z,\ a)+\varepsilon}{k(z,\ a)-\varepsilon}$$

$$\leqq\frac{k(z,\ a)+(\sqrt{C}-1)(\sqrt{C}+1)^{-1}k(z,\ a)}{k(z,\ a)-(\sqrt{C}-1)(\sqrt{C}+1)^{-1}k(z,\ a)}=\sqrt{C}<C\,.$$

Now we set $K_a=\partial A_\mu$. Applying this inequality to the integral formula of a solution of Dirichlet problem of $Lu=0$ on Ω_a with the normal derivative of P-Green's function on $\partial\Omega_a$, we have

$$C(K_a\,;\ \Omega_a,\ L)\leqq C \tag{4}$$

for any a in $(0,\ 1]$.

3. Assume next that $\limsup_{\zeta\to\delta\Omega}e(\zeta)\equiv\alpha>0$. There exists a closed set E thin at $\delta\Omega$ in Ω such that $e(\zeta)\to\alpha$ as $\zeta\to\delta\Omega$ with $\zeta\notin E$ (cf. Brelot [**1**]).[*] Then we may take a decreasing sequence $\{\lambda_n\}$ in $(0,\ 1)$ with $E\cap\bigcup_{n=1}^{\infty}\partial\Omega_{\lambda_n}=\emptyset$ and $\lim_{n\to\infty}\lambda_n=0$. Since $e(\zeta)\to\alpha$ as $\zeta\to\delta\Omega$ with ζ in $\bigcup_{n=1}^{\infty}\partial\Omega_{\lambda_n}$, there exists a positive integer $m=m(C,\ a)$ such that $\lambda_m<\delta$ and

$$|e(\zeta)-\alpha|<\frac{\sqrt{C}-1}{\sqrt{C}+1}\alpha$$

for any ζ in $\bigcup_{n=m}^{\infty}\partial\Omega_{\lambda_n}$. Then from this inequality and (3) it follows that for

---

[*] This can be deduced, e.g., by using the following results successively in this order: Theorem VII, 2 in p. 55, Theorem III, 1 in p. 18, and Theorem I, 3 in p. 3.

any ζ_1, ζ_2 in $\bigcup_{n=m}^{\infty}\partial\Omega_{\lambda_n}$ and z in $\partial\Omega_a$

$$\frac{\frac{\partial}{\partial n_z}G_P^{\Omega_a}(z, \zeta_1)}{\frac{\partial}{\partial n_z}G_P^{\Omega_a}(z, \zeta_2)}=\frac{\frac{\partial}{\partial n_z}G_P^{\Omega_a}(z, \zeta_1)}{e(\zeta_1)}\frac{e(\zeta_2)}{\frac{\partial}{\partial n_z}G_P^{\Omega_a}(z, \zeta_2)}\frac{e(\zeta_1)}{e(\zeta_2)}$$

$$<\sqrt{C}\frac{\alpha+(\sqrt{C}-1)(\sqrt{C}+1)^{-1}\alpha}{\alpha-(\sqrt{C}-1)(\sqrt{C}+1)^{-1}\alpha}=C.$$

Now setting $K_a=\partial\Omega_{\lambda_m}$ we have

(5) $$C(K_a\,;\,\Omega_a,\,L)\leqq C.$$

4. We compile (4) and (5) in the following form: If the Picard principle is valid for P, then for every λ in $(0, 1]$ there exists a Jordan curve K_λ in Ω separating $\delta\Omega$ from $\partial\Omega$ such that for every constant $C>1$ and a in $(0, 1]$ we may take $\nu=\nu(C, a)$ in $(0, 1]$ with $K_\lambda\subset\Omega_a$ and $C(K_\lambda\,;\,\Omega_a,\,L)<C$ for any λ in $(0, \nu]$. Then we have the following

THEOREM. *For every density on Ω following three principles at $\delta\Omega$ are all equivalent to each other: The boundary Harnack principle for L_P, the boundary Harnack principle for $\hat{L}_P$, and the Picard principle for P.*

§ 2. Normal derivative of the P-Green's function.

