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# On mod odd prime Brown-Peterson cohomology groups of exceptional Lie groups

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## § 1. Introduction.

Let $p$ be *an odd prime* and $BP^*(-;Z_p)$ be the mod $p$ Brown-Peterson cohomology theory with the coefficient $BP^*/p=Z_p[v_1, v_2, \cdots]$. Note that $BP^*(-;Z_p)$ has a commutative associative multiplication [3]. In this paper we shall study $BP(G;Z_p)$ for *a simple simply-connected compact Lie group* $G$. If $H^*(G;Z)$ is $p$-torsion free, there is a $BP^*$-algebra isomorphism $BP^*(G;Z_p)\cong BP^*\otimes H^*(G;Z_p)$. Hence we shall consider only cases when $H^*(G;Z)$ has $p$-torsions, i.e., exceptional Lie groups $F_4$, $E_6$, $E_7$ and $E_8$.

THEOREM 1.1. *There are $BP^*$-algebra isomorphisms*

(a) $BP^*(F_4;Z_3)\cong(BP^*/(3)\otimes\Lambda(w_{19})\oplus BP^*/(3, v_1)\otimes(Z_3[x_8]/(x_8^3)-\{1\}))$
$$\otimes\Lambda(x_7, x_{11}, x_{15}),$$

(b) $BP^*(E_6;Z_3)\cong BP^*(F_4;Z_3)\otimes\Lambda(x_9, x_{17})$,

(c) $BP^*(E_7;Z_3)\cong BP^*(F_4;Z_3)\otimes\Lambda(x_{19}, x_{27}, x_{35})$,

(d) $BP^*(E_8;Z_5)\cong(BP^*/(5)\otimes\Lambda(w_{51})\oplus BP^*/(5, v_1)\otimes(Z_5[x_{12}]/(x_{12}^5)-\{1\}))$
$$\otimes\Lambda(x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$$

*where $w_{19}\cdot x_8=0$, $w_{51}\cdot x_{12}=0$, and the dimension of $x_i$ or $w_i$ is $i$.*

THEOREM 1.2. *There is a $BP^*$-module isomorphism*

(e) $BP^*(E_8;Z_3)\cong(BP^*/(3)\{1, w_{15}, w_{74}\}\oplus BP^*/(3)[Z_3\{w_{55}, w_{43}, x_{20}, x_{20}^2\}$
$$\otimes Z_3\{1, x_8, x_8^2\}]/(v_1w_{43}=v_2w_{55}, v_1x_8=v_2x_{20}, v_1x_{20}=0))$$
$$\otimes\Lambda(x_7, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}).$$

*Moreover* $x_{20}^3=0$, $w_{43}\cdot x_{20}=w_{55}\cdot x_8$, $w_{15}\cdot w_{55}=v_1w_{74}$, $w_{55}\cdot w_{43}=0$, $w_{15}\cdot x_{20}=0$ *and* $w_{55}\cdot x_{20}=0$.

In this paper we use the cohomology theory $P(n)^*(-)$ with the coefficient $P(n)^*=BP^*/(p, v_1, \cdots, v_{n-1})$ and the cohomology theory $K(n)^*(-)$ with the coefficient $K(n)^*=Z_p[v_n, v_n^{-1}]$. We recall the facts that $K(n)^*(G)$ is *a Hopf* *algebra*, since $K(n)^*(G)$ is $K(n)^*$-free, and *the Conner-Floyd type theorem*; $K(n)^*(G) \cong K(n)^* \underset{P(n)^*}{\otimes} P(n)^*(G)$. To compute $BP^*(G; Z_p)$ from $H^*(G; Z_p)$, the following tower of these cohomology theories is important.

$$BP^*(-; Z_p) = P(1)^*(-) \xrightarrow[i]{} P(2)^*(-) \longrightarrow \cdots\cdots \longrightarrow P(\infty)^*(-) = H^*(-; Z_p)$$

$$P(1)^*(-) \xrightarrow{v_1} P(1)^*(-), \qquad P(2)^*(-) \xrightarrow{\delta} P(1)^*(-)$$

It seems that $P(n)^*(-)$ theories are useful to know the $BP^*$-module structure of $BP^*(X; Z_p)$ when the cohomology operations of $H^*(X; Z_p)$ are known.

## §2. Preliminary results.

Let $P(n)^*(-)$ be the cobordism theory with the coefficient $P(n)^* = BP^*/(p, \cdots, v_{n-1})$. Note that $P(1)^*(-) = BP^*(-; Z_p)$ and $P(\infty)^*(-) = H^*(-; Z_p)$. (For details see [6], [8].) Let $K(n)^*(-)$ be the Morava $K$-theory with the coefficient $K(n)^* = Z_p[v_n, v_n^{-1}]$. Note that $K(1)^*(-)$ is the $p-1$ component of the mod $p$ $K$-theory $K^*(-; Z_p)$, and we have the Conner-Floyd type theorem; $K(n)^*(-) \cong K(n)^* \underset{P(n)^*}{\otimes} P(n)^*(-)$ ([6], [7]).

We first recall that simple simply-connected compact Lie groups $(G, p)$ having odd prime torsions are given by (, see [2], [5],)

$$p=3, \quad G = F_4, E_6, E_7, E_8, \qquad (2.1)$$
$$p=5, \quad G = E_8.$$

The cohomology rings are known by Araki, Borel and others ([2], [5]);

(a) $H^*(F_4; Z_3) \cong Z_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15})$

(b) $H^*(E_6; Z_3) \cong Z_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17})$

(c) $H^*(E_7; Z_3) \cong Z_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35})$

(e) $H^*(E_8; Z_3) \cong Z_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$.

