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Equivariant embeddings and isotopies of a sphere in a representation

Dedicated to Professor Kentaro Murata on his 60 th birthday

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§1. Introduction and statement of results.

Let G be a finite group. Let M, N be smooth (i.e., infinitely differentiable) G-manifolds, and R the real line with trivial G-action. A level preserving smooth G-embedding

$$H: M\times R \longrightarrow N\times R$$

defines smooth G-embeddings H_t of M in N, for all $t\in R$, by the relation

$$H(x, t)=(H_t(x), t) \qquad \text{for any} \quad x\in M.$$

Let $f, g: M\to N$ be smooth G-embeddings. If, for some $a<b$,

$$H_t=f \qquad \text{for any} \quad t\leqq a,$$

$$H_t=g \qquad \text{for any} \quad t\geqq b,$$

then H is called a smooth G-isotopy between f and g, and f, g are called to be G-isotopic. The isotopy class $[f]$ is the set of all smooth G-embeddings which are G-isotopic to f. Denote by $\mathrm{Iso}^G(M, N)$ the set of all isotopy classes of smooth G-embeddings of M in N.

Let U be a finite dimensional representation of G. $S(U)$ denotes the unit sphere in U with respect to some G-invariant inner product. Then $S(U)$ is a smooth G-manifold. The purpose of this paper is to enumerate $\mathrm{Iso}^G(S(U), V)$ for finite dimensional representations U, V of G. In this paper we restrict ourselves to the case

$$\text{(C)} \qquad 0<\dim U^G<\dim U,$$

where U^G is the fixed point set in U by the G-action. All representations considered are real representations, and dim denotes the real dimension.

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Let $\mathrm{Hom}^G(U, V)$ be the set of all G-equivariant R-linear homomorphisms from U to V. This is a vector space over R. Let $\{W_j \mid j \in J(G)\}$ be a complete set of nontrivial, nonisomorphic, irreducible representations of G. Define

$$J(G; U)=\{j \in J(G) \mid \dim \mathrm{Hom}^G(W_j, U) \neq 0\}.$$

Then positive integers m_j for all $j \in J(G; U)$ are defined by splitting U into

$$U=U^G \oplus \oplus_{j \in J(G; U)} m_j W_j,$$

where $m_j W_j$ is the direct sum of m_j copies of W_j. Similarly positive integers n_j for all $j \in J(G; V)$ are defined by splitting V into

$$V=V^G \oplus \oplus_{j \in J(G; V)} n_j W_j.$$

For the case (C), there is a smooth G-embedding of $S(U)$ in V if and only if U is a subrepresentation of V. For such representations U, V we see that

$$J(G; U) \subset J(G; V),$$

$$m_j \leqq n_j \qquad \text{for any} \quad j \in J(G; U).$$

Define

$$\nu(S(U^G))=(\tau(S(U)) \mid S(U^G))/\tau(S(U^G)),$$

$$\nu(V^G)=(\tau(V) \mid V^G)/\tau(V^G),$$

the normal bundles of $S(U^G)$ in $S(U)$ and of V^G in V, respectively. Then there are smooth G-vector bundle isomorphisms

$$\Psi^U : \nu(S(U^G)) \cong S(U^G) \times \oplus_{j \in J(G; U)} m_j W_j,$$

$$\Psi^V : \nu(V^G) \cong V^G \times \oplus_{j \in (G; V)} n_j W_j.$$

Let $f: S(U) \to V$ be a smooth G-embedding, and $df: \tau(S(U)) \to \tau(V)$ the differential of f. Since $(df)^{-1}(\tau(V^G))=\tau(S(U^G))$, df induces a smooth G-vector bundle embedding $\bar{d}f: \nu(S(U^G)) \to \nu(V^G)$. A map

$$\Phi(f): S(U^G) \longrightarrow \mathrm{Mon}^G(\oplus_{j \in J(G; U)} m_j W_j, \oplus_{j \in J(G; V)} n_j W_j)$$

is defined by the relation

$$\Psi^V \circ \bar{d}f \circ (\Psi^U)^{-1}(x, w)=(f(x), \Phi(f)(x)(w))$$

for $x \in S(U^G)$ and $w \in \oplus_{j \in J(G; U)} m_j W_j$, where $\mathrm{Mon}^G(-, -)$ denotes the subspace of $\mathrm{Hom}^G(-, -)$ consisting of all G-equivariant R-linear monomorphisms. If f and g are G-isotopic smooth G-embeddings of $S(U)$ in V, $\Phi(f)$ and $\Phi(g)$ are homotopic. Thus the transformation

$$\Phi : \mathrm{Iso}^G(S(U), V) \longrightarrow [S(U^G), \mathrm{Mon}^G(\oplus_{j \in J(G; U)} m_j W_j, \oplus_{j \in J(G; V)} n_j W_j)]$$

is defined, where [-, -] denotes the homotopy set.

For any $j\in J(G)$, define $d_j=\dim \mathrm{Hom}^G(W_j, W_j)$. Then $d_j=2$ if W_j is the real restriction of a complex representation, $d_j=4$ if W_j is the real restriction of a quaternionic representation, and $d_j=1$ for other W_j. Let F_j be the real numbers R if $d_j=1$, the complex numbers C if $d_j=2$, and the quaternionic numbers Q if $d_j=4$.

For positive integers $m\leqq n$, consider the vector space nF_j over F_j and an ordered mF_j-linearly independent vectors $\{v_1, \cdots, v_m\}$ in nF_j. Such $\{v_1, \cdots, v_m\}$ is called an m-frame in nF_j. Denote by $V(m, n; F_j)$ the space of all m-frames in nF_j. $V(m, n; F_j)$ is identified with $GL(n; F_j)/GL(n-m; F_j)$, and is called the *open* Stiefel manifold. The *closed* Stiefel manifold consists of all orthonormal m-frames in nF_j, and is identified with $O(n)/O(n-m)$, $U(n)/U(n-m)$ or $Sp(n)/Sp(n-m)$. $V(m, n; F_j)$ is $(d_j(n-m+1)-2)$-connected, since the closed Stiefel manifold is a deformation retract of the open Stiefel manifold.

Since W_j is the real restriction of a representation over F_j, the scalar multiplication gives an isomorphism

$$\mathrm{Hom}^G(W_j, W_j)\cong F_j .$$

Thus we see that $\mathrm{Mon}^G(m_jW_j, n_jW_j)$ and $V(m_j, n_j; F_j)$ are identified. By Schur's lemma

$$\mathrm{Mon}^G(\oplus_{j\in J(G;U)}m_jW_j, \oplus_{j\in J(G;V)}n_jW_j)\approx\prod_{j\in J(G;U)} \mathrm{Mon}^G(m_jW_j, n_jW_j) .$$

Thus Φ gives a transformation

$$\tilde{\Phi} : \mathrm{Iso}^G(S(U), V) \longrightarrow \prod_{j\in J(G;U)}[S(U^G), V(m_j, n_j; F_j)] .$$

We obtain

THEOREM 1. *Let V be a finite dimensional representation of a finite group G, and let U be a subrepresentation of V with $0<\dim U^G<\dim U$. Suppose the G-action on V is semifree. Then*

(1) *$\tilde{\Phi}$ is surjective, if*
$\dim U+\max\{\dim U-1, \dim V^G\}\leqq\dim V$,

(2) *$\tilde{\Phi}$ is bijective, if*
$3\dim U^G<2\dim V^G$, *and*
$\dim U+\max\{\dim U-1, \dim V^G\}<\dim V$.

