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Linear evolution equations $du/dt+A(t)u=0$: a case where $A(t)$ is strongly uniform-measurable

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§ 1. Introduction.

Kato [**1, 2**] studied the Cauchy problem for a linear evolution equation of hyperbolic type in a Banach space X:

$$(d/dt)u(t)+A(t)u(t)=0, \quad u(s)=y\in Y, \quad 0\leqq s\leqq t\leqq T<\infty,$$

where Y is a Banach space dense in X and $-A(t)$ is the generator of a (C_0) semigroup of bounded linear operators on X for each t. He proved a basic existence theorem (Theorem 4.1 of [**1**]) of the solution for the Cauchy problem when the family $A=\{A(t)\}$ is stable (see P. 244 of [**1**]) and $A(\cdot)$ is (Y, X) norm-continuous, i.e., $A(t)$ belongs to $\boldsymbol{B}(Y, X)$ and it is continuous in the norm of $\boldsymbol{B}(Y, X)$. Here $\boldsymbol{B}(Y, X)$ denotes the set of all bounded linear operators on Y to X. Though he used Cauchy's method in the proof, the author gave another proof by means of the Yosida approximation in [**6**]. Kato also proposed to solve the Cauchy problem when $A(\cdot)$ is (Y, X) strongly continuous.

In this paper we prove an existence theorem (Theorem 2.1) directly by the Yosida approximation method for a case where $A(\cdot)$ is (Y, X) strongly uniform-measurable. Since our method involves no process of step function approximations of time-dependent operators, it is distinguished from Cauchy's method as well as from the usual Yosida approximation method for evolution equations (see [**7, 8**]). Some additional assumption ((A4) (c) in § 2) is needed for the proof but we hope it is not so restrictive. We remark that Kobayasi [**9**] obtained a similar result by Cauchy's method with no additional assumptions when $A(\cdot)$ is (Y, X) strongly continuous but it seems difficult to extend his result to a case where $A(\cdot)$ is (Y, X) strongly measurable.

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§ 2. Theorem.

In this section we state our theorem with some preliminary considerations. Our assumptions are the following.

Let $0 \leqq t \leqq T < \infty$. Further assume (A1) to (A4).

(A1) Y is a Banach space densely and continuously embedded in a real Banach space X.

(A2) $-A(t)$ is the generator of a (C_0) semigroup on X for a. e. t. A is quasi-stable with index $\{M, \beta(\cdot)\}$:

$$\|(I+\lambda_k A(t_k))^{-1} \cdots (I+\lambda_1 A(t_1))^{-1}\|_X \leqq M \prod_{j=1}^{k} (1-\lambda_j \beta(t_j))^{-1},$$

for $0 \leqq t_1 \leqq \cdots \leqq t_k \leqq T$, $1 > \lambda_j \beta(t_j)$, $1 \leqq j \leqq k$, $k \in \boldsymbol{N}$, where M is a constant, β is a real-valued upper-integrable function (in the Lebesgue sense) on $[0, T]$ and $\|\cdot\|_X$ denotes the norm of $\boldsymbol{B}(X) = \boldsymbol{B}(X, X)$.

(A3) $Y \subset D(A(t))$ a. e., so that $A(t) \in \boldsymbol{B}(Y, X)$ a. e. $\|A(\cdot)\|_{Y,X}$ is upper-integrable on $[0, T]$ and $A(\cdot)$ is (Y, X) strongly uniform-measurable on $[0, T]$, i. e., there is a sequence of finite partitions $\{I_{nk} : k=1, \cdots, k(n)\}$, $n=1, 2, \cdots$, of $[0, T]$ into subintervals and Riemann step functions A_n, such that A_n takes a constant value $A(t_{nk})$ on I_{nk} for some $t_{nk} \in I_{nk}$, $\sup_k |I_{nk}| \to 0$, and $A_n(t) \to A(t)$ strongly for a. e. t.

(A4) There is a family $\{S(t)\}$ of isomorphisms from Y onto X such that:

(a) $S(t)A(t)S(t)^{-1} = A(t) + B(t)$, $B(t) \in \boldsymbol{B}(X)$ for a. e. t, where $B(\cdot)$ is (X) strongly measurable with $\|B(\cdot)\|_X$ upper-integrable on $[0, T]$.

(b) There is a strongly measurable function $\dot{S} : [0, T] \to \boldsymbol{B}(Y, X)$ a. e., with $\|\dot{S}(\cdot)\|_{Y,X}$ upper-integrable on $[0, T]$, such that S is equal to an indefinite strong integral of $\dot{S}$, where $\|\cdot\|_{Y,X}$ denotes the norm of $\boldsymbol{B}(Y, X)$.

(c) $(I+\alpha S(t))^{-1}$ is uniformly bounded in $\boldsymbol{B}(X)$ for $0 \leqq t \leqq T$, $0 < \alpha \leqq \alpha_0$, where α_0 is some positive constant.