In cases (a)-(c), (e) we have the relations

$$\mathcal{P}^1 x_3 = x_7, \quad \delta x_7 = x_8 \quad \text{while in case (e) alone}$$

$$\mathcal{P}^1 x_{15} = -x_{19}, \quad \delta x_{19} = x_{20}, \quad \mathcal{P}^3 x_3 = 0, \quad \mathcal{P}^3 x_8 = x_{20}.$$

(d) $H^*(E_8; Z_5) \cong Z_5[x_{12}]/(x_{12}^5) \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$

and $\mathcal{P}^1 x_3 = x_{11}$, $\delta x_{11} = x_{12}$.

Here, recall that $Q_0$=Bockstein operation and $Q_n = \mathcal{P}^{p^n} Q_{n-1} - Q_{n-1} \mathcal{P}^{p^n}$.

Next we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*}=H^*(G;Z_p)\otimes K(n)^*\Longrightarrow K(n)^*(G).$$

Recall that $d_{2p^n-1}=v_n\otimes Q_n$ (Lemma 2.1 in [**9**]) and $d_r=0$ for $r\neq 2s(p^n-1)+1$, and moreover we note

$$E_r^{*,*}\cong K(n)^*\otimes H(E_{r-1}^{0,*},\ v_n^{-s}d_r).$$

Since $K(n)^*(X)$ is a free $K(n)^*$-module for all $X$, $K(n)^*(X\wedge Y)\cong K(n)^*(X)\underset{K(n)^*}{\otimes}K(n)^*(Y)$. Hence as $G$ is a compact Lie group, each $E_r$ is a *Hopf algebra.*

To compute $P(n)^*(X)$ from $K(n)^*(X)$, the following lemma is useful.

LEMMA 2.1. *Let $X$ be a finite complex. Let $x_i\in H^*(X;Z_p)$ be permanent cycles in the Atiyah-Hirzebruch spectral sequences of both $P(n)^*$-theory and $K(n)^*$-theory. Then in $E_\infty^{*,*}$ of $P(n)^*$-theory, the $P(n)^*$-module generated by $\{x_i\}$ is $P(n)^*$-free.*

PROOF. Assume that there is a relation

$$R=ax_1-\sum_{j\geq 2}a_jx_j=0,\quad a\neq 0,\quad a_j\in P(n)^*,\quad \text{in } E_\infty^{*,*}.$$

Let $a=(\lambda v_n^{\alpha_n}\cdots v_k^{\alpha_k}+c)$ with $\lambda\neq 0$ where $(\alpha_n,\cdots,\alpha_k)$ is the largest sequence by the right lexicographical order.

The associated filtration $F_m$ is defined such as

$$F_m=\operatorname{Ker}(P(n)^*(X)\longrightarrow P(n)^*(X^m))\quad \text{and}\quad E^{m,*}\cong F_m/F_{m+1}.$$

We take the Quillen-Novikov operation ([6], [8]) in $P(n)^*(X)$. That $R=0$ in $E_\infty^{*,*}$ means $R\in F_{|x_1|+1}$ and the naturality of the operation implies $r_\alpha(R)\in F_{|x_1|+1}$. Let $\gamma=(p^n\alpha_{n+1},\cdots,p^n\alpha_k)$. Since $r_\beta x_i\in F_{|x_1|+1}$ for $|\beta|>0$,

$$\begin{aligned} r_\gamma(R)&=r_\gamma(a)x_1+ar_\gamma(x_1) && \operatorname{mod}(F_{|x_1|+1},\ x_2,\ x_3,\ \cdots)\\ &=\lambda v_n^{\alpha_n+\cdots+\alpha_k}x_1 && \operatorname{mod}(F_{|x_1|+1},\ x_2,\ x_3,\ \cdots). \end{aligned}$$

Hence $v_n^\alpha x_1=0 \operatorname{mod}(x_2,x_3,\cdots)$ in $E_\infty^{*,*}$.

The natural map $i: P(1)\to K(n)$ induces a map $i_\infty: E_\infty^{P(n)}\to E_\infty^{K(n)}$ of $E_\infty$-terms. From the assumption of this theorem $i_\infty(v_n^\alpha x_1)\neq 0 \operatorname{mod}(x_2,x_3,\cdots)$ in $E_\infty^{K(n)}$. This is a contradiction. q.e.d.

## § 3. Cases (a)-(d).

In this section we consider only cases (a)-(d) and throughout this section $(G,p)$ *are assumed as* (a)-(d) in (2.1).

LEMMA 3.1. *There is a $K(2)^*$-module isomorphism*

$$K(2)^*(G)\cong K(2)^*\otimes H^*(G;Z_p).$$

PROOF. Consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*}=H^*(G;Z_p)\otimes K(2)^*\Longrightarrow K(2)^*(G).$$

Let $d_r$ be its differential and $v_2^{-s}d_{2s(p^2-1)+1}=\phi_s$. We need only prove all $\phi_s=0$.

Let $B(E_r)$ be the biprimitive form of $E_r$ (for details see [2], [4]). Since all $p$-th powers of elements of positive dimensional in $E_r$ is zero, as algebras, $B(E_r)$ is isomorphic to $E_r$ (Theorem 2.1, Theorem 2.7 in [4]).