Now let G be the cyclic group of prime power order p^r. For each integer s with $0\leqq s\leqq r$, G has just one subgroup $H(s)$ of order p^s such that

$$\{1\}=H(0)\subset H(1)\subset \cdots \subset H(r)=G .$$

For each s let $\{W_j | j\in J(H(s))\}$ be a complete set of nontrivial, nonisomorphic, irreducible representations of $H(s)$. Let U, V be finite dimensional representations of G. By restricting the G-actions on U, V to $H(s)$-actions, we consider U, V as representations of $H(s)$. As before, define

$$J(H(s); U)=\{j\in J(H(s)) \mid \dim \mathrm{Hom}^{H(s)}(W_j, U)\neq 0\},$$

and define positive integers m_j for all $j\in J(H(s); U)$ by the relation

$$U=U^{H(s)}\oplus\bigoplus_{j\in J(H(s); U)} m_j W_j .$$

Similarly positive integers n_j for all $j\in J(H(s); V)$ are defined by the relation

$$V=V^{H(s)}\oplus\bigoplus_{j\in J(H(s); V)} n_j W_j .$$

Also define $d_j=\dim \mathrm{Hom}^{H(s)}(W_j, W_j)$ for any $j\in J(H(s))$.

We obtain

THEOREM 2. *Let G be the cyclic group of prime power order p^r. Let V be a finite dimensional representation of G, and U a subrepresentation of V with* $0<\dim U^G<\dim U$. *Then*

(1) $\tilde{\Phi}$ *is surjective, if*
$\dim U^{H(s)}+\max\{\dim U^{H(s)}-1, \dim V^{H(s+1)}\}\leqq \dim V^{H(s)}$ *for any s with* $0\leqq s<r$, *and*
$\dim U^{H(s)}\leqq d_j(n_j-m_j+1)$ *for any $j\in J(H(s); U)$ and any s with $0<s<r$,*

(2) $\tilde{\Phi}$ *is bijective, if*
$3\dim U^G<2\dim V^G$,
$\dim U^{H(s)}+\max\{\dim U^{H(s)}-1, \dim V^{H(s+1)}\}<\dim V^{H(s)}$ *for any s with* $0\leqq s<r$, *and*
$\dim U^{H(s)}<d_j(n_j-m_j+1)$ *for any $j\in J(H(s); U)$ and any s with $0<s<r$.*

In § 2 we provide some lemmas for the proofs of our theorems. We prove the surjectivity of $\tilde{\Phi}$ in § 3, and the injectivity of $\tilde{\Phi}$ in § 4.

NOTE. K. Abe [1] studied $\mathrm{Iso}^G(S(U), S(V))$, and obtained the triviality and the infiniteness of the set for very restricted U, V. Since the one-point compactification S^V of V is identified with $S(R\oplus V)$, there is a bijective transformation

$$\mathrm{Iso}^G(S(U), V)\approx \mathrm{Iso}^G(S(U), S(R\oplus V)),$$

if $\dim U^G<\dim V^G$. Thus $\mathrm{Iso}^G(S(U), S(R\oplus V))$ is enumerated from our theorems. Theorem A and Theorem B (2) in [1] are corollaries of our Theorem 1.

§ 2. Lemmas.

In this section we provide four lemmas. Lemma 3 and Lemma 4 are fundamental facts in differential topology.

LEMMA 3. *Let M, N be smooth manifolds with boundary, and M compact. Let $f: M\to N$ be a continuous map such that (1) $f^{-1}(\partial N)=\partial M$, and (2) f is a* *smooth embedding on some neighborhood of* ∂M *in* M. *If*

$$2\dim M+1\leqq\dim N,$$

then there is a smooth embedding $g: M\to N$ *such that* (1) $g=f$ *on some neighborhood of* ∂M *in* M, *and* (2) $g\simeq f$ *relative to the neighborhood of* ∂M *in* M.

LEMMA 4. *Let* M, N *be smooth manifolds with boundary, and* M *compact. Let* f, $g: M\to N$ *be smooth embeddings, and* $H: M\times R\to N\times R$ *a level preserving continuous map such that*

(1) $H_t=f$ *for* $t\leqq -1$,
$H_t=g$ *for* $t\geqq 1$,
(2) $H^{-1}(\partial N\times R)=\partial M\times R$,
(3) H *is a smooth embedding on some neighborhood of* $\partial M\times R$ *in* $M\times R$.

If

$$2\dim M+2\leqq\dim N,$$

then, for any $\lambda>1$, *there is a smooth isotopy* $K: M\times R\to N\times R$ *such that*

(1) $K_t=f$ *for* $t\leqq -\lambda$,
$K_t=g$ *for* $t\geqq\lambda$,
(2) $K=H$ *on some neighborhood of* $\partial M\times R$ *in* $M\times R$.

LEMMA 5. *Let* G *be a finite group. Let* M *be a compact smooth free* G-*manifold with boundary, and* N *a* $(\dim M-1)$-*connected smooth* G-*manifold with boundary. If* f *is a smooth* G-*map from a* G-*invariant neighborhood of* ∂M *in* M *to* N, *then there is a smooth* G-*map* $g: M\to N$ *such that* $g=f$ *on some neighborhood of* ∂M *in* M.

PROOF. Consider the smooth fibre bundle $M\times_G N\to M/G$ with fibre N. The smooth cross sections $s: M/G\to M\times_G N$ and the smooth G-maps $h: M\to N$ are in bijective correspondence by the relation

$$s([x])=[x, h(x)] \quad \text{for any} \quad x\in M.$$

Let A be a G-invariant open neighborhood of ∂M in M where f is defined. Let $s^{(1)}: A/G\to A\times_G N$ be the smooth cross section corresponding to f. By the assumption that the fibre N is $(\dim M-1)$-connected and by the differentiable approximation theorem [4; 6.7], we obtain a smooth cross section $s^{(2)}: M/G\to M\times_G N$ such that $s^{(2)}=s^{(1)}$ on a neighborhood of $\partial M/G$ in M/G. Let $g: M\to N$ be the smooth G-map corresponding to $s^{(2)}$. Then $g=f$ on a neighborhood of ∂M in M. Q.E.D.

LEMMA 6. *Let* H *be a subgroup of a finite group* G. *Let* M, N *be smooth* G-*manifolds with boundary, and* M *compact. Let* $f: A\to N$ *be a smooth* G-*embedding with* $f(\partial M)\subset\partial N$, *where* A *is a* G-*invariant open neighborhood of* ∂M *in* M. *Suppose that* $\tilde{d}f: \nu(M^H)|A^H\to\nu(N^H)$ *extends to a smooth* G-*vector bundle embedding*

$\zeta: \nu(M^H)\to\nu(N^H)$. *Then there are a G-invariant tubular neighborhood T of M^H in M and a smooth G-embedding $g: T\to N$ such that*

(1) *$g=f$ on $T\cap B$, where B is some neighborhood of ∂M in M, and*
(2) $\tilde{d}g=\zeta: \nu(M^H) \longrightarrow \nu(N^H)$.