Remark 2.1. If $D(A(t))$ is equal to Y for all t with the graph norm and $A(\cdot)$ is (Y, X) strongly continuously differentiable, then A(3) and A(4) are satisfied by taking $S(t) = I + \alpha A(t)$ for some $\alpha > 0$.

Next we state the definition of evolution operators.

Definition 2.1. A family $U = \{U(t, s) : 0 \leqq s \leqq t \leqq T\}$ of bounded linear operators in $\boldsymbol{B}(X)$ is called an evolution operator for A if the following conditions are satisfied.

(a) $U(\cdot, \cdot)$ is (X) strongly continuous.

(b) $U(t, s)U(s, r) = U(t, r)$, $U(s, s) = I$, $0 \leqq r \leqq s \leqq t \leqq T$.

(c) $U(t, s)Y \subset Y$ and $U(\cdot, \cdot)$ is (Y) strongly continuous.

(d) For each $y \in Y$, $U(\cdot, \cdot)y$ satisfies the following:

$$U(t, s)y - y = -\int_s^t A(\sigma)U(\sigma, s)y \, d\sigma,$$

$$U(t, s)y - y = -\int_s^t U(t, \sigma)A(\sigma)y \, d\sigma.$$

So that $U(\cdot, \cdot)y$ is strongly absolutely continuous in X and satisfies

$$\frac{\partial}{\partial t}U(t, s)y=-A(t)U(t, s)y \quad \text{a.e. } t,$$

$$\frac{\partial}{\partial s}U(t, s)y=U(t, s)A(s)y \quad \text{a.e. } s,$$

which exist in the strong sense in X.

Now we can state our theorem.

THEOREM 2.1. *Under the assumptions* (A1) *to* (A4) *there is a unique evolution operator U for A.*

In the proof of Theorem 2.1 we often use the following lemmas.

LEMMA 2.1. 1) *Let $P(t)\in \boldsymbol{B}(X_1)$ and $Q(t)\in \boldsymbol{B}(X_2)$ be uniformly bounded for $t_1\leqq t\leqq t_2$, where X_1, X_2 are Banach spaces with X_2 continuously embedded in X_1. If $Q(\cdot)y$, $y\in X_2$, is strongly absolutely continuous in X_1 and $P(\cdot)$ is strongly absolutely continuous in $\boldsymbol{B}(X_2, X_1)$-norm, then $P(\cdot)Q(\cdot)y$ is strongly absolutely continuous in X_1.*

2) *Let X_1 be a Banach space. If $P(\cdot)$, $Q(\cdot)\in \boldsymbol{B}(X_1)$ are strongly absolutely continuous in $\boldsymbol{B}(X_1)$-norm, then $P(\cdot)Q(\cdot)\in \boldsymbol{B}(X_1)$ is also strongly absolutely continuous in $\boldsymbol{B}(X_1)$-norm.*

3) *Let $f(t)$ be strongly absolutely continuous in X_1 for $t_1\leqq t\leqq t_2$, where X_1 is a Banach space. If $f(\cdot)$ is a.e. differentiable and $(d/dt)f(t)$ is strongly integrable, then*

$$f(t_2)-f(t_1)=\int_{t_1}^{t_2}\frac{df(t)}{dt}dt.$$

PROOF. 1) It suffices to note the following.

$$\begin{aligned}&\|P(a)Q(a)y-P(b)Q(b)y\|_{X_1}\\ &\quad\leqq\|P(a)-P(b)\|_{X_2, X_1}\|Q(a)\|_{X_2}\|y\|_{X_2}+\|P(b)\|_{X_1}\|Q(a)y-Q(b)y\|_{X_1}\\ &\quad\leqq \mathrm{Const}\,(\|P(a)-P(b)\|_{X_2, X_1}\|y\|_{X_2}+\|Q(a)y-Q(b)y\|_{X_1}),\\ &\qquad\qquad t_1\leqq a\leqq b\leqq t_2, \quad y\in X_2.\end{aligned}$$

The proof of 2) and 3) is straightforward.

LEMMA 2.2. *Let $P(\cdot)\in B(X_1, X_2)$ and $Q(\cdot)\in B(X_2, X_3)$ be strongly measurable, where X_1, X_2 and X_3 are Banach spaces. Then $Q(\cdot)P(\cdot)\in B(X_1, X_3)$ is strongly measurable* [**2**: Lemma A4].

Hereafter we assume, without loss of generality, that β is Lebesgue integrable with some positive constant $\beta_0<\beta(t)$ a.e., if necessary, by replacing β with a dominating integrable function. Then we define the Yosida approximation A_λ of A by the relation:

(2.1) $$A_\lambda(t)=(I-J_\lambda(t))/\lambda(t), \quad \text{a.e. } t,$$

where

$$J_\lambda(t)=(I+\lambda(t)A(t))^{-1}, \qquad \text{a.e. } t, \tag{2.2}$$

$$\lambda(t)=\lambda/(\beta(t)+Mb(t)), \qquad \text{a.e. } t, \quad 0<\lambda<1, \tag{2.3}$$

and $b(\cdot)$ is a Lebesgue integrable function such that $\|B(t)\|_X \leqq b(t)$ a.e.