We consider the biprimitive spectral sequence of $E_r$ (Theorem 3.4 in [4]), where the first term is $B(E_r)$ and the $E_\infty$-term is $B(H(E_r, \phi_s))$. In $B(E_r)\cong E_r$, there are no generators $x_i$, $x_j$ such that

$$|x_i|-|x_j|=\dim\phi_s=16s+1 \qquad \text{(or } 48s+1 \text{ for (d))}.$$

Note that the dimension of the differential $d_t$ in the biprimitive spectral sequence is $\dim\phi_s$. Hence $d_t(x_i)$ is decomposed and hence the spectral sequence collapses by Theorem 3.9 in [4]. This shows $B(E_r)\cong B(H(E_r, \phi_s))$ and $E_r\cong H(E_r, \phi_s)$ as algebras. Therefore we have $\phi_s=0$ and $E_r\cong E_{r+1}$. q. e. d.

LEMMA 3.2. *There is a $P(2)^*$-algebra isomorphism*

$$P(2)^*(G)\cong P(2)^*\otimes H^*(G;Z_p).$$

PROOF. Consider the spectral sequence

$$E_2^{*,*}=H^*(G;Z_p)\otimes P(2)^*\Longrightarrow P(2)^*(G).$$

First we prove that each element $a\in E_{2p}{}^{s,*}$ is a permanent cycle, by the induction on $s$. Assume that for all $s>t$, $u\in E_{2p}{}^{s,*}$ are permanent cycles. Let $y\in E_r{}^{t,0}$. Then $d_ry\in E_r{}^{s,*}$, $s>t$ and hence $d_ry=0$, that is, $y$ is a permanent cycle.

From Lemma 3.1, $y$ is also permanent in the spectral sequence of $K(2)^*$-theory. From Lemma 2.1, the $P(2)^*$-module generated by $y$ in $E_\infty^{t,*}$ is $P(2)^*$-free. This implies all elements in $E_r{}^{t,*}$ are permanent cycles, because if $d_rb=c\neq 0$ in $E_r{}^{t,*}$ then $c=0$ in $E_\infty^{t,*}$ and $E_\infty^{t,*}$ is not $P(2)^*$-free. Therefore we complete the induction and this spectral sequence collapses. This shows the $P(2)^*$-module isomorphism in this lemma.

For the $P(2)^*$-algebra isomorphism in this lemma, we only need to prove $x_8{}^3=0$. Since $P(2)^*(G)$ is $P(2)^*$-free by the $P(2)^*$-module isomorphism, we have

$$P(2)^*(G\wedge G)\cong P(2)^*(G)\otimes_{P(2)^*}P(2)^*(G).$$

Hence $P(2)^*(G)$ is a Hopf algebra. Let $\psi$ be the coproduct map and

$$\psi(x_8)=x_8\otimes 1+1\otimes x_8+v_2(\lambda x_8{}^2\otimes x_8+\mu x_8\otimes x_8{}^2)+a.$$

Here $a$ is a sum of tensor products, as factors, contain odd dimensional generators. From (2.1) (a)-(c), $|x_i|=4m-1$, $i\neq 8$ and the number of odd dimensional generators is less than 8. Since $|a|=4m'$, we can write

$$a=\sum c_I x_{i_1}\cdots x_{i_4} \qquad \text{where } c_I\in P(2)^*\{x_8,\ x_8^2,\ 1\} \qquad |x_{i_j}|=\text{odd}.$$

This implies

$$a^2=\sum c_I c_J x_{i_1}\cdots x_{i_4}x_{j_1}\cdots x_{j_4}=0\,, \qquad \text{since } x_i^2=0\,.$$

Now consider

$$\psi(x_8^3)=x_8^3\otimes 1+1\otimes x_8^3+v_2^3(\lambda^3 x_8^6\otimes x_8^3+\mu^3 x_8^3\otimes x_8^6)+a^3\,.$$

Here $a^3=0$ and by the $P(1)^*$-module isomorphism, $x_8^3$ also contains odd dimensional generators and $(x_8^3)^2=0$. Hence $x_8^3$ is primitive.

Let write $x_8^3=\sum c_I x_{i_1}\cdots x_{i_4}$. Let $J$ be the largest sequence by the left lexicographic order so that $|c_J|$ is largest in $I$.

$$\psi(x_8^3)=1\otimes x_8^3+x_8^3\otimes 1+c_J x_{j_1}\cdot x_{j_3}\otimes x_{j_4}+\cdots\,.$$

The primitivity implies that all $c_I=0$ and so $x_8^3=0$.

When the case (d) we can also prove $x_{12}^5=0$ by the similar reason. Therefore we have the lemma. q. e. d.

LEMMA 3.3 (Hodgkin [5]). *Let $A$ be the $Q_1$-subalgebra*

*in cases* (a)-(c) $\quad A=\Lambda(x_3)\otimes Z_3[x_8]/(x_8^3)\,,$

*in* (d) $\quad A=\Lambda(x_3)\otimes Z_5[x_{12}]/(x_{12}^5)\,,$

*and let $B$ be the subalgebra generated by all $x_i$ not listed in $A$. Then there is a $K(1)^*$-module isomorphism*

$$K(1)^*(G)\cong K(1)^*\otimes H(A,\ Q_1)\otimes B$$

*and in cases* (a)-(c) $H(A,\ Q_1)\cong\Lambda(\{x_3x_8^2\})$, *in* (d) $H(A,\ Q_1)\cong\Lambda(\{x_3x_{12}^4\})$.