PROOF. Give a G-invariant smooth Riemannian metric $\langle\,,\rangle_1$ on the tangent bundle $\tau(M)$ of M, and let $\nu^1(M^H)$ be the orthogonal complement of $\tau(M^H)$ in $\tau(M)|M^H$ with respect to the metric $\langle\,,\rangle_1$. Let

$$p_M: \tau(M)|M^H \longrightarrow (\tau(M)|M^H)/\tau(M^H)=\nu(M^H)$$

be the projection. Then

$$p'_M=p_M|\nu^1(M^H): \nu^1(M^H) \longrightarrow \nu(M^H)$$

is a smooth G-vector bundle isomorphism. We can give $\tau(N)$ a G-invariant smooth Riemannian metric $\langle\,,\rangle_2$ such that

$$df(\nu^1(M^H)|\tilde{A}^H)\subset\nu^2(N^H),$$

where $\tilde{A}$ is a neighborhood, contained in A, of ∂M in M, and where $\nu^2(N^H)$ is the orthogonal complement of $\tau(N^H)$ in $\tau(N)|N^H$ with respect to the metric $\langle\,,\rangle_2$. Let

$$p'_N=p_N|\nu^2(N^H): \nu^2(N^H) \longrightarrow \nu(N^H)$$

be the smooth G-vector bundle isomorphism which is the restriction of the projection

$$p_N: \tau(N)|N^H \longrightarrow (\tau(N)|N^H)/\tau(N^H)=\nu(N^H).$$

For small ε, $\delta>0$, define

$$\nu^1_\varepsilon(M^H)=\{v\in\nu^1(M^H)|\langle v, v\rangle_1\leqq\varepsilon^2\},$$

$$\nu^2_\delta(N^H)=\{v\in\nu^2(N^H)|\langle v, v\rangle_2\leqq\delta^2\}.$$

There are the exponential maps

$$\exp: \nu^1_\varepsilon(M^H) \longrightarrow M,$$

$$\exp: \nu^2_\delta(N^H) \longrightarrow N.$$

These exponential maps are smooth G-embeddings onto tubular neighborhoods of M^H, N^H in M, N, respectively. The sequence of smooth G-embeddings

$$M \xleftarrow{\exp} \nu^1_\varepsilon(M^H) \xrightarrow{p'_M} \nu(M^H) \xrightarrow{\zeta} \nu(N^H) \xleftarrow{p'_N} \nu^2_\delta(N^H) \xrightarrow{\exp} N$$

gives a desired smooth G-embedding $g: T\to N$. Q. E. D.

§ 3. Surjectivity of $\tilde{\Phi}$.

[I] Proof of the surjectivity of $\tilde{\Phi}$ for Theorem 1. Let U, V be such representations of G as in Theorem 1. Let

$$[g]\in[S(U^G),\ \mathrm{Mon}^G(\oplus_{j\in J(G;U)}m_jW_j,\ \oplus_{j\in J(G;V)}n_jW_j)]$$

be an arbitrary class, whose representative g can be taken to be a smooth map. To prove the surjectivity of $\tilde{\Phi}$ it suffices to show that there exists a smooth G-embedding $f: S(U)\to V$ with $\Phi(f)=g$.

Since U^G is a subspace of V^G, there is a smooth embedding $\iota: S(U^G)\to V^G$. Define

$$h: S(U^G)\times\oplus_{j\in J(G;U)}m_jW_j \longrightarrow V^G\times\oplus_{j\in J(G;V)}n_jW_j$$

by, for $x\in S(U^G)$ and $w\in\oplus_{j\in J(G;U)}m_jW_j$,

$$h(x,\ w)=(\iota(x),\ g(x)(w)).$$

Then h is a smooth embedding. By Lemma 6 the smooth G-vector bundle embedding

$$(\Psi^V)^{-1}\circ h\circ\Psi^U: \nu(S(U^G)) \longrightarrow \nu(V^G)$$

induces a smooth G-embedding

$$f^{(1)}: T_\varepsilon(S(U^G)) \longrightarrow V$$

with $\tilde{\partial} f^{(1)}=(\Psi^V)^{-1}\circ h\circ\Psi^U$, where $T_\varepsilon(S(U^G))$ is a G-invariant closed tubular neighborhood of $S(U^G)$ in $S(U)$ with radius $\varepsilon>0$. Let $T_\delta(V^G)$ be a G-invariant closed tubular neighborhood of V^G in V with radius $\delta>0$. Take δ such as

$$\mathrm{Int}\, T_\varepsilon(S(U^G))\supset(f^{(1)})^{-1}(T_\delta(V^G)).$$

The boundary of $T_\delta(V^G)$ and the image of $f^{(1)}$ intersect transversally, as we see in the proof of Lemma 6. Therefore

$$M=S(U)-\mathrm{Int}\,(f^{(1)})^{-1}(T_\delta(V^G))$$

is a smooth free G-manifold with boundary. Also

$$N=V-\mathrm{Int}T_\delta(V^G)$$

is a smooth free G-manifold with boundary. N has the same homotopy type as the $(\dim V-\dim V^G-1)$-dimensional sphere. By the assumption

$$\dim U+\dim V^G\leqq\dim V,$$

we see that N is $(\dim M-1)$-connected. By Lemma 5 we obtain a G-map $f^{(2)}: M\to N$ which coincides with $f^{(1)}$ on some neighborhood of ∂M in M. Making use of collars of ∂M and ∂N, we may take $f^{(2)}$ such as $(f^{(2)})^{-1}(\partial N)=\partial M$. Passing to the orbit spaces, we obtain the map

$$f^{(2)}/G : M/G \longrightarrow N/G$$

which coincides with the smooth embedding $f^{(1)}/G$ on a neighborhood of $\partial M/G$ in M/G. The assumption $2\dim U-1\leqq \dim V$ implies

$$2\dim M/G+1\leqq \dim N/G .$$

Thus, by Lemma 3, we obtain a smooth embedding

$$f^{(3)} : M/G \longrightarrow N/G$$

such that

(1) $f^{(3)}=f^{(1)}/G$ on a neighborhood of $\partial M/G$ in M/G, and
(2) $f^{(3)}\simeq f^{(2)}/G$ relative to the neighborhood.

By the covering homotopy property for $M\to M/G$ and $N\to N/G$, $f^{(3)}$ induces a smooth G-embedding $f^{(4)} : M\to N$ which coincides with $f^{(1)}$ on a neighborhood of ∂M in M. Pasting $f^{(1)}$ and $f^{(4)}$, we obtain a smooth G-embedding $f : S(U)\to V$ with

$$\tilde{d}f=\tilde{d}f^{(1)}=(\Psi^V)^{-1}\circ h\circ \Psi^U .$$

This implies $\Phi(f)=g$, and completes the proof of surjectivity of $\tilde{\Phi}$ for Theorem 1.

[II] Proof of the surjectivity of $\tilde{\Phi}$ for Theorem 2. Let G, U, V be as in Theorem 2. Let

$$[g]\in[S(U^G),\ \mathrm{Mon}^G(\oplus_{j\in J(G;U)} m_jW_j,\ \oplus_{j\in J(G;V)} n_jW_j)]$$

be an arbitrary class, whose representative g can be taken to be a smooth map. It suffices to show that there exists a smooth G-embedding $f : S(U)\to V$ with $\Phi(f)=g$.