LEMMA 2.3. *Let $0\leqq t\leqq T$ and $0<\alpha<\alpha_0$. Under the assumptions above we have the following.*

(B1) *For each $y\in Y$, $(d/dt)S(t)y=\dot{S}(t)y$ a.e.*

(B2) *$S(t)$ is strongly absolutely continuous in $\boldsymbol{B}(Y, X)$-norm. Hence $\|S(t)\|_{Y,X}$ is uniformly bounded in t.*

(B3) *$S(\cdot)^{-1}$ is strongly absolutely continuous in $\boldsymbol{B}(X, Y)$-norm. Hence $\|S(t)^{-1}\|_{X,Y}$ is uniformly bounded in t.*

(B4) *$(I+\alpha S(\cdot))^{-1}$ is strongly absolutely continuous in $\boldsymbol{B}(X, Y)$-norm for each α. $(I+\alpha S(t))^{-1}$ is uniformly bounded for t, α both in $\boldsymbol{B}(X)$ and in $\boldsymbol{B}(Y)$.*

(B5) $$J_\lambda(t)=S(t)^{-1}(I+\lambda(t)A_1(t))^{-1}S(t) \qquad a.e.\ t,$$

where $A_1=A+B$.

$$\|J_\lambda(t)\|_X\leqq\frac{M}{1-\lambda}, \qquad a.e.\ t, \quad 0<\lambda<1.$$

$$\|J_\lambda(t)\|_Y\leqq\frac{M}{1-\lambda}\cdot\sup\|S(t)^{-1}\|_{X,Y}\cdot\sup\|S(t)\|_{Y,X}, \quad a.e.\ t, \quad 0<\lambda<1.$$

(B6) *$J_\lambda(\cdot)$ is (X) strongly measurable for $0<\lambda<1$ and (Y) strongly measurable for $0<\lambda<1/2$.*

(B7) *A_1 is quasi-stable with index $\{M, \beta(\cdot)+M\|B(\cdot)\|_X\}$. Since $A_1(t)=S(t)A(t)S(t)^{-1}$, Y is $A(t)$-admissible (a.e. t), i.e., the semi-group generated by $-A(t)$ leaves Y invariant and forms a (C_0) semi-group on Y.*

(B8) $$\|(I+\lambda(t)A_1(t))^{-1}\|_X\leqq\frac{M}{1-\lambda} \qquad a.e.\ t, \quad 0<\lambda<1.$$

(B9) *$(I+\lambda(\cdot)A_1(\cdot))^{-1}$ is (X) strongly measurable for $0<\lambda<1/2$.*

(B10) *$A_\lambda(\cdot)$ is (X) strongly measurable for $0<\lambda<1$ and (Y) strongly measurable for $0<\lambda<1/2$.*

(B11) $$\|A_\lambda(t)\|_X\leqq\frac{1}{\lambda}(\beta(t)+Mb(t))\Big(1+\frac{M}{1-\lambda}\Big) \quad a.e.\ t, \quad 0<\lambda<1.$$

$$\|A_\lambda(t)\|_Y\leqq\frac{1}{\lambda}(\beta(t)+Mb(t))\Big(1+\frac{M}{1-\lambda}\sup\|S(t)^{-1}\|_{X,Y}\sup\|S(t)\|_{Y,X}\Big),$$
$$a.e.\ t, \quad 0<\lambda<1.$$

PROOF. (B1) to (B3) is a simple consequence of (A4) (b). To prove (B4) we note the following.

$$\|(I+\alpha S(t))^{-1}\|_Y \leqq \sup\|S(t)^{-1}\|_{X,Y} \sup\|S(t)\|_{Y,X} \sup\|(I+\alpha S(t))^{-1}\|_X .$$

In fact,

$$\begin{aligned}\|(I+\alpha S(t))^{-1}y\|_Y &= \|S(t)^{-1}S(t)(I+\alpha S(t))^{-1}y\|_Y \\ &\leqq \|S(t)^{-1}\|_{X,Y}\|S(t)(I+\alpha S(t))^{-1}y\|_X \\ &= \|S(t)^{-1}\|_{X,Y}\|(I+\alpha S(t))^{-1}S(t)y\|_X \\ &\leqq \|S(t)^{-1}\|_{X,Y}\|(I+\alpha S(t))^{-1}\|_X\|S(t)\|_{Y,X}\|y\|_Y .\end{aligned}$$

Hence $(I+\alpha S(t))^{-1}$ is uniformly bounded for t, α in $\boldsymbol{B}(Y)$ by (A4) (c), (B2) and (B3). Similarly we have

$$\|(I+\alpha S(t))^{-1}\|_{X,Y} \leqq \frac{1}{\alpha} \sup\|S(t)^{-1}\|_{X,Y}(1+\sup\|(I+\alpha S(t))^{-1}\|_X) .$$

Thus $(I+\alpha S(\cdot))^{-1}$ is strongly absolutely continuous in $\boldsymbol{B}(X, Y)$-norm by (B2), completing the proof of (B4).