PROOF. Consider the spectral sequence

$$E_2^{*,*}=H^*(G\,;\,Z_p)\otimes K(1)^*\Longrightarrow K(1)^*(G)\,.$$

Then $d_{2p-1}=v_1\otimes Q_1$, and $H(A,\ Q_1)$ is easily calculated as listed. Thus $H(A\otimes B,\ Q_1)\cong H(A,\ Q_1)\otimes B$ is an exterior algebra of odd dimensional. Consider the biprimitive spectral sequence as in the proof of Lemma 3.1. Since all dimensions of differentials are odd, the spectral sequence collapses, and we have the lemma.

q. e. d.

Now we shall consider the spectral sequence

$$E_2^{*,*}=H^*(G\,;\,Z_p)\otimes P(1)^*\Longrightarrow P(1)^*(G)\,.$$

LEMMA 3.4. *Let $E_r^{*,*}$ be the above spectral sequence. Then for cases* (a)-(c)

$$E_\infty^{*,*}\cong E_{2p}^{*,*}\cong(P(1)^*\otimes H(A,\ Q_1)\oplus P(1)^*/(v_1)\otimes Z_3\{x_8,\ x_8^2\})\otimes B\ . \qquad (3.4)$$

PROOF. Since for $*>-2(p^2-1)$ $P(1)^*\cong k(1)^*$, we have $d_{2p-1}=Q_1\otimes v_1$. Therefore $E_{2p}^{*,*}$ is isomorphic to the right hand side of (3.4).

We use the argument similar to that in the proof of Lemma 3.2. By induction, we assume that for all $s>t$, $u\in E_{sp}{}^{s,*}$ are permanent cycles. By the assumption $x\in E_{2p}{}^{t,0}$ are permanent cycles.

Let $dw=\sum a_ix_i+\sum b_jy_j$, $|x_i|=|y_j|=t$ where $a_i\in P(1)^*$, $b_j\in P(2)^*$, $x_i\in H(A,\ Q_1)\otimes B$ and $y_j\in Z_3\{x_8,\ x_8^2\}\otimes B$. From Lemma 3.3 and Lemma 2.1, $x_i$ is $P(1)^*$-free and $a_i$ must be zero. But $i_\infty(y_j)$ is $P(2)^*$-free where $i_\infty\colon E^{P(1)^*}\to E^{P(2)^*}$ is the induced map from the natural map $i\colon P(1)\to P(2)$. Hence $b_j$ must be also zero.

Therefore all $u\in E_{2p}^{t,*}$ are permanent cycles. q. e. d.

THEOREM 1.1. *For cases* (a)-(c), *there are* $P(1)^*$-*algebra isomorphisms*

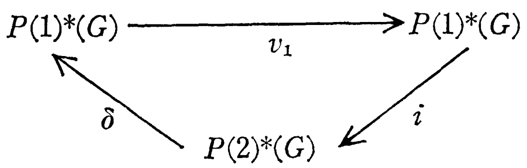
$$P(1)^*(G)\cong(P(1)^*\otimes\Lambda(w_{19})\oplus P(2)^*(Z_3[y_8]/(y_8^3)-\{1\}))\otimes B\ .$$

PROOF. To prove the $P(1)^*$-module isomorphism, from Lemma 3.4, we only need to prove the sequence of the associated filtration of $P(1)^*(G)$

$$0\longrightarrow F^{r+1,*}\longrightarrow F^{r,*}\longrightarrow E_\infty^{r,*}\longrightarrow 0$$

are split for all $r$. That each element in $E_\infty^{*,*}$, except for $P(2)^*\{x_8,\ x_8^2\}\otimes B$, is $P(1)^*$-free follows that we only need to show there is $y_8$ in $P(1)^*(G)$ which represents $x_8$ and $v_1y_8=0$.

Consider the Sullivan exact sequence [8]




Let $\delta x_3=y_8$. Then $i\delta x_3=Q_1x_3=x_8 \bmod (v_2,\ v_3,\ \cdots)$, and by the exact sequence, $v_1y_8=0$. Therefore we have the $P(1)^*$-module isomorphism in this theorem.

Note that $i(y_8^3)=0$ in $P(2)^*(G)$, and so $y_8^3=v_1a$ in $P(1)^*(G)$ and $v_1y_8^3=0$. From the $P(1)^*$-module isomorphism in this theorem, there is no element $b\neq0\in P(1)^*(G)$ so that $b\in\text{image}\,v_1$ and $v_1b=0$. Thus we have $y_8^3=0$.

Since $\delta(x_3i(y_8^2))=y_8^3=0$ in $P(1)^*(G)$, there is $w_{19}$ such that $i(w_{19})=x_3i(y_8^2)$ in $P(2)^*(G)$. By the same reason as the case $y_8^3=0$, that $i(w_{19}y_8)=i(y_8^3)x_3=0$ implies $w_{19}y_8=0$ in $P(1)^*(G)$. These show the $P(1)^*$-algebra isomorphisms.

q. e. d.

For $G$ as (d) we can prove the main theorem by exchanging $x_8$ for $x_{12}$.