Let $\iota : S(U^G)\to V^G$ be a smooth embedding, and define a smooth G-embedding

$$h : S(U^G)\times \oplus_{j\in J(G;U)} m_jW_j \longrightarrow V^G\times \oplus_{j\in J(G;V)} n_jW_j$$

by, for $x\in S(U^G)$ and $w\in \oplus_{j\in J(G;U)} m_jW_j$,

$$h(x,\ w)=(\iota(x),\ g(x)(w)) .$$

As in the proof of surjectivity for Theorem 1, we obtain a smooth G-embedding $\tilde{f} : T(S(U^G))\to V$ with $\tilde{d}\tilde{f}=(\Psi^V)^{-1}\circ h\circ \Psi^U$, where $T(S(U^G))$ is a G-invariant closed tubular neighborhood of $S(U^G)$ in $S(U)$.

Consider the following assertion $\mathcal{A}(s)$ for any s with $0\leqq s\leqq r$:

$\mathcal{A}(s)$. *There exist a G-invariant compact smooth submanifold M_s of $S(U)$ and a smooth G-embedding $f^{(s)} : M_s\to V$ such that*

(1) $\dim M_s = \dim S(U)$,
(2) $\operatorname{Int} M_s \supset S(U^{H(s)})$, *and*
(3) $f^{(s)} = \tilde{f}$ *on some neighborhood of* $S(U^G)$ *in* $S(U)$.

In the assertion $\mathcal{A}(0)$, M_0 must be $S(U)$, and $f^{(0)}$ is the required embedding. We can prove all $\mathcal{A}(s)$ by induction descending on s. First $\mathcal{A}(r)$ is insured by the smooth G-embedding $\tilde{f}: T(S(U^G)) \to V$. Assuming $\mathcal{A}(s+1)$ for $0 \leqq s < r$, we will prove $\mathcal{A}(s)$.

Take a G-invariant closed tubular neighborhood $T(V^{H(s+1)})$ of $V^{H(s+1)}$ in V such as

$$\operatorname{Int} M_{s+1} \supset (f^{(s+1)})^{-1}(T(V^{H(s+1)}))\,.$$

If the radius of $T(V^{H(s+1)})$ is appropriately small, then

$$L = S(U) - \operatorname{Int}(f^{(s+1)})^{-1}(T(V^{H(s+1)}))$$

is a smooth G-manifold with boundary. Also

$$N = V - \operatorname{Int} T(V^{H(s+1)})$$

is a smooth G-manifold with boundary. $f^{(s+1)}$ is defined on a G-invariant open neighborhood of ∂L in L. Let A be such a neighborhood.

We split U and V as representations of $H(s)$ into

$$U = U^{H(s)} \oplus U_1\,, \quad V = V^{H(s)} \oplus V_1\,,$$

where

$$U_1 = \oplus_{j \in J(H(s);U)} m_j W_j\,,$$

$$V_1 = \oplus_{j \in J(H(s);V)} n_j W_j\,.$$

We may consider U_1 and V_1 as G-invariant subspaces of U and V, respectively. Let $\nu(L^{H(s)})$, $\nu(N^{H(s)})$ be the normal bundles of $L^{H(s)}$, $N^{H(s)}$ in L, N, respectively. Then there are smooth G-vector bundle isomorphisms

$$\alpha: \nu(L^{H(s)}) \cong L^{H(s)} \times U_1\,,$$

$$\beta: \nu(N^{H(s)}) \cong N^{H(s)} \times V_1\,.$$

Since $A^{H(s)} = A \cap L^{H(s)}$,

$$\tilde{\partial} f^{(s+1)}: \nu(L^{H(s)}) \,|\, A^{H(s)} \longrightarrow \nu(N^{H(s)})$$

is defined. $\operatorname{Mon}^{H(s)}(U_1, V_1)$ admits the smooth G-action such that $H(s)$ acts trivially on it. Define a smooth G-map

$$k^{(1)}: A^{H(s)} \longrightarrow \operatorname{Mon}^{H(s)}(U_1, V_1)$$

by, for $x \in A^{H(s)}$ and $u \in U_1$,

$$\beta\circ\partial f^{(s+1)}\circ\alpha^{-1}(x, u)=(f^{(s+1)}(x), k^{(1)}(x)(u)).$$

As in §1 we see

$$\mathrm{Mon}^{H(s)}(U_1, V_1)\approx\prod_{j\in J(H(s);U)}V(m_j, n_j; F_j).$$

The assumption

$$\dim U^{H(s)}\leqq d_j(n_j-m_j+1)$$

implies that $\mathrm{Mon}^{H(s)}(U_1, V_1)$ is $(\dim L^{H(s)}-1)$-connected. We may consider $L^{H(s)}$ and $\mathrm{Mon}^{H(s)}(U_1, V_1)$ as $G/H(s)$-manifolds, and $k^{(1)}$ as a $G/H(s)$-map. Then the $G/H(s)$-action on $L^{H(s)}$ is free. Thus, by Lemma 5, we obtain a smooth $G/H(s)$-map

$$k^{(2)}: L^{H(s)}\longrightarrow \mathrm{Mon}^{H(s)}(U_1, V_1)$$

which coincides with $k^{(1)}$ on a neighborhood of $\partial L^{H(s)}$ in $L^{H(s)}$. We reconsider $k^{(2)}$ as a smooth G-map.

Since $N^{H(s)}=V^{H(s)}-\mathrm{Int}\,T(V^{H(s+1)})^{H(s)}$, and $T(V^{H(s+1)})^{H(s)}$ is a tubular neighborhood of $V^{H(s+1)}$ in $V^{H(s)}$, then $N^{H(s)}$ has the same homotopy type as the $(\dim V^{H(s)}-\dim V^{H(s+1)}-1)$-dimensional sphere. Thus the assumption

$$\dim U^{H(s)}+\dim V^{H(s+1)}\leqq\dim V^{H(s)}$$

implies that $N^{H(s)}$ is $(\dim L^{H(s)}-1)$-connected. So, by Lemma 5, we obtain a G-map

$$k^{(3)}: L^{H(s)}\longrightarrow N^{H(s)}$$

which coincides with $f^{(s+1)}$ on a neighborhood of $\partial L^{H(s)}$. Making use of collars of $\partial L^{H(s)}$ and $\partial N^{H(s)}$, we may take $k^{(3)}$ such as $(k^{(3)})^{-1}(\partial N^{H(s)})=\partial L^{H(s)}$. Passing to the orbit spaces, we obtain the map

$$k^{(3)}/G: L^{H(s)}/G\longrightarrow N^{H(s)}/G$$

which coincides with the smooth embedding $f^{(s+1)}/G$ on a neighborhood of $\partial L^{H(s)}/G$. The assumption

$$2\dim U^{H(s)}-1\leqq\dim V^{H(s)}$$

implies

$$2\dim L^{H(s)}/G+1\leqq\dim N^{H(s)}/G.$$

Thus we can apply Lemma 3 to $k^{(3)}/G$, and obtain a smooth embedding

$$k^{(4)}: L^{H(s)}/G\longrightarrow N^{H(s)}/G$$

such that

(1) $k^{(4)}=f^{(s+1)}/G$ on a neighborhood of $\partial L^{H(s)}/G$, and

(2) $k^{(4)}\simeq k^{(3)}/G$ relative to the neighborhood of $\partial L^{H(s)}/G$.