For the proof of (B7) we refer to Proposition 2.4 of [**1**]. Then (B5) and (B8) are obtained from (A2) and (B7) since $J_\lambda(t)=S(t)^{-1}\cdot(I+\lambda(t)A_1(t))^{-1}S(t)$ a. e.

Now we prove the strong measurability of $J_\lambda(\cdot)$. Since $\lambda(\cdot)$ is measurable by definition (2.3), we can take a sequence of Riemann step functions $\lambda_n(\cdot)$ such that $\lambda_n(t)\to\lambda(t)$ a. e. A sequence of Riemann step functions $(I+\lambda_n(t)A_n(t))^{-1}y$ strongly converges in X to $(I+\lambda(t)A(t))^{-1}y$ by (B5) where $y\in Y$, $0<\lambda<1$ and A_n is defined in (A3). Thus $J_\lambda(\cdot)y$ is strongly measurable in X for each $y\in Y$ and $0<\lambda<1$, so that $J_\lambda(\cdot)x$ is strongly measurable in X by continuity for each $x\in X$.

(B9) is verified as follows. First we note that $(I+\lambda(t)A_1(t))^{-1}=(I+\lambda(t)J_\lambda(t)B(t))^{-1}\cdot J_\lambda(t)$ for a. e. t, $0<\lambda<1$ and $(I+\lambda(\cdot)J_\lambda(\cdot)B(\cdot))^{-1}$ is (X) strongly measurable for $0<\lambda<1/2$. The latter is obtained by development into series for $0<\lambda<1/2$ since $\lambda(\cdot)J_\lambda(\cdot)B(\cdot)$ is (X) strongly measurable by Lemma 2.2 with the estimate:

$$\|\lambda(t)J_\lambda(t)B(t)\|_X \leqq \frac{\lambda}{\beta(t)+Mb(t)}\cdot\|J_\lambda(t)\|_X\|B(t)\|_X \leqq \frac{\lambda}{1-\lambda} \quad \text{a. e. } t, \quad 0<\lambda<1 .$$

Thus we complete the proof of (B9) by Lemma 2.2. Hence we can also get (B6) by $J_\lambda(t)=S(t)^{-1}(I+\lambda(t)A_1(t))^{-1}S(t)$ since $J_\lambda(\cdot)$ is (Y) strongly measurable for $0<\lambda<1/2$.

(B10) and (B11) are simple results of (B5), (B6) since $\lambda(\cdot)^{-1}$ is measurable.

REMARK 2.2. If $A(\cdot)$ is (Y, X) strongly piecewise continuous, then it is (Y, X) strongly uniform-measurable. In case X is separable (so that Y is also separable by (A4)) or $A(t)$ is uniformly bounded in $\boldsymbol{B}(X)$, strong measurability of $J_\lambda(\cdot)$ is implied by that of $A(\cdot)$ for small $\lambda>0$ (see Lemma A2 of [**2**]).

REMARK 2.3. If we assume (X) strong measurability of $J_\lambda(\cdot)$, $A(\cdot)$ is (Y, X) strong measurable as the limit of strongly measurable function $A_\lambda(\cdot)$ in $\boldsymbol{B}(X)$. But this assumption seems to be difficult to verify because of the complicated structure of $J_\lambda(\cdot)$, $A_\lambda(\cdot)$.

§ 3. Proof of Theorem 2.1.

We use the Yosida approximation method to construct an evolution operator for A. We will show that a family of evolution operators U_λ for the Yosida approximation A_λ of A has a unique strong limit U as $\lambda\searrow 0$, which corresponds to a unique evolution operator for A.

Since $J_\lambda(\cdot)$ is (X) strongly measurable by (B6) and $\|J_\lambda(t)\|_X\leqq M/(1-\lambda)$ for a. e. t and $0<\lambda<1$ by (B5), we can define an operator U_λ:

$$U_\lambda(t,s)=\exp\left[-\int_s^t\frac{d\tau}{\lambda(\tau)}\right]\cdot\left[I+\int_s^t\frac{J_\lambda(\tau)}{\lambda(\tau)}d\tau+\cdots+\int_s^t\left(\int_s^{t_1}\cdots\left(\int_s^{t_{n-1}}\frac{J_\lambda(t_1)}{\lambda(t_1)}\cdots\frac{J_\lambda(t_n)}{\lambda(t_n)}dt_n\right)\cdots\right)dt_1+\cdots\right],\quad 0\leqq s\leqq t\leqq T,\quad 0<\lambda<1, \tag{3.1}$$

in the strong sense in $\boldsymbol{B}(X)$. Now we will show that U_λ is the evolution operator for A_λ. First we note that this operator is estimated by (A2) as follows.

$$\|U_\lambda(t,s)\|_X\leqq M\cdot\exp\left[\frac{1}{1-\lambda}\int_s^t\beta(\tau)d\tau\right],\qquad 0<\lambda<1. \tag{3.2}$$