## § 4. Case (e).

In this section we consider the case $(G, p)=(E_8, 3)$. By the same argument as in § 3, we can prove the following lemmas.

LEMMA 4.1. *There is a $K(3)^*$-module isomorphism*

$$K(3)^*(E_8)\cong K(3)^*\otimes H^*(E_8; Z_3).$$

LEMMA 4.2. *There is a $P(3)^*$-algebra isomorphism*

$$P(3)^*(E_8)\cong P(3)^*\otimes H^*(E_8; Z_3).$$

LEMMA 4.3. *There is a $K(2)^*$-module isomorphism*

$$K(2)^*(E_8)\cong K(2)^*\otimes \Lambda(\{x_3x_{20}^2\})\otimes Z_3[x_8]/(x_8^3)\otimes \Lambda(x_{15})\otimes B$$

*where $B=\Lambda(x_7, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$.*

PROOF. Consider the spectral sequence

$$H^*(E_8; Z_p)\otimes K(2)^* \Longrightarrow K(2)^*(E_8).$$

Here we note that $d_{2(p^2-1)+1}=v_2\otimes Q_2$, and $Q_2x_3=-x_{20}$ and $Q_2z=0$ for other ring generators $z\in H^*(E_8; Z_3)$. Then we have

$$E_{2p^2}\cong K(2)^*\otimes \Lambda(\{x_3x_{20}^2\}, x_{15})\otimes B\otimes Z_3[x_8]/(x_8^3).$$

By using the same argument as the proof of Lemma 3.3, we can prove $E_{2(p^2-1)+2}\cong E_\infty$. q. e. d.

By the arguments similar to that in the proof of Theorem 1.1, we can prove the following proposition.

PROPOSITION 4.4. *There is a $P(2)^*$-algebra isomorphism*

$$P(2)^*(E_8)\cong (P(2)^*\Lambda(\bar{w}_{43})\oplus P(2)^*/(v_2)\{y_{20}, y_{20}^2\})\otimes \Lambda(x_{15})\otimes B\otimes Z_3[x_8]/(x_8^3).$$

We recall the boundary operation $Q_1'$ in $P(2)^*(-)$ theory ([8]), which has the property such that

$$iQ_1'=Q_1i$$

where $i: P(2)\to KZ_p$ is the natural map.

LEMMA 4.5. *Let $x\in K(2)^*(E_8)$. Then $Q_1'x=0$ in $K(2)^*(E_8)$.*

PROOF. Let $x_i'$ be the element in $K(2)^*(E_8)$ which corresponds to the $P(2)^*$-algebra generator $x_i$ in $P(2)^*(E_8)$. By the derivativity of $Q_1'$ we need only prove $Q_1'x_i'=0$.

Assume $Q_1'x_i'\neq 0$ for some $i$. Since $iQ_1'x_i=Q_1ix_i=0$ in $H^*(E_8; Z_3)$ for $i\neq 15$ and $i(Q_1x_{15}-y_{20})=0$, we can write in $K(2)^*(E_8)$

$$Q_1'x_i'=v_2^sy+\sum_{s'\geqq s} v_2^{s'}z, \qquad s>0,$$

where $y$, $z$ are $K(2)^*$-module generators $x'_{j_1}\cdots x'_{j_k}$. Choose $x'_i$, $s$ so that $s$ is the least positive integer with $y\neq 0$, and then denote $x'_i$, $s$ by $w$, $t$.

Let the coproduct $\psi$ be

$$\psi(x'_i)=\sum\lambda_{\alpha,\beta}x'_\alpha\otimes x'_\beta+\sum_{s\neq 0}v_2^s x'_\gamma\otimes x'_\delta, \qquad \lambda_{\alpha,\beta}\in Z_3 .$$

Since $\psi$ is the induced map from

$$\psi_{P(2)^*}: P(2)^*(E_8)\longrightarrow P(2)^*(E_8\wedge E_8),$$

all $s$ in the right terms of the above equation are positive. Denote $\sum\lambda_{\alpha,\beta}x'_\alpha\otimes x'_\beta$ by $\tilde{\psi}(x_i)$. Let $d=v_2^{-t}Q_1$ and

$$E=\Lambda(x_3x_{20}^2)\otimes Z_3[x_8]/(x_8^3)\otimes\Lambda(x_{15})\otimes B .$$

Then $E$ is a differential Hopf algebra by $\tilde{\psi}$ and the differential $d$.

Consider the homology group $H(E, d)$. Note that

$$\text{dimension } d=2(p^2-1)t+2(p-1)+1=16t+5 .$$

There is no generators such that

$$|x_i|-|x_j|=16t+5 .$$

Hence by the same arguments as in the proof of Lemma 3.1, we have

$$H(E, d)\cong E , \qquad \text{and so } d=0 .$$

Thus $dw=0$ and this is a contradiction. q. e. d.

LEMMA 4.6 (Hodgkin [5]). *There is a* $K(1)^*$*-module isomorphism* $K(1)^*(E_8)\cong K(1)^*\otimes\Lambda(\{x_3x_8^2\},\ \{x_{15}x_{20}^2\})\otimes B$.

PROOF. Since $Q_1x_3=-x_8$ and $Q_1x_{15}=x_{20}$, the lemma is proved by the same arguments as Lemma 3.3. q. e. d.