By the covering homotopy property for $L^{H(s)}\to L^{H(s)}/G$ and $N^{H(s)}\to N^{H(s)}/G$, $k^{(4)}$ induces a smooth G-embedding

$$k^{(5)}: L^{H(s)} \longrightarrow N^{H(s)}$$

which coincides with $f^{(s+1)}$ on a neighborhood of $\partial L^{H(s)}$.

Define

$$k^{(6)}: L^{H(s)} \times U_1 \longrightarrow N^{H(s)} \times V_1$$

by, for $x \in L^{H(s)}$ and $u \in U_1$,

$$k^{(6)}(x, u) = (k^{(5)}(x), k^{(2)}(x)(u)).$$

This is a smooth G-embedding. Consider the smooth G-vector bundle embedding

$$\beta^{-1} \circ k^{(6)} \circ \alpha : \nu(L^{H(s)}) \longrightarrow \nu(N^{H(s)}).$$

This coincides with $\tilde{d}f^{(s+1)}$ on $\nu(L^{H(s)}) | \tilde{A}^{H(s)}$, where $\tilde{A}$ is some G-invariant open neighborhood of ∂L in L which is contained in A. By Lemma 6 there is a G-invariant tubular neighborhood $T(L^{H(s)})$ of $L^{H(s)}$ in L and a smooth G-embedding

$$k^{(7)}: T(L^{H(s)}) \longrightarrow N$$

which coincides with $f^{(s+1)}$ on $T(L^{H(s)}) \cap B$, where B is a neighborhood of ∂L in L.

Take M_s as a G-invariant closed tubular neighborhood of $S(U^{H(s)})$ in $S(U)$ which is contained in

$$(f^{(s+1)})^{-1}(T(V^{H(s+1)})) \cup T(L^{H(s)}).$$

Define $f^{(s)}: M_s \to V$ by

$$f^{(s)} = f^{(s+1)} \quad \text{on} \quad M_s \cap (f^{(s+1)})^{-1}(T(V^{H(s+1)})),$$

$$f^{(s)} = k^{(7)} \quad \text{on} \quad M_s \cap T(L^{H(s)}).$$

This is a smooth G-embedding, and coincides with $\tilde{f}$ on some neighborhood of $S(U^G)$ in $S(U)$. Thus we see that the assertion $\mathcal{A}(s+1)$ implies the assertion $\mathcal{A}(s)$.

§4. Injectivity of $\tilde{\Phi}$.

[I] Proof of the injectivity of $\tilde{\Phi}$ for Theorem 1. Let U, V be such representations of G as in Theorem 1. For

$$[f], [g] \in \mathrm{Iso}^G(S(U), V),$$

assume that

$$\Phi(f), \Phi(g): S(U^G) \longrightarrow \mathrm{Mon}^G(\oplus_{j \in J(G;U)} m_j W_j, \oplus_{j \in J(G;V)} n_j W_j)$$

are homotopic. We will construct a smooth G-isotopy between f and g.

There is a level preserving smooth map

$$H^{(1)}: S(U^G)\times R \longrightarrow \mathrm{Mon}^G(\oplus_{j\in J(G;U)} m_j W_j,\ \oplus_{j\in J(G;V)} n_j W_j)\times R$$

such that

$$H_t^{(1)}=\Phi(f) \quad \text{for} \quad t\leqq -1\,,$$
$$H_t^{(1)}=\Phi(g) \quad \text{for} \quad t\geqq 1\,.$$

It is known that any two smooth embeddings of n-sphere in R^m are smoothly isotopic if $m>3(n+1)/2$. (See Haefliger [2], [3].) Thus our assumption $3\dim U^G <2\dim V^G$ implies that there is a smooth isotopy

$$H^{(2)}: S(U^G)\times R \longrightarrow V^G\times R$$

such that

$$H_t^{(2)}=f\,|\,S(U^G) \quad \text{for} \quad t\leqq -1\,,$$
$$H_t^{(2)}=g\,|\,S(U^G) \quad \text{for} \quad t\geqq 1\,.$$

Define

$$H^{(3)}: (S(U^G)\times \oplus_{j\in J(G;U)} m_j W_j)\times R \longrightarrow (V^G\times \oplus_{j\in J(G;V)} n_j W_j)\times R$$

by, for $x\in S(U^G)$, $w\in \oplus_{j\in J(G;U)} m_j W_j$ and $t\in R$,

$$H^{(3)}(x,\ w,\ t)=(H_t^{(2)}(x),\ H_t^{(1)}(x)(w),\ t)\,.$$

Then $H^{(3)}$ is a smooth G-isotopy such that

$$H_t^{(3)}=\Psi^V\circ\partial f\circ(\Psi^U)^{-1} \quad \text{for} \quad t\leqq -1\,,$$
$$H_t^{(3)}=\Psi^V\circ\partial g\circ(\Psi^U)^{-1} \quad \text{for} \quad t\geqq 1\,.$$

Let $\nu(S(U^G)\times R)$ and $\nu(V^G\times R)$ be the normal bundles of $S(U^G)\times R$ in $S(U)\times R$, and of $V^G\times R$ in $V\times R$, respectively. Let

$$\zeta: \nu(S(U^G)\times R) \longrightarrow \nu(V^G\times R)$$

be the smooth G-vector bundle embedding composed of the bundle embeddings in the diagram:

$$\begin{array}{ccc} \nu(S(U^G)\times R)\cong \nu(S(U^G))\times R & \xrightarrow{\Psi^U\times \mathrm{id}} & (S(U^G)\times \oplus_{j\in J(G;U)} m_j W_j)\times R \\ & & \downarrow H^{(3)} \\ \nu(V^G\times R)\cong \nu(V^G)\times R & \xrightarrow{\Psi^V\times \mathrm{id}} & (V^G\times \oplus_{j\in J(G;V)} n_j W_j)\times R\,. \end{array}$$

Give $\tau(S(U))$ and $\tau(V)$ G-invariant smooth Riemannian metrics $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$, respectively. Let $\nu^1(S(U^G))$ and $\nu^2(V^G)$ be the orthogonal complements of $\tau(S(U^G))$ in $\tau(S(U))\,|\,S(U^G)$, and of $\tau(V^G)$ in $\tau(V)\,|\,V^G$, respectively. For small ε, $\delta>0$, the exponential maps

$$\exp: \nu_\varepsilon^1(S(U^G))=\{v\in \nu^1(S(U^G))\,|\,\langle v,\ v\rangle_1\leqq \varepsilon^2\} \longrightarrow S(U)\,,$$

$$\exp: \nu_\delta^2(V^G)=\{v\in \nu^2(V^G)\,|\,\langle v,\ v\rangle_2\leqq \delta^2\} \longrightarrow V$$

are defined. Let

$$T_\varepsilon(S(U^G))=\exp(\nu_\varepsilon^1(S(U^G)))\,,$$

$$T_\delta(V^G)=\exp(\nu_\delta^2(V^G))\,.$$

Applying the same method as in the proof of Lemma 6 to

$$\zeta: \nu(S(U^G)\times R) \longrightarrow \nu(V^G\times R)\,,$$

we obtain a smooth G-isotopy, for appropriate ε, $\delta>0$,

$$H^{(4)}: T_\varepsilon(S(U^G))\times R \longrightarrow T_\delta(V^G)\times R\subset V\times R$$

such that

$$H_t^{(4)}=f\,|\,T_\varepsilon(S(U^G)) \qquad \text{for} \quad t\leqq -1\,,$$

$$H_t^{(4)}=g\,|\,T_\varepsilon(S(U^G)) \qquad \text{for} \quad t\geqq 1\,.$$

For nonzero $v\in\nu^1(S(U^G))$, let

$$\theta_v=\exp(\{\lambda v\,|\,\lambda\in R,\ \lambda\geqq 0\}\cap\nu_\varepsilon^1(S(U^G)))\,.$$