By definition $U_\lambda(\cdot,\cdot)$ satisfies the integral equations:

$$U_\lambda(t,s)x-x=-\int_s^t A_\lambda(\tau)U_\lambda(\tau,s)x\,d\tau,$$

$$U_\lambda(t,s)x-x=-\int_s^t U_\lambda(t,\tau)A_\lambda(\tau)x\,d\tau,\qquad x\in X,$$

so that U_λ is strongly absolutely continuous in $\boldsymbol{B}(X)$-norm for a fixed λ and in $\boldsymbol{B}(Y,X)$-norm uniformly for λ by (3.2) and (B11). It also satisfies the relation:

$$\frac{\partial}{\partial t}U_\lambda(t,s)x=-A_\lambda(t)U_\lambda(t,s)x\quad\text{a. e. } t,\quad x\in X, \tag{3.3}$$

$$\frac{\partial}{\partial s}U_\lambda(t,s)x=U_\lambda(t,s)A_\lambda(s)x\quad\text{a. e. } s,\quad x\in X, \tag{3.4}$$

and

$$U_\lambda(s,s)=I\qquad\text{for}\quad 0\leqq s\leqq T.$$

The relation $U_\lambda(t,s)U_\lambda(s,r)=U_\lambda(t,r)$, $0\leqq r\leqq s\leqq t\leqq T$, is verified by Lemma 2.1 (3), if we use strong absolute continuity of $U_\lambda(t,\cdot)\cdot U_\lambda(\cdot,r)$ in $B(X)$-norm and the relation $(\partial/\partial s)[U_\lambda(t,s)U_\lambda(s,r)x]=0$ a. e. s, $x\in X$.

Moreover U_λ satisfies the following lemma as $\boldsymbol{B}(Y)$-valued operator.

LEMMA 3.1. *$U_\lambda(t,s)$ is uniformly bounded in $\boldsymbol{B}(Y)$ for $0<\lambda<1/2$, $0\leqq s\leqq t\leqq T$, and strongly absolutely continuous in $\boldsymbol{B}(Y)$-norm for each λ.*

REMARK 3.1. Since $J_\lambda(\cdot)$ is (Y) strongly measurable by (B6), the operator U_λ in (3.1) is well defined also in $\boldsymbol{B}(Y)$, if we notice the stability of A restricted in Y (see Proposition 4.4 of [1]). But we prove Lemma 3.1 only by the essential boundedness of $J_\lambda(t)$ in $\boldsymbol{B}(Y)$ (see (B5)).

PROOF OF LEMMA 3.1. Let $0 \leqq r \leqq t \leqq T$. We note that $U_\lambda(t, \cdot)S(\cdot)^{-1}U_\lambda(\cdot, r)$ is strongly absolutely continuous in $\boldsymbol{B}(X)$-norm for $0<\lambda<1$. Consider the relation:

$$\frac{\partial}{\partial s}[U_\lambda(t, s)S(s)^{-1}U_\lambda(s, r)x]$$

$$=U_\lambda(t, s)\Big[A_\lambda(s)S(s)^{-1}-S(s)^{-1}A_\lambda(s)+\frac{d}{ds}S(s)^{-1}\Big]U_\lambda(s, r)x\,, \quad \text{a.e. } s\,, \quad x\in X\,.$$

The right hand side of this equation is strongly integrable for s in X, so we get

$$(3.5)\quad S(t)^{-1}U_\lambda(t, r)x-U_\lambda(t, r)S(r)^{-1}x$$
$$=\int_r^t U_\lambda(t, s)\Big[A_\lambda(s)S(s)^{-1}-S(s)^{-1}A_\lambda(s)-S(s)^{-1}\frac{dS(s)}{ds}S(s)^{-1}\Big]U_\lambda(s, r)x\,ds\,.$$

Omitting the argument s, we notice the following:

$$(3.6)\qquad \begin{aligned} A_\lambda S^{-1}-S^{-1}A_\lambda &= \lambda^{-1}(I-J_\lambda)S^{-1}-S^{-1}\lambda^{-1}(I-J_\lambda) \\ &= \lambda^{-1}(S^{-1}J_\lambda-J_\lambda S^{-1}) \\ &= \lambda^{-1}J_\lambda[(I+\lambda A)S^{-1}-S^{-1}(I+\lambda A)]J_\lambda \\ &= J_\lambda(AS^{-1}-S^{-1}A)J_\lambda \\ &= J_\lambda S^{-1}BJ_\lambda\,. \end{aligned}$$