Now we consider the spectral sequence

$$E_2=P(1)^*\otimes H(E_8;\ Z_3)\Longrightarrow P(1)^*(E_8) .$$

LEMMA 4.7. *In the above spectral sequence*

$$\begin{aligned}E_{2p^2}\cong[&P(1)^*\{v_1\otimes x_3x_8^2,\ x_3x_8^2x_{15}x_{20}^2,\ x_{15}x_{20}^2\}\\ &\oplus P(2)^*\{x_3x_8^2x_{20}^2,\ x_{15}x_{20}^2(x_8,\ x_8^2)\}\\ &\oplus P(2)^*\{(x_3x_{20}-x_8x_{15})(x_{20},\ x_{20}x_8)\}\\ &\oplus P(2)^*\{x_8,\ x_8^2,\ x_{20},\ x_8x_{20}\}\\ &\oplus P(3)^*\{x_{20}^2,\ x_8x_{20}^2,\ x_8^2x_{20}^2,\ x_8^2x_{20}\}]\otimes B .\end{aligned}$$

PROOF. From Lemma 4.6, in the spectral sequence $E_{K(1)^*}$ of $K(1)^*$-theory, the differentials $d_{2p-1}=v_1\otimes Q_1$ and $d_r=0$ for $r\neq 2p-1$. Therefore the spectral sequence $E_{k(n)*}$ of $k(n)^*$-theory (, $k(n)^*=Z_p[v_n]$ and for details see [6], [9],) have such property, i.e., $d_{2p-1}=Q_1$ and $d_r=0$ for $r\neq 2p-1$.

Since $k(1)^*\cong P(1)^*$ for $*>-2(p^2-1)$, we have the isomorphism

$$\begin{aligned} E_{2p^2-1}&\cong[P(1)^*\otimes H(A, Q_1)\oplus P(2)^*\otimes \text{Image}(A, Q_1)]\otimes B\\ &\cong[P(1)^*\otimes \Lambda(x_3x_8^2, x_{15}x_{20}^2)\\ &\quad\oplus P(2)^*\{x_3x_8^2(x_{20}, x_{20}^2), x_{15}x_{20}^2(x_8, x_8^2)\}\\ &\quad\oplus P(2)^*\{(x_3x_{20}-x_8x_{15})\otimes(1, x_{20}, x_8, x_8x_{20})\}\\ &\quad\oplus P(2)^*\otimes(Z_3[x_8, x_{20}]/(x_8^3, x_{20}^3)-\{1\})]\otimes B\,. \end{aligned}$$

Now compare the spectral sequences of $P(1)^*$-theory and $P(2)^*$-theory. Let

$$i_E^{*,*}: E_{2p^2-1}{}^{P(1)^{*,*}}\longrightarrow E_{2p^2-1}{}^{P(2)^{*,*}}$$

be the induced map from the natural map. Since $E_{2p^2-1}{}^{P(2)^{*,*}}\cong E_2{}^{P(2)^{*,*}}$, $i_E^{*,0}$ is injective. Hence $d_{2p^2-1}{}^{*,0}=v_2\otimes Q_2 \bmod (v_1)$ in $E_{2p^2-1}{}^{P(1)^{*,*}}$ since $d_{2p^2-1}=v_2\otimes Q_2$ in $E_{2p^2-1}{}^{P(2)^{*,*}}$.

The operation $d_{2p^2-1}=d_r=v_2\otimes Q_2$ is calculated as follows.

(1) $d_r(x_3x_8^2)=v_2x_{20}x_8^2$ and $v_1\cdot x_{20}x_8^2=0$ in $E_{2p^2-1}$. Hence we have $0\neq v_1\cdot x_{20}x_8^2\in H(E_r, d_r)$, and $P(3)^*\{x_{20}x_8^2\}\subset H(E_r, d_r)$.

(2) $d_r(x_3x_{20}-x_8x_{15})=v_2x_{20}^2$, $d_r(x_3x_{20}-x_8x_{15})x_8=v_2x_{20}^2x_8$.
Here $(x_3x_{20}-x_8x_{15})$ and $x_{20}^2$, $x_{20}^2x_8$ are also $P(2)^*$-free, we have

$$P(3)^*\{x_{20}^2, x_{20}^2x_8\}\subset H(E_r, Q_r)\,.$$

(3) $d_r(x_3x_{20}-x_8x_{15})x_{20}=0$, and hence

$$P(2)^*\{(x_3x_{20}-x_8x_{15})(x_{20}, x_{20}x_8)\}\subset H(E_r, d_r)\,.$$

(4) $d_rx_3x_8^2x_{20}=v_2\cdot x_{20}^2x_8$, hence we have $P(3)^*\{x_{20}^2x_8^2\}\subset H(E_r, d_r)$.

From (1)-(4) we can easily calculate the $2p^2$-term as in the lemma. q.e.d.

LEMMA 4.8. *In the spectral sequence in Lemma* 4.7, $E_{2p^2}\cong E_\infty$.

PROOF. Assume that there is $x$ such that $d_rx\neq 0$ for some $r\geqq 2p^2$. Then we can take a $P(1)^*$-module generator $w\in E_{2p^2}{}^{t-r,*}$ such that

(1) for all $s>t$, $u\in E_{2p^2}{}^{s,*}$ are permanent cycles,
(2) $d_rw\neq 0$ in $E_{2p^2}{}^{t,*}$,
(3) for $2p^2\leqq t-r<s$, $u\in E_{2p^2}{}^{s,*}$, all differentials $d_{t-s}u=0$.