5. In this section we consider a *rotation free* density P on Ω, i. e. a density satisfying $P(z)=P(|z|)$ on $\bar{\Omega}$. For every nonnegative integer n $P_n(z)=P(z)+n^2/|z|^2$ is also a rotation free density on Ω. Since for every a in $(0, 1]$ the P_n-unit $e_n(z, a)=e_{P_n}(z, a)$ on Ω_a, i. e. the unique bounded solution of $L_{P_n}u=0$ on Ω_a with continuous boundary values 1 on $\partial\Omega_a$, is also rotation free, every $e_n(z, a)$ may be viewed as a function in r in $(0, a]$. In other words, $e_n(r, a)$ is considered as the unique bounded solution of

$$l_n\psi(r)\equiv l_{P_n}\psi(r)\equiv\frac{d^2}{dr^2}\psi(r)+\frac{1}{r}\frac{d}{dr}\psi(r)-P_n(r)\psi(r)=0$$

on $(0, a)$ with continuous boundary values 1 at $r=a$, where we follow the convention $P_0=P$ and $e_0(r, a)=e(r, a)=e_P(r, a)$. We recall some of fundamental properties of $e_n(r, a)$ (cf. Nakai [7]): For every r in $(0, a]$, ρ in $[r, a]$, and nonnegative integer n

(6) $$e_n(r, \rho)=\frac{e_n(r, a)}{e_n(\rho, a)}\,;$$

(7) $$e_n(r, a)\geqq e_{n+1}(r, a)\,,$$

where the equality is valid if and only if $r=a$;

(8) $$\frac{e_{n+1}(r,\ a)}{e_n(r,\ a)} \geqq \frac{e_{n+2}(r,\ a)}{e_{n+1}(r,\ a)}\,;$$

(9) $$\left[\frac{e_{n+1}(r,\ a)}{e_n(r,\ a)}\right]^3 \leqq \frac{e_{n+2}(r,\ a)}{e_{n+1}(r,\ a)}\,;$$

the Picard principle is valid for P at $\partial\Omega$ if and only if

(10) $$\lim_{r\to 0}\frac{e_1(r)}{e_0(r)}=0\,,$$

where we simply denote by $e_n(r)$ the P_n-unit $e_n(r,\ 1)$ on Ω.

6. For a continuous function w on $\bar{\Omega}_a$ the Fourier coefficients

$$\begin{cases} c_0(r\,;\,w)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} w(re^{i\theta})d\theta\,, \\ a_n(r\,;\,w)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} w(re^{i\theta})\cos n\theta d\theta\,, \\ b_n(r\,;\,w)=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} w(re^{i\theta})\sin n\theta d\theta \end{cases}$$

of w are continuous function in r in $(0,\ a]$. If w is further a bounded solution of $Lu=0$ on Ω_a, then the Fourier coefficients of w satisfy that

$$\frac{d^2}{dr^2}c_0(r\,;\,w)+\frac{1}{r}\frac{d}{dr}c_0(r\,;\,w)=c_0\Big(r\,;\,\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\Big)$$

$$=c_0\Big(r\,;\,\Delta w-\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\Big)=P(r)c_0(r\,;\,w)\,,$$

$$\frac{d^2}{dr^2}a_n(r\,;\,w)+\frac{1}{r}\frac{d}{dr}a_n(r\,;\,w)=a_n\Big(r\,;\,\Delta w-\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\Big)$$

$$=P(r)a_n(r\,;\,w)-\frac{n^2}{r^2}b_n\Big(r\,;\,\frac{\partial w}{\partial\theta}\Big)=P_n(r)a_n(r\,;\,w)\,,$$

and similarly

$$\frac{d^2}{dr^2}b_n(r\,;\,w)+\frac{1}{r}\frac{d}{dr}b_n(r\,;\,w)=P_n(r)b_n(r\,;\,w)$$

on $(0,\ a)$. Therefore they are bounded solutions of $l_0\phi=0$ or $l_n\phi=0$ and hence, we have

(11) $$\begin{cases} c_0(r\,;\,w)=c_0(a\,;\,w)e_0(r,\ a)\,, \\ a_n(r\,;\,w)=a_n(a\,;\,w)e_n(r,\ a)\,, \\ b_n(r\,;\,w)=b_n(a\,;\,w)e_n(r,\ a) \end{cases}$$

on $(0, a]$. For every τ in $(0, 2\pi]$ we consider a sequence $\{\Phi_{\tau,n}\}$ of C^2 functions on $\partial\Omega_a$ such that $0\leqq\Phi_{\tau,n}\leqq\Phi_{\tau,n+1}\leqq 1$ $(n=1, 2, \cdots)$ and $\lim_{n\to\infty}\Phi_{\tau,n}(ae^{i\theta})=1$ $(0<\theta<\tau)$, 0 $(\tau\leqq\theta\leqq 2\pi)$. Since $\delta\Omega$ is of parabolic character, the unique bounded solution $w_{\tau,n}$ of $Lu=0$ on Ω_a with continuous boundary values $\Phi_{\tau,n}$ on $\partial\Omega_a$ is represented in the following form:

$$w_{\tau,n}(\zeta)=\frac{1}{2\pi}\int_0^{2\pi}-\frac{\partial}{\partial r}G_P^{\Omega_a}(re^{i\theta}, \zeta)|_{r=a}\Phi_{\tau,n}(ae^{i\theta})ad\theta \tag{12}$$

for any ζ in Ω_a, where $G_P^{\Omega_a}(z, \zeta)$ is the P-Green's function on Ω_a with pole at ζ. On the other hand, applying (11) to $w_{\tau,n}$ we have also

$$\begin{aligned} w_{\tau,n}(se^{i\sigma})=&c_0(a\,;w_{\tau,n})e_0(s, a)\\ &+\sum_{m=1}^{\infty}\{a_m(a\,;w_{\tau,n})\cos m\sigma+b_m(a\,;w_{\tau,n})\sin m\sigma\}e_m(s, a)\end{aligned} \tag{13}$$

for any s in $(0, a)$ and σ in $[0, 2\pi)$. Observe the facts that $|c_0(a\,;w_{\tau,n})|$, $|a_m(a\,;w_{\tau,n})|$, $|b_m(a\,;w_{\tau,n})|$ are less than 2 and by (7), (8) we have

$$\frac{e_1(s, a)}{e_0(s, a)}<1, \tag{14}$$

$$e_m(s, a)=e_0(s, a)\frac{e_1(s, a)}{e_0(s, a)}\cdots\frac{e_m(s, a)}{e_{m-1}(s, a)}\leqq e_0(s, a)\left\{\frac{e_1(s, a)}{e_0(s, a)}\right\}^m. \tag{15}$$

Then on making $n\to\infty$ in (13) we obtain

$$\begin{aligned}\lim_{n\to\infty}w_{\tau,n}(se^{i\sigma})=&\lim_{n\to\infty}c_0(a\,;w_{\tau,n})e_0(s, a)\\ &+\sum_{m=1}^{\infty}\lim_{n\to\infty}\{a_m(a\,;w_{\tau,n})\cos m\sigma+b_m(a\,;w_{\tau,n})\sin m\sigma\}e_m(s, a)\\ =&\frac{e_0(s, a)}{2\pi}\int_0^{\tau}1d\theta\\ &+\sum_{m=1}^{\infty}\frac{e_m(s, a)}{\pi}\left\{\cos m\sigma\int_0^{\tau}\cos m\theta\,d\theta+\sin m\sigma\int_0^{\tau}\sin m\theta\,d\theta\right\}.\end{aligned}$$

Therefore from (14) and (15) it follows that

$$\begin{aligned}\frac{\partial}{\partial\tau}\lim_{n\to\infty}w_{\tau,n}(se^{i\sigma})=&\frac{e_0(s, a)}{2\pi}\frac{\partial}{\partial\tau}\int_0^{\tau}1d\theta\\ &+\sum_{m=1}^{\infty}\frac{e_m(s, a)}{\pi}\frac{\partial}{\partial\tau}\left\{\cos m\sigma\int_0^{\tau}\cos m\theta\,d\theta+\sin m\sigma\int_0^{\tau}\sin m\theta\,d\theta\right\}\\ =&\frac{e_0(s, a)}{2\pi}+\sum_{m=1}^{\infty}\frac{e_m(s, a)}{\pi}\cos m(\sigma-\tau).\end{aligned}$$

On the other hand from (12) it also follows that

$$\frac{\partial}{\partial\tau}\lim_{n\to\infty} w_{\tau,n}(\zeta)=-\frac{a}{2\pi}\frac{\partial}{\partial r}G_P^{\Omega_a}(re^{i\tau},\zeta)|_{r=a}.$$

Thus identifying ζ with $se^{i\sigma}$ we have the following

LEMMA. *Let P be a rotation free density on Ω. Then for every a in* (0, 1], *s in* (0, a), *and σ, τ in* $[0, 2\pi)$

$$-\frac{\partial}{\partial r}G_P^{\Omega_a}(re^{i\tau}, se^{i\sigma})|_{r=a}=a^{-1}\left\{e_0(s, a)+2\sum_{m=1}^{\infty}e_m(s, a)\cos m(\sigma-\tau)\right\}.$$

Estimating the right hand side of the above equality by using (14) and (15), we obtain the following inequalities:

$$-\frac{\partial}{\partial r}G_P^{\Omega_a}(re^{i\tau}, se^{i\sigma})|_{r=a}\leqq a^{-1}e_0(s, a)\left\{1+\frac{e_1(s, a)}{e_0(s, a)}\right\}\left\{1-\frac{e_1(s, a)}{e_0(s, a)}\right\}^{-1} \tag{16}$$

and

$$-\frac{\partial}{\partial r}G_P^{\Omega_a}(re^{i\tau}, se^{i\sigma})|_{r=a}\geqq a^{-1}e_0(s, a)\left\{1-3\frac{e_1(s, a)}{e_0(s, a)}\right\}\left\{1-\frac{e_1(s, a)}{e_0(s, a)}\right\}^{-1}. \tag{17}$$

§ 3. An application of Lemma in § 2.