Hence from (3.5) we have

$$(3.7)\qquad V_\lambda(t, r)x=S(t)^{-1}U_\lambda(t, r)x+\int_r^t V_\lambda(t, s)C_\lambda(s)U_\lambda(s, r)x\,ds\,,$$

where

$$(3.8)\qquad V_\lambda(t, r)=U_\lambda(t, r)S(r)^{-1}\,,$$

$$(3.9)\qquad C_\lambda(s)=\frac{dS(s)}{ds}S(s)^{-1}-S(s)J_\lambda(s)S(s)^{-1}B(s)J_\lambda(s)\,,$$

with the estimate

$$\|C_\lambda(s)\|_X \leqq \|\dot{S}(s)\|_{Y,X}\|S(s)^{-1}\|_{X,Y}+\frac{M}{1-\lambda}\|B(s)\|_X\cdot\frac{M}{1-\lambda}\,, \qquad 0<\lambda<1/2\,.$$

Since $\|C_\lambda(\cdot)\|_X$ is upper-integrable and $C_\lambda(\cdot)$ is (X) strongly measurable, we can define a family $\{W_\lambda\}$ of bounded linear operators in X:

$$(3.10)\quad W_\lambda(t, r)=U_\lambda(t, r)+U_\lambda*(C_\lambda U_\lambda)(t, r)+U_\lambda*(C_\lambda U_\lambda)*(C_\lambda U_\lambda)(t, r)+\cdots\,,$$

where $U_1*(PU_2)(t, r)=\int_r^t U_1(t, s)P(s)U_2(s, r)ds$. By use of the estimate:

$$\|W_\lambda(t, r)\|_X \leqq \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int^*_{(r,t)} \|C_\lambda(s)\|_X\, ds\right)^k (\sup \|U_\lambda(t, s)\|_X)^{k+1},$$

where $\int^*$ denotes upper-integral, we can conclude that $W_\lambda(t, r)$ is uniformly bounded in $\boldsymbol{B}(X)$ for λ, t, r and strongly absolutely continuous for t, r in $\boldsymbol{B}(X)$-norm as the limit of a uniformly convergent sequence since $U_\lambda(\cdot, \cdot)$ is strongly absolutely continuous. Moreover W_λ satisfies the following relation by definition.

$$W_\lambda(t, r) = U_\lambda(t, r) + \int_r^t W_\lambda(t, s) C_\lambda(s) U_\lambda(s, r) ds. \tag{3.11}$$

Since the solution of (3.7) is unique, we obtain (see [**4**, **5**])

$$V_\lambda(t, r)x = S(t)^{-1} W_\lambda(t, r)x, \qquad x \in X.$$

Thus by (3.8) we have

$$U_\lambda(t, r)y = S(t)^{-1} W_\lambda(t, r) S(r) y, \qquad y \in Y. \tag{3.12}$$

This relation implies that $U_\lambda(t, r)$ is uniformly bounded in $\boldsymbol{B}(Y)$ for λ, t, r and strongly absolutely continuous for t, r in $\boldsymbol{B}(Y)$-norm. The lemma is proved.

By using this lemma, we can conclude that U_λ is a unique evolution operator for A_λ.

To show that the family $\{U_\lambda(t, s)x : \lambda \searrow 0\}$ has a strong limit in X for each $x \in X$, we use the lemma.

Lemma 3.2.

$$\|U_\mu(t, r)y - U_\lambda(t, r)y\|_X \leqq C\|y\|_Y\left[\alpha + \int_E a(s)ds + \frac{\lambda+\mu}{\alpha\delta\beta_0}\right], \qquad y \in Y,$$

where $a(\cdot)$ is a Lebesgue integrable function on $[0, T]$ with $\|A(s)\|_{Y,X} \leqq a(s)$ a.e., $E = \{s : a(s) \geqq \delta^{-1}, 0 \leqq s \leqq T\}$ is a measurable set, β_0 is a constant with $\beta_0 \leqq \beta(t)$ a.e. and C is a constant independent of $0 \leqq r \leqq t \leqq T$, $y \in Y$, $0 < \alpha \leqq \alpha_0$, $0 < \lambda$, $\mu < 1/2$ and δ, $\beta_0 > 0$.

Proof. We begin with the relation obtained from (3.3), (3.4):

$$\frac{\partial}{\partial s}[U_\lambda(t, s) K_\alpha(s) U_\mu(s, r)y] \tag{3.13}$$

$$= U_\lambda(t, s)\left[A_\lambda(s)K_\alpha(s) - K_\alpha(s)A_\mu(s) - \alpha K_\alpha(s)\frac{dS(s)}{ds}K_\alpha(s)\right]U_\mu(s, r)y,$$

a.e. s, where

$$K_\alpha(s) = (I + \alpha S(s))^{-1}, \tag{3.14}$$

$y \in Y$, $0 < \alpha \leqq \alpha_0$, $0 < \lambda$, $\mu < 1/2$, $0 \leqq r \leqq t \leqq T$. Since the right hand side of (3.13) is strongly integrable for s in X and $U_\lambda(t, \cdot)K_\alpha(\cdot)U_\mu(\cdot, r)y$ is strongly absolutely continuous in X, we have