From Lemma 4.7 and (3), we decompose

$$E_r{}^{t,*}\cong E_{2p^2}{}^{t,*}\cong P(1)^*(\mathrm{I})\oplus P(2)^*(\mathrm{II})\oplus P(3)^*(\mathrm{III})\oplus P(2)^*(\mathrm{IV})$$

where

(I) $Z_3\{1, v_1x_3x_8^2\cdot x_3x_8^2x_{15}x_{20}^2, x_{15}x_{20}^2\}\otimes B$,

(II) $Z_3\{x_8, x_8^2, x_3x_{20}^2x_8^2, (x_3x_{20}-x_8x_{15})(x_{20}, x_{20}x_8)\}\otimes B$,

(III) $Z_3\{x_{20}^2, x_8x_{20}^2, x_8^2x_{20}^2, x_8^2x_{20}\}\otimes B$,

(IV) $Z_3\{x_{20}, x_8x_{20}, x_{15}x_{20}^2(x_8, x_8^2)\}\otimes B$.

Let $dw=\sum a_ix_i+\sum b_jy_j+\sum c_kz_k+\sum e_mu_m$ where $a_i, b_j, c_k, e_m\in P(1)^*$ and $x_i\in$(I), $y_j\in$(II), $z_k\in$(III), $u_m\in$(IV).

First suppose $a_i\neq 0\in P(1)^*$. From Lemma 4.6, $x_i$ is $K(1)^*$-free. This contradicts to Lemma 2.1. Hence $a_i=0$.

Secondly suppose $b_j\neq 0\in P(2)^*$. From Proposition 4.4, $i(y_j)$ is $P(2)^*$-free in $P(2)^*(E_8)$. Contradiction follows and $b_j=0$.

Thirdly, we also see $c_k=0$, because $z_k$ is $P(3)^*$-free in $P(2)^*(E_8)$ from Proposition 4.4.

Lastly, suppose $d_rw=\sum e_mu_m\neq 0$. We note $i(u_m)$ is $P(3)^*$-free in $P(2)^*(G)$. Before we consider the differential $dw$, we need some facts about elements in (II), (III), (IV).

By the assumption (1), we note all elements $x\in E_{2p^2}^{s,*}$, $s\geqq t-2p^2+1$, are permanent cycles. Let $w\in$(III) or (IV) and $|w|\geqq t-2p^2-1$. From Sullivan exact sequence, we can take $w\in P(1)^*(E_8)$ such that

$$v_1w=0 \quad \text{in } P(1)^*(E_8),$$

because if $w=x_{20}\overline{w}$, we take $w=\delta x_{15}\overline{w}$, and if $w=x_8x_{20}^2\overline{w}$, we take $\delta(x_3x_{20}\overline{w})=w$.

Let $u\in$(IV) and $|u|\geqq t$. From Lemma 4.7, $v_2u\neq 0$ in $E_r^{*,*}$. But from Proposition 4.4, $i(v_2u)=0$ in $P(2)^*(E_8)$ and $v_2u\in\text{image}\, v_1$. Hence we can write

$$v_2u=\sum_{k'>k} v_1^k(y+v_1^{k'}y_{k'}) \tag{i}$$

where $|u|>|y|$, $|y_k|>|v_2u|=t-2p^2+2$.

Since $v_1(v_2u)=0$, $y, y_{k'}\notin$(I). But $v_1y=0$ for $y\in$(III) or (IV), and hence we can take $y, y_k\in$(II).

Let $y\in$(II), $|y|\geqq t-2p^2+1$. If $v_1y=0$ in $P(1)^*(X)$, then $y\in\text{image}\,\delta$ and $i(y)\in\text{image}\, Q_1$ in $P(2)^*(E_8)$, and this is not valid. Hence $v_1y\neq 0$ in $P(1)^*(E_8)$. That $v_1y=0$ in $E_r^{*,*}$ means $v_1y$ is contained in the associated filtration $F^s$, $s>|y|$. Hence we can write

$$v_1y=\sum a_\alpha y_\alpha+\sum b_\beta w_\beta$$

with $|y_\alpha|>|y|$, $|w_\beta|>|y|$, $y\in$(II), $w\in$(III) or (IV), and moreover, $a\in\text{ideal}\, v_1$, since $i(v_1y)=0$ in $P(2)^*(E_8)$ and $i(y_\alpha)$ is $P(2)^*$-free.

Next we apply $v_1y$ the same argument, i.e.,

$$v_1y=\sum(a_\alpha/v_1)(\sum\tilde{a}_\alpha\tilde{y}_\alpha+\sum\tilde{b}_\beta\tilde{w}_\beta)+\sum b_\beta w_\beta$$
$$=\sum a'_\alpha y'_\alpha+\sum b'_\beta w'_\beta$$

where $|y'_\alpha|>|y|+1$, $y'_\alpha\in$(II) and $w'_\beta\in$(III) or (IV). Continuing this argument, we can ascend dimensions of each generators $y_\alpha$. Consequently, we can eliminate $y_\alpha$. Therefore

$$v_1y=\sum b''_\beta w''_\beta, \qquad w''_\beta\in(\mathrm{II}).$$

Since $v_1w''_\beta=0$, $v_1^2y=0$ and hence $k$ in (i) must be 1, i.e.,

$$v_2u=v_1y.$$

By the assumption (2), $\sum e_mu_m=0$ in $E_{r+1}$. This means

$$\sum e_mu_m=\sum c_\alpha h_\alpha \qquad \text{in } P(1)^*(E_8),$$

where $|h_\alpha|>|u_m|$, $c_\alpha\in P(1)^*$.