Choose so small $\varepsilon>0$ that, for any nonzero $v\in\nu^1(S(U^G))$, any γ with $0\leqq\gamma\leqq\delta$, and any $t\in R$,

(1) $H_t^{(4)}(\theta_v)\cap S_\gamma(V^G)=\emptyset$, or
(2) $H_t^{(4)}(\theta_v)$ and $S_\gamma(V^G)$ intersect transversally,

where

$$S_\gamma(V^G)=\exp(\{v\in\nu^2(V^G)\,|\,\langle v, v\rangle_2=\gamma^2\})\,.$$

Also choose γ with $0<\gamma\leqq\delta$ such that

(1) $\mathrm{Int}\, T_{\varepsilon/2}(S(U^G))\supset f^{-1}(T_\gamma(V^G))\cup g^{-1}(T_\gamma(V^G))$, and
(2) $\mathrm{Int}\, T_{\varepsilon/2}(S(U^G))\times R\supset(H^{(4)})^{-1}(T_\gamma(V^G)\times R)$.

For such ε, γ, let

$$\eta: S(U)\times R \longrightarrow S(U)\times R$$

be a level preserving G-diffeomorphism such that

(1) $\eta(T_\varepsilon(S(U^G))\times R)=T_\varepsilon(S(U^G))\times R$, and
(2) $\eta(T_{\varepsilon/2}(S(U^G))\times R)=(H^{(4)})^{-1}(T_\gamma(V^G)\times R)$.

Such η is obtained by regulating lengths of normal vectors in $T_\varepsilon(S(U^G))$.

Let

$$M=S(U)-\mathrm{Int}\, T_{\varepsilon/2}(S(U^G))\,,$$

$$N=V-\mathrm{Int}\, T_\gamma(V^G)\,.$$

These are compact smooth free G-manifolds with boundary. Consider the G-invariant subspace of $M\times[-2, 2]$,

$$A=(T_{\varepsilon}(S(U^G))-\text{Int}\,T_{\varepsilon/2}(S(U^G)))\times[-2, 2]\cup M\times[-2, -1]\cup M\times[1, 2].$$

Define a G-map $k^{(1)}: A\to N$ by

$$k^{(1)}=f\circ\eta_t \quad \text{on} \quad M\times\{t\},\ -2\leqq t\leqq -1,$$

$$k^{(1)}=g\circ\eta_t \quad \text{on} \quad M\times\{t\},\ 1\leqq t\leqq 2,$$

$$k^{(1)}=\pi\circ H^{(4)}\circ\eta \quad \text{on} \quad (T_{\varepsilon}(S(U^G))-\text{Int}\,T_{\varepsilon/2}(S(U^G)))\times[-2, 2],$$

where $\pi: V\times R\to V$ is the projection. The assumption

$$\dim U+\dim V^G<\dim V$$

implies that N is $(\dim M\times[-2, 2]-1)$-connected. Thus, by Lemma 5, there is a G-map

$$k^{(2)}: M\times[-2, 2]\longrightarrow N$$

such that

(1) $k^{(2)}=k^{(1)}$ on some neighborhood of $\partial(M\times[-2, 2])$, and
(2) $(k^{(2)})^{-1}(\partial N)=(\partial M)\times[-2, 2]$.

Define a level preserving G-map

$$H^{(5)}: M\times R\longrightarrow N\times R$$

by, for $x\in M$ and $t\in R$,

$$H_t^{(5)}(x)=f\circ\eta_t(x) \quad \text{if} \quad t\leqq -2,$$

$$H_t^{(5)}(x)=k^{(2)}(x, t) \quad \text{if} \quad -2\leqq t\leqq 2,$$

$$H_t^{(5)}(x)=g\circ\eta_t(x) \quad \text{if} \quad t\geqq 2.$$

This is well-defined. The level preserving map

$$H^{(5)}/G: M/G\times R\longrightarrow N/G\times R$$

coincides with $H^{(4)}\circ\eta/G$ on some neighborhood of $\partial M/G\times R$ in $M/G\times R$, and we see

$$(H^{(5)}/G)^{-1}(\partial N/G\times R)=\partial M/G\times R.$$

The assumption $2\dim U-1<\dim V$ implies

$$2\dim M/G+2\leqq\dim N/G.$$

Thus we can apply Lemma 4 to $H^{(5)}/G$, and obtain a smooth isotopy

$$H^{(6)}: M/G\times R\longrightarrow N/G\times R.$$

By the covering homotopy property and the unique lifting property for $M\to M/G$ and $N\to N/G$, $H^{(6)}$ induces a smooth G-isotopy

$$H^{(7)}: M\times R \longrightarrow N\times R$$

such that

$$H_t^{(7)}=f\circ\eta_t|M \quad \text{for } t\leqq -3,$$

$$H_t^{(7)}=g\circ\eta_t|M \quad \text{for } t\geqq 3,$$

$$H^{(7)}=H^{(4)}\circ\eta \text{ on a neighborhood of } \partial M\times R \text{ in } M\times R.$$

Define $H^{(8)}: S(U)\times R\to V\times R$ by

$$H^{(8)}=H^{(4)}\circ\eta \quad \text{on} \quad T_{\varepsilon/2}(S(U^G))\times R,$$

$$H^{(8)}=H^{(7)} \quad \text{on} \quad M\times R.$$

Then $H^{(9)}=H^{(8)}\circ\eta^{-1}$ is a smooth G-isotopy such that

$$H_t^{(9)}=f \quad \text{for } t\leqq -3,$$

$$H_t^{(9)}=g \quad \text{for } t\geqq 3.$$

Thus this is a smooth G-isotopy between f and g, and completes the proof of injectivity of $\tilde{\Phi}$ for Theorem 1.

[II] Proof of the injectivity of $\tilde{\Phi}$ for Theorem 2. Let G, U, V be as in Theorem 2. For

$$[f],\ [g]\in \mathrm{Iso}^G(S(U),\ V),$$

assume that

$$\Phi(f),\ \Phi(g): S(U^G) \longrightarrow \mathrm{Mon}^G(\oplus_{j\in J(G;U)}m_jW_j,\ \oplus_{j\in J(G;V)}n_jW_j)$$

are homotopic. Consider the following assertion $\mathcal{A}(s)$ for any s with $0\leqq s\leqq r$:

$\mathcal{A}(s)$. *There exist a G-invariant compact smooth submanifold M_s of $S(U)$ and a smooth G-isotopy*

$$K^{(s)}: M_s\times R \longrightarrow V\times R$$

such that

(1) $\dim M_s=\dim S(U)$,
(2) $\mathrm{Int}\, M_s\supset S(U^{H(s)})$,
(3) $K_t^{(s)}=f|M_s$ for $t\leqq -(r-s+1)$, and
(4) $K_t^{(s)}=g|M_s$ for $t\geqq r-s+1$.