(3.15) $$U_\mu(t, r)y - U_\lambda(t, r)y$$
$$= [I - K_\alpha(t)]U_\mu(t, r)y - U_\lambda(t, r)[I - K_\alpha(r)]y$$
$$+ \int_r^t U_\lambda(t, s)\Big[A_\lambda(s)K_\alpha(s) - K_\alpha(s)A_\mu(s) - \alpha K_\alpha(s)\frac{dS(s)}{ds}K_\alpha(s)\Big]$$
$$\cdot U_\mu(s, r)y\, ds\,.$$

We notice the decomposition

$$J = \int_r^t U_\lambda(t, r)[A_\lambda(s)K_\alpha(s) - K_\alpha(s)A_\mu(s)]U_\mu(s, r)y\, ds$$
$$= \Big(\int_{E\cap(r, t)} + \int_{(r, t)\setminus E}\Big)\cdots y\, ds\,, \qquad y \in Y\,.$$

Then each term of J is estimated as follows,

(3.16) $$\Big\|\int_{E\cap(r, t)} \cdots y\, ds\Big\| \leqq \int_E \|\cdots y\|\, ds \leqq C\|y\|_Y \int_E a(s)ds\,.$$

(3.17) $$\Big\|\int_{(r, t)\setminus E} \cdots y\, ds\Big\|_X \leqq C\int^*_{(r, t)\setminus E} \|A_\lambda(s)K_\alpha(s) - K_\alpha(s)A_\mu(s)\|_{Y, X}\|y\|_Y ds\,.$$

To estimate (3.17) we observe, with argument s omitted,

$$A_\lambda K_\alpha - K_\alpha A_\mu = (A_\lambda - A_\mu)K_\alpha + (A_\mu K_\alpha - K_\alpha A_\mu)$$
$$= A(J_\lambda - J_\mu)K_\alpha + \mu^{-1}(K_\alpha J_\mu - J_\mu K_\alpha)$$
$$= (\mu - \lambda)A_\lambda A_\mu K_\alpha + \mu^{-1}K_\alpha[J_\mu(I + \alpha S) - (I + \alpha S)J_\mu]K_\alpha$$
$$= (\mu - \lambda)A_\lambda J_\mu A S^{-1}\cdot SK_\alpha + \alpha\mu^{-1}K_\alpha(J_\mu S - SJ_\mu)K_\alpha$$
$$= (\mu - \lambda)A_\lambda J_\mu S^{-1}(A + B)SK_\alpha + \alpha\cdot K_\alpha J_\mu BSJ_\mu K_\alpha\,.$$

Thus we have

$$\|A_\lambda(s)K_\alpha(s) - K_\alpha(s)A_\mu(s)\|_{Y, X}$$
$$\leqq \frac{\lambda + \mu}{\beta_0}\|A_\lambda(s)J_\mu(s)\|_{Y, X}\|S^{-1}(s)(A(s) + B(s))\|_Y \cdot \alpha^{-1}\|I - K_\alpha(s)\|_Y$$
$$+ \alpha\|K_\alpha(s)J_\mu(s)B(s)S(s)J_\mu(s)K_\alpha(s)\|_{Y, X}$$
$$\leqq \text{Const}\Big[\frac{\lambda + \mu}{(1 - \lambda)(1 - \mu)}\cdot\frac{M^2}{\alpha\delta\beta_0}\cdot(a(s) + b(s)) + \frac{M^2\alpha b(s)}{(1 - \mu)^2}\Big]\,, \qquad s \in (r, t)\setminus E\,,$$

where we used the estimate $\lambda(s) = \lambda/(\beta(s) + Mb(s)) \leqq \lambda/\beta_0$. Hence (3.17) is estimated as follows.

(3.18) $$\Big\|\int_{(r, t)\setminus E} \cdots y\, ds\Big\|_X \leqq C\|y\|_Y\Big(\alpha + \frac{\lambda + \mu}{\alpha\delta\beta_0}\Big)\,.$$

The lemma is verified by (3.16) and (3.18).

Lemma 3.2 implies that $U_\lambda(t, r)y$ has a strong limit in X uniformly for t, r, if $y \in Y$, since $0<\alpha \leqq \alpha_0$, $\delta>0$ are arbitrary and the measure of E is not greater than $\delta\int_0^T a(s)ds$. Then by continuity $U_\lambda(t, r)y$ converges to some $U(t, r)y$ strongly in X uniformly in t, r for each $y \in X$, so that $U(t, r)x$ is strongly continuous in X.

Moreover U has the following properties as the limit of U_λ.

$$U(t, t)=I\,, \quad U(t, s)U(s, r)=U(t, r)\,, \quad 0\leqq r\leqq s\leqq t\leqq T\,,$$

$$\|U(t, s)\|_X \leqq M\cdot\exp\left[\int_s^t \beta(\tau)d\tau\right].$$

$U_\lambda(\cdot, \cdot)y$, $y \in Y$, is strongly absolutely continuous in X uniformly for λ and so is $U(\cdot, \cdot)y$.

To check the regularity of $U(t, s)$ we use the next lemma.