7. In this section we consider rotation free densities P and R on Ω with $P\leqq R$ on $\bar{\Omega}$. For rotation free densities $P_n(z)=P(z)+n^2/|z|^2$ and $R_n(z)=R(z)+n^2/|z|^2$ ($n=0, 1, \cdots$) on Ω we denote by $e_n(z, a)$ and $f_n(z, a)$ the P_n-unit and R_n-unit on Ω_a ($0<a\leqq 1$), respectively. As a fundamental relation between the functions $e_n(r, a)$ and $f_n(r, a)$ in r in $(0, a]$, it is known that

$$\frac{e_{n+1}(r, a)}{e_n(r, a)}\leqq\frac{f_{n+1}(r, a)}{f_n(r, a)} \quad (n=0, 1, \cdots) \tag{18}$$

(cf. Imai [4]). In this no. we give some relations between $e_n(r, a)$ and $f_n(r, a)$ in the case $R(z)=P(z)+9k^2/|z|^2$ on $\bar{\Omega}$ for a positive integer k. From (9) it follows that

$$\frac{f_{4k}(r, a)}{f_0(r, a)}=\frac{f_{4k}(r, a)}{f_{4k-1}(r, a)}\cdots\frac{f_1(r, a)}{f_0(r, a)}$$
$$\geqq\left\{\frac{f_1(r, a)}{f_0(r, a)}\right\}^{3^{4k-1}+3^{4k-2}+\cdots+1}=\left\{\frac{f_1(r, a)}{f_0(r, a)}\right\}^{(81^k-1)/2}$$

and

$$\frac{e_{3k}(r, a)}{e_0(r, a)}\geqq\left\{\frac{e_1(r, a)}{e_0(r, a)}\right\}^{3^{3k-1}+\cdots+1}=\left\{\frac{e_1(r, a)}{e_0(r, a)}\right\}^{(27^k-1)/2}.$$

Observe that $R=P_{3k}$, $R_{4k}=P_{5k}$ and hence, $f_0(r, a)=e_{3k}(r, a)$, $f_{4k}(r, a)=e_{5k}(r, a)$.

Then above two inequalities are rewritten in the following forms:

$$\frac{f_1(r, a)}{f_0(r, a)} \leqq \left\{\frac{e_{5k}(r, a)}{e_{3k}(r, a)}\right\}^{2/(81^k-1)} \tag{19}$$

and

$$\frac{e_1(r, a)}{e_0(r, a)} \leqq \left\{\frac{f_0(r, a)}{e_0(r, a)}\right\}^{2/(27^k-1)}, \tag{20}$$

respectively. On the other hand by (8) we have

$$\frac{e_{5k}(r, a)}{e_{3k}(r, a)} = \frac{e_{5k}(r, a)}{e_{5k-1}(r, a)} \cdots \frac{e_{3k+1}(r, a)}{e_{3k}(r, a)} \leqq \left\{\frac{e_1(r, a)}{e_0(r, a)}\right\}^{2k}. \tag{21}$$

Therefore we compile (19), (20), and (21) in the following inequality:

$$\frac{f_1(r, a)}{f_0(r, a)} \leqq \left\{\frac{f_0(r, a)}{e_0(r, a)}\right\}^{\alpha_k}, \quad \text{i.e.} \quad \frac{e_0(r, a)}{f_0(r, a)} \leqq \left\{\frac{f_0(r, a)}{f_1(r, a)}\right\}^{\alpha_k^{-1}} \tag{22}$$

for any a in $(0, 1]$ and r in $(0, a]$, where $\alpha_k = 8k(81^k-1)^{-1}(27^k-1)^{-1}$.