In the assertion $\mathcal{A}(0)$, M_0 must be $S(U)$, and $K^{(0)}$ is a smooth G-isotopy between f and g. Thus $\mathcal{A}(0)$ implies the injectivity of $\tilde{\Phi}$. We can prove all $\mathcal{A}(s)$ by induction descending on s. Taking M_r as a G-invariant closed tubular neighborhood of $S(U^G)$ in $S(U)$, $\mathcal{A}(r)$ is proved as in the proof of injectivity of $\tilde{\Phi}$ for Theorem 1. Assuming $\mathcal{A}(s+1)$ for $0\leqq s<r$, we will prove $\mathcal{A}(s)$.

Let $T(S(U^{H(s+1)}))$ be a G-invariant closed tubular neighborhood of $S(U^{H(s+1)})$ in $S(U)$ which is contained in $\mathrm{Int}\, M_{s+1}$. Let $T(V^{H(s+1)})$ be a G-invariant closed tubular neighborhood of $V^{H(s+1)}$ in V such as

$$\mathrm{Int}\, T(S(U^{H(s+1)})) \supset f^{-1}(T(V^{H(s+1)})) \cup g^{-1}(T(V^{H(s+1)})),$$

$$\mathrm{Int}\, T(S(U^{H(s+1)})) \times R \supset (K^{(s+1)})^{-1}(T(V^{H(s+1)}) \times R).$$

If the radius of $T(S(U^{H(s+1)}))$ is appropriately small, as in the proof for Theorem 1, we obtain a level preserving G-diffeomorphism

$$\eta : S(U) \times R \longrightarrow S(U) \times R$$

such that

(1) $\eta(M_{s+1} \times R) = M_{s+1} \times R$,
(2) $\eta(T(S(U^{H(s+1)})) \times R) = (K^{(s+1)})^{-1}(T(V^{H(s+1)}) \times R)$,
(3) $\eta_t = \eta_{-(r-s)}$ for $t \leqq -(r-s)$, and
(4) $\eta_t = \eta_{r-s}$ for $t \geqq r-s$.

Let

$$L = S(U) - \mathrm{Int}\, T(S(U^{H(s+1)})),$$

$$N = V - \mathrm{Int}\, T(V^{H(s+1)}).$$

Define G-invariant subspaces A, B of $S(U) \times R$ by

$$A = L \times \left[-\left(r-s+\frac{1}{3}\right), r-s+\frac{1}{3}\right],$$

$$B = (\mathrm{Int}\, M_{s+1} - \mathrm{Int}\, T(S(U^{H(s+1)}))) \times (-(r-s+1), r-s+1)$$
$$\cup L \times (-(r-s+1), -(r-s)) \cup L \times (r-s, r-s+1).$$

Define a smooth G-map $E^{(1)} : B \to N \times R$ by

$$E^{(1)} = K^{(s+1)} \circ \eta \quad \text{on} \quad (\mathrm{Int}\, M_{s+1} - \mathrm{Int}\, T(S(U^{H(s+1)})) \times (-(r-s+1), r-s+1),$$

$$E^{(1)} = (f \times \mathrm{id}) \circ \eta \quad \text{on} \quad L \times (-(r-s+1), -(r-s)),$$

$$E^{(1)} = (g \times \mathrm{id}) \circ \eta \quad \text{on} \quad L \times (r-s, r-s+1).$$

This is well-defined.

We split U and V as representations of $H(s)$ into

$$U = U^{H(s)} \oplus U_1, \quad V = V^{H(s)} \oplus V_1,$$

where

$$U_1 = \oplus_{j \in J(H(s); U)} m_j W_j,$$

$$V_1 = \oplus_{j \in J(H(s); V)} n_j W_j.$$

We may consider U_1 and V_1 as G-invariant subspaces of U and V, respectively.

Let $\nu(L^{H(s)}\times R)$ and $\nu(N^{H(s)}\times R)$ be the normal bundles of $L^{H(s)}\times R$ in $L\times R$, and of $N^{H(s)}\times R$ in $N\times R$, respectively.

$$\tilde{d}E^{(1)}: \nu(L^{H(s)}\times R)|B^{H(s)} \longrightarrow \nu(N^{H(s)}\times R)$$

is defined. There are smooth G-vector bundle isomorphisms

$$\alpha: \nu(L^{H(s)}\times R)\cong(L^{H(s)}\times R)\times U_1,$$
$$\beta: \nu(N^{H(s)}\times R)\cong(N^{H(s)}\times R)\times V_1.$$

$\mathrm{Mon}^{H(s)}(U_1, V_1)$ admits the smooth G-action such that $H(s)$ acts trivially. Define a smooth G-map

$$h^{(1)}: B^{H(s)} \longrightarrow \mathrm{Mon}^{H(s)}(U_1, V_1)$$

by, for $x\in B^{H(s)}$ and $u\in U_1$,

$$\beta\circ\tilde{d}E^{(1)}\circ\alpha^{-1}(x, u)=(E^{(1)}(x), h^{(1)}(x)(u)).$$

Since

$$\mathrm{Mon}^{H(s)}(U_1, V_1)\approx\prod_{j\in J(H(s);U)}V(m_j, n_j; F_j),$$

the assumption

$$\dim U^{H(s)}<d_j(n_j-m_j+1)$$

implies that $\mathrm{Mon}^{H(s)}(U_1, V_1)$ is $(\dim A^{H(s)}-1)$-connected. Thus, by Lemma 5, we obtain a smooth G-map

$$h^{(2)}: A^{H(s)} \longrightarrow \mathrm{Mon}^{H(s)}(U_1, V_1)$$

which coincides with $h^{(1)}$ on a neighborhood of $\partial A^{H(s)}$. Consider the smooth G-embeddings

$$(f\times\mathrm{id})\circ\eta,\ (g\times\mathrm{id})\circ\eta: L\times R \longrightarrow N\times R,$$

and the smooth G-vector bundle embeddings

$$\tilde{d}((f\times\mathrm{id})\circ\eta),\ \tilde{d}((g\times\mathrm{id})\circ\eta): \nu(L^{H(s)}\times R) \longrightarrow \nu(N^{H(s)}\times R).$$

Smooth G-maps

$$h^{(3)},\ h^{(4)}: L^{H(s)}\times R \longrightarrow \mathrm{Mon}^{H(s)}(U_1, V_1)$$

are defined by the relations, for $x\in L^{H(s)}\times R$ and $u\in U_1$,

$$\beta\circ\tilde{d}((f\times\mathrm{id})\circ\eta)\circ\alpha^{-1}(x, u)=((f\times\mathrm{id})\circ\eta(x), h^{(3)}(x)(u)),$$
$$\beta\circ\tilde{d}((g\times\mathrm{id})\circ\eta)\circ\alpha^{-1}(x, u)=((g\times\mathrm{id})\circ\eta(x), h^{(4)}(x)(u)).$$

Then define

$$h^{(5)}: L^{H(s)}\times R \longrightarrow \mathrm{Mon}^{H(s)}(U_1, V_1)$$

by

$$h^{(5)}=h^{(3)} \quad \text{on} \quad L^{H(s)}\times\left(-\infty, -\left(r-s+\frac{1}{3}\right)\right],$$

$$h^{(5)}=h^{(2)} \quad \text{on} \quad A^{H(s)}=L^{H(s)}\times\left[-\left(r-s+\frac{1}{3}\right),\ r-s+\frac{1}{3}\right],$$

$$h^{(5)}=h^{(4)} \quad \text{on} \quad L^{H(s)}\times\left[r-s+\frac{1}{3},\ \infty\right).$$

This is a well-defined smooth G-map.