Lemma 3.3. *$U(t, r)$ is uniformly bounded in $\boldsymbol{B}(Y)$ for t, r and it is (Y) strongly continuous for t, r. Moreover $U_\lambda(t, r)$ converges strongly to $U(t, r)$ as $\lambda \searrow 0$ in $\boldsymbol{B}(Y)$ uniformly for t, r.*

Proof. We get the following from (3.7), (3.10) by the dominated convergence theorem.

$V_\lambda(t, r)$, $W_\lambda(t, r)$ converges strongly to $V(t, r)$, $W(t, r)$, respectively, as $\lambda \searrow 0$ in $\boldsymbol{B}(X)$ uniformly for t, r, and following is satisfied:

$$V(t, r)x=S(t)^{-1}\cdot U(t, r)x+\int_r^t V(t, s)C(s)U(s, r)x\,ds\,, \qquad x \in X\,,$$

$$W(t, r)=U(t, r)+U*(C\cdot U)(t, r)+U*(C\cdot U)*(C\cdot U)(t, r)+\cdots\,,$$

where

$$V(t, r)=U(t, r)S(r)^{-1}\,,$$

$$C(s)=\frac{dS(s)}{ds}\cdot S(s)^{-1}-B(s)\,.$$

Then we can conclude as in Lemma 3.1 that

$$U(t, r)=S(t)^{-1}\cdot W(t, r)S(r) \qquad \text{in} \quad \boldsymbol{B}(Y)$$

so $U(t, r)$ is uniformly bounded in $\boldsymbol{B}(Y)$ for t, r and it is (Y) strongly continuous for t, r. Thus $U_\lambda(t, r)$ converges strongly to $U(t, r)$ as $\lambda \searrow 0$ in $\boldsymbol{B}(Y)$ uniformly for t, r. Proof is completed.

Since U satisfies the following integral equations:

$$U_\lambda(t, s)-I=-\int_s^t U_\lambda(t, \sigma)A_\lambda(\sigma)d\sigma\,,$$

$$U_\lambda(t, s)-I=-\int_s^t A_\lambda(\sigma)U_\lambda(\sigma, s)d\sigma\,,$$

we can prove by the dominated convergence theorem and Lemma 3.3:

$$U(t, s)y-y=-\int_{s}^{t}U(t, \sigma)A(\sigma)y\,d\sigma\,, \qquad y\in Y\,,$$

$$U(t, s)y-y=-\int_{s}^{t}A(\sigma)U(\sigma, s)y\,d\sigma\,, \qquad y\in Y\,.$$

Thus U is an evolution operator for A and strongly absolutely continuous both in $\boldsymbol{B}(Y, X)$-norm and in $\boldsymbol{B}(X)$. U also satisfies the following:

$$\frac{\partial}{\partial t}U(t, s)y=-A(t)U(t, s)y \quad \text{a.e. } t\,, \quad y\in Y\,,$$

$$\frac{\partial}{\partial s}U(t, s)y=U(t, s)A(s)y \quad \text{a.e. } s\,, \quad y\in Y\,,$$

which exist in the strong sense in X.

The uniqueness of the evolution operator is verified as follows. If U' is another evolution operator for A,

$$\begin{aligned}\frac{\partial}{\partial s}[U(t, s)U'(s, r)y]&=U(t, s)[A(s)-A(s)]U'(s, r)y\\ &=0\,, \qquad y\in Y\,, \quad \text{a.e. } s\,.\end{aligned}$$

Since $U(t, \cdot)\cdot U'(\cdot, r)y$ is strongly absolutely continuous in X, $U'(t, r)x=U(t, r)x$ for each $x\in X$ by continuity. This completes the proof of Theorem 2.1.

References

[1] T. Kato, Linear evolution equations of "hyperbolic" type, J. Fac. Sci. Univ. Tokyo Sect. IA, **17** (1970), 241-258.

[2] T. Kato, Linear evolution equations of "hyperbolic" type II, J. Math. Soc. Japan, **25** (1973), 648-666.

[3] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in Spectral Theory and Differential Equations, Lecture Notes in Math., **448**, Springer-Verlag, Berlin-Heidelberg-New York, 1975, 25-70.

[4] K. Masuda, Evolution equations, Kinokuniya, Tokyo, 1975, in Japanese.

[5] J.R. Dorroh, A simplified proof of a theorem of Kato on linear evolution equations, J. Math. Soc. Japan, **27** (1975), 474-478.

[6] S. Ishii, An approach to linear hyperbolic evolution equations by the Yosida approximation method, Proc. Japan Acad. Ser. A, **54** (1978), 17-20.

[7] J. Kisyński, Sur les opérateurs de Green des problèmes de Cauchy abstraits, Studia Math., **23** (1963/4), 285-328.

[8] E. Heyn, Die Differentialgleichung $dT/dt=P(t)T$ für Operator-funktionen, Math. Nachr., **24** (1962), 281-330.

[9] K. Kobayasi, On a theorem for linear evolution equations of hyperbolic type, J. Math. Soc. Japan, **31** (1979), 647-654.

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