8. Consider a *general* density Q on Ω with $P(z) \leqq Q(z) \leqq P(z) + C/|z|^2$ on $\bar{\Omega}$ for some nonnegative constant C. We take a positive integer k with $9k^2 \geqq C$ and denote by R the rotation free density P_{3k} on Ω. In view of (16), (17), and (18) the inner normal derivative of the Q-Green's function $G_Q^{\Omega_a}(z, \zeta)$ on Ω_a with pole at ζ satisfies that

$$\begin{aligned} \frac{\partial}{\partial n_z} G_Q^{\Omega_a}(z, \zeta) &\leqq \frac{\partial}{\partial n_z} G_P^{\Omega_a}(z, \zeta) \\ &\leqq a^{-1} e_0(|\zeta|, a)\left\{1 + \frac{e_1(|\zeta|, a)}{e_0(|\zeta|, a)}\right\}\left\{1 - \frac{e_1(|\zeta|, a)}{e_0(|\zeta|, a)}\right\}^{-1} \\ &\leqq a^{-1} e_0(|\zeta|, a)\left\{1 + \frac{f_1(|\zeta|, a)}{f_0(|\zeta|, a)}\right\}\left\{1 - \frac{f_1(|\zeta|, a)}{f_0(|\zeta|, a)}\right\}^{-1} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \frac{\partial}{\partial n_z} G_Q^{\Omega_a}(z, \zeta) &\geqq \frac{\partial}{\partial n_z} G_R^{\Omega_a}(z, \zeta) \\ &\geqq a^{-1} f_0(|\zeta|, a)\left\{1 - 3\frac{f_1(|\zeta|, a)}{f_0(|\zeta|, a)}\right\}\left\{1 - \frac{f_1(|\zeta|, a)}{f_0(|\zeta|, a)}\right\}^{-1} \end{aligned} \tag{24}$$

for any z in $\partial\Omega_a$. By (6), (19), and (21) we have

$$\frac{f_1(r, a)}{f_0(r, a)} \leqq \left\{\frac{e_1(r, a)}{e_0(r, a)}\right\}^{4k/(81^k-1)} = \left\{\frac{e_1(r)}{e_0(r)}\,\frac{e_0(a)}{e_1(a)}\right\}^{4k/(81^k-1)},$$

where $e_n(r) = e_n(r, 1)$ and $f_n(r) = f_n(r, 1)$ $(n = 0, 1, \cdots)$. Assume that the Picard principle is valid for P at $\delta\Omega$. Then from (10) it follows that there exists $s=s(C, a)$ in $(0, a)$ with $f_1(s, a)/f_0(s, a)=1/4$. Applying (23) and (24) to ζ_1 and ζ_2 in $\partial\Omega_s$, respectively we obtain

$$\frac{\frac{\partial}{\partial n_z}G_Q^{\Omega_a}(z, \zeta_1)}{\frac{\partial}{\partial n_z}G_Q^{\Omega_a}(z, \zeta_2)} \leqq \frac{e_0(s, a)}{f_0(s, a)} \frac{1+\frac{f_1(s, a)}{f_0(s, a)}}{1-3\frac{f_1(s, a)}{f_0(s, a)}} = 5\frac{e_0(s, a)}{f_0(s, a)}.$$

Therefore in view of (22) we have the following estimate:

$$C(\partial\Omega_s ; \Omega_a, Q) \leqq 5\cdot 4^{\alpha_k^{-1}}.$$

Thus we deduce the following

THEOREM. *Let P be a rotation free density on Ω for which the Picard principle is valid at $\delta\Omega$. Then the Picard principle is valid for every density Q on Ω at $\delta\Omega$ with $P(z)\leqq Q(z)\leqq P(z)+C/|z|^2$ on $\bar{\Omega}$ for a nonnegative constant C.*

References

[1] M. Brelot, On topologies and boundaries in potential theory, Lecture Notes in Math., **175**, Springer-Verlag, 1971.

[2] K. Hayashi, Les solutions positives de l'équation $\Delta u=Pu$ sur une surface de Riemann, Kōdai Math. Sem. Rep., **13** (1961), 20-24.

[3] M. Heins, Riemann surfaces of infinite genus, Ann. of Math., **55** (1952), 296-317.

[4] H. Imai, On singular indices of rotation free densities, Pacific J. Math., **80** (1979), 179-190.

[5] S. Itô, Martin boundary for linear elliptic differential operator of second order in a manifold, J. Math. Soc. Japan, **16** (1964), 307-334.

[6] M. Kawamura, On a conjecture of Nakai on Picard principle, J. Math. Soc. Japan, **31** (1979), 359-372.

[7] M. Nakai, Martin boundary over isolated singularity of rotation free density, J. Math. Soc. Japan, **26** (1974), 483-507.

[8] M. Nakai, Picard principle and Riemann theorem, Tôhoku Math. J., **28** (1976), 277-292.

Toshimasa TADA
Department of Mathematics
Daido Institute of Technology
Daido, Minami, Nagoya 457
Japan