Consider the G-map $\pi\circ E^{(1)}: B^{H(s)}\to N^{H(s)}$, where $\pi: N\times R\to N$ is the projection. The assumption

$$\dim U^{H(s)}+\dim V^{H(s+1)}<\dim V^{H(s)}$$

implies that $N^{H(s)}$ is ($\dim A^{H(s)}-1$)-connected. Thus, by Lemma 5, we obtain a G-map

$$h^{(6)}: A^{H(s)} \longrightarrow N^{H(s)}$$

such that

(1) $h^{(6)}=\pi\circ E^{(1)}$ on some neighborhood of $\partial A^{H(s)}$, and

(2) $(h^{(6)})^{-1}(\partial N^{H(s)})=(\partial L^{H(s)})\times\left[-\left(r-s+\frac{1}{3}\right),\ r-s+\frac{1}{3}\right]$.

Define a level preserving G-map

$$E^{(2)}: L^{H(s)}\times R \longrightarrow N^{H(s)}\times R$$

by, for $x\in L^{H(s)}$ and $t\in R$,

$$E_t^{(2)}(x)=f\circ\eta_t(x) \quad \text{if} \quad t\leqq-\left(r-s+\frac{1}{3}\right),$$

$$E_t^{(2)}(x)=h^{(6)}(x, t) \quad \text{if} \quad -\left(r-s+\frac{1}{3}\right)\leqq t\leqq r-s+\frac{1}{3},$$

$$E_t^{(2)}(x)=g\circ\eta_t(x) \quad \text{if} \quad t\geqq r-s+\frac{1}{3}.$$

The level preserving map

$$E^{(2)}/G: L^{H(s)}/G\times R \longrightarrow N^{H(s)}/G\times R$$

coincides with $(K^{(s+1)}\circ\eta)/G$ on some neighborhood of $\partial L^{H(s)}/G\times R$ in $L^{H(s)}/G \times R$, and we see

$$(E^{(2)}/G)^{-1}(\partial N^{H(s)}/G\times R)=\partial L^{H(s)}/G\times R.$$

The assumption $2\dim U^{H(s)}-1<\dim V^{H(s)}$ implies

$$2\dim L^{H(s)}/G+2\leqq\dim N^{H(s)}/G.$$

Thus we can apply Lemma 4 to $E^{(2)}/G$, and obtain a smooth isotopy

$$E^{(3)}: L^{H(s)}/G\times R \longrightarrow N^{H(s)}/G\times R.$$

By the covering homotopy property and the unique lifting property for $L^{H(s)} \to L^{H(s)}/G$ and $N^{H(s)} \to N^{H(s)}/G$, $E^{(3)}$ induces a smooth G-isotopy

$$E^{(4)}: L^{H(s)} \times R \longrightarrow N^{H(s)} \times R$$

such that

(1) $E_t^{(4)} = f \circ \eta_t | L^{H(s)}$ for $t \leqq -\left(r-s+\frac{1}{2}\right)$,

(2) $E_t^{(4)} = g \circ \eta_t | L^{H(s)}$ for $t \geqq r-s+\frac{1}{2}$, and

(3) $E^{(4)} = K^{(s+1)} \circ \eta$ on some neighborhood of $\partial L^{H(s)} \times R$ in $L^{H(s)} \times R$.

Define

$$E^{(5)}: (L^{H(s)} \times R) \times U_1 \longrightarrow (N^{H(s)} \times R) \times V_1$$

by, for $x \in L^{H(s)}$, $t \in R$, and $u \in U_1$,

$$E^{(5)}(x, t, u) = (E^{(4)}(x, t), h^{(5)}(x, t)(u)).$$

Consider the smooth G-vector bundle embedding

$$\zeta = \beta^{-1} \circ E^{(5)} \circ \alpha : \nu(L^{H(s)} \times R) \longrightarrow \nu(N^{H(s)} \times R),$$

and see that

(1) $\zeta = \tilde{d}((f \times \mathrm{id}) \circ \eta)$ on $\nu(L^{H(s)} \times R) | L^{H(s)} \times \left(-\infty, -\left(r-s+\frac{1}{2}\right)\right]$,

(2) $\zeta = \tilde{d}((g \times \mathrm{id}) \circ \eta)$ on $\nu(L^{H(s)} \times R) | L^{H(s)} \times \left[r-s+\frac{1}{2}, \infty\right)$, and

(3) $\zeta = \tilde{d}(K^{(s+1)} \circ \eta)$ on $\nu(L^{H(s)} \times R) | (\text{nbd of } \partial L^{H(s)} \text{ in } L^{H(s)}) \times R$.

Applying to ζ the same method as in the proof of Lemma 6, we obtain a G-invariant tubular neighborhood $T(L^{H(s)})$ of $L^{H(s)}$ in L, and obtain a smooth G-isotopy

$$E^{(6)}: T(L^{H(s)}) \times R \longrightarrow N \times R$$

such that

(1) $E_t^{(6)} = f \circ \eta_t | T(L^{H(s)})$ for $t \leqq -(r-s+1)$,

(2) $E_t^{(6)} = g \circ \eta_t | T(L^{H(s)})$ for $t \geqq r-s+1$, and

(3) $E^{(6)} = K^{(s+1)} \circ \eta$ on $T(L^{H(s)}) \cap C$, where C is some neighborhood of ∂L in L.

We can take M_s as a G-invariant closed tubular neighborhood of $S(U^{H(s)})$ in $S(U)$ such that

$$M_s \times R \subset (K^{(s+1)})^{-1}(T(V^{H(s+1)}) \times R) \cup \eta(T(L^{H(s)}) \times R).$$

Define $K^{(s)}: M_s \times R \longrightarrow V \times R$ by

$$K^{(s)} = K^{(s+1)} \quad \text{on} \quad (M_s \times R) \cap (K^{(s+1)})^{-1}(T(V^{H(s+1)}) \times R),$$

$$K^{(s)} = E^{(6)} \circ \eta^{-1} \quad \text{on} \quad (M_s \times R) \cap \eta(T(L^{H(s)}) \times R).$$

This is a well-defined smooth G-isotopy such that

$$K_t^{(s)}=f|M_s \quad \text{for} \quad t\leqq -(r-s+1),$$

$$K_t^{(s)}=g|M_s \quad \text{for} \quad t\geqq r-s+1.$$

Thus the assertion $\mathcal{A}(s)$ is proved.

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