Taking the adequate cohomology operation $r_\gamma$, as in the proof of Lemma 2.1,

$$v_2^su=\sum c'_\alpha h'_\alpha \qquad \text{in } P(1)^*(E_8),\text{ where } |u|=|u_m|<|h'_\alpha|.$$

That $v_2u=v_1y$ implies

$$v_1v_2^{s-1}y=\sum c'_\alpha h'_\alpha. \tag{ii}$$

Therefore $i(\sum c'_\alpha h'_\alpha)=0$ in $P(2)^*(E_8)$. Since $v_1u=0$, $h'_\alpha\notin$(I). If $h'_\alpha\in$(II), then $i(h'_\alpha)$ is $P(2)^*$-free and we have $v_1\in c_\alpha$. If $h'_\alpha\in$(III) or (IV), then since $i(h'_\alpha)$ is $P(3)^*$-free and we have $v_2\in c_\alpha$. But if $h'_\alpha\in$(III), then since $v_2h'_\alpha=0$ in $E_r^{*,*}$, we can write

$$v_2h'_\alpha=\sum c'_\beta h''_\alpha, \qquad |h''_\alpha|>|h_\alpha| \text{ in } P(2)^*(E_8).$$

Hence we can eliminate $v_2h'_\alpha$ for $h'_\alpha\in$(III). When $h'_\alpha\in$(IV), then there is a $P(1)^*$-module generator $y'_\alpha$ such that $v_1y'_\alpha=v_2h'_\alpha$. Here we note $|y'_\alpha|>|y|$, since $|h'_\alpha|>|u|$.

Therefore the equation (ii) is

$$v_1(v_2^{s-1}y-\sum_{h'_\alpha\in(\mathrm{IV})}(c_\alpha/v_2)y'_\alpha-\sum_{h'_\alpha\in(\mathrm{II})}(c_\alpha/v_1)h'_\alpha)=0.$$

That is, $v_1(v_2^{s-1}y-\sum c''_\alpha y''_\alpha)=0$, $|y''_\alpha|>|y'|$. Then

$$0\neq i(v_2^{s-1}y-\sum c''_\alpha y''_\alpha)\in \text{image } Q_1 \qquad \text{in } P(2)^*(E_8).$$

But this shows also Image $Q_1(K(2)^*(E_8))\neq 0$ and this contradicts Lemma 4.5. Hence the last supposition is not valid.

Therefore the first assumption is not valid and we can complete the lemma.

q.e.d.

Proof of the $P(1)^*$-module isomorphism of Theorem 1.2 (e). First, by the dimensional reason we have

$$v_1x_8=\lambda v_2x_{20}\,, \qquad \lambda \neq 0\,.$$

Here we note the following facts;

$P(1)^*(x_8)\cong P(1)^*/(v_1^2)$ in $P(1)^*(E_8)$ because $v_1^2x_8=v_1v_2x_{20}=0$,

$P(1)^*(x_8^2)\cong P(1)^*/(v_1^2)$,

$P(1)^*(x_{20})\cong P(2)^*$,

$P(1)^*(x_{20}^2)\cong P(3)^*$ because $v_2x_{20}^2=v_1x_8x_{20}=0$,

$P(1)^*(x_8x_{20}^2)\cong P(1)^*(x_8^2x_{20}^2)\cong P(3)^*$,

$P(1)^*(x_8^2x_{20})\cong P(3)^*$ because $v_2x_8^2x_{20}=v_1x_8^3=0$, since $i(x_8^3)=0$ in $P(2)^*(E_8)$ and so $x_8^3\in \text{Image}\, v_1$.

By the arguments as (IV) in the proof of Lemma 4.8, there is a generator $g'$ such that

$$v_2x_{15}x_{20}^2=v_1g'\,.$$

By the dimensional reason, $g'$ must be

$$g'=\mu(x_3x_{20}-x_8x_{15})x_{20}, \qquad \mu\neq 0\in Z_3\,.$$

By the same arguments, we have

$$v_2x_{15}x_{20}^2x_8=\mu v_1(x_3x_{20}-x_8x_{15})x_{20}x_8\,,$$

$$v_2x_{15}x_{20}^2x_8^2=\mu v_1x_3x_{20}^2x_8^2\,.$$

Therefore by Lemma 4.8, we have the $P(1)^*$-module isomorphism in Theorem 1.2. q. e. d.

Proof of the multiplications in Theorem 1.2. First we note that if $y\neq 0$ and $y\in \text{Image}\, v_1$, then by the $P(1)^*$-isomorphism of Theorem 1.2, $v_2y\neq 0$ and $v_1y=0$.

(1) $x_{20}^3=0$. Since $i(x_{20}^3)=0$ in $P(2)^*(E_8)$, $x_{20}^3\in \text{Image}\, v_1$. Acting $v_2$, we have $v_2(x_{20}^3)=v_1x_8x_{20}^2=0$.

(2) $w_{43}x_{20}=w_{55}x_8$. $i(w_{43}x_{20}-w_{55}x_8)=(x_8x_{15}x_{20}-x_3x_{20}^2)x_{20}-x_{15}x_{20}^2x_8=0$ and $v_2w_{43}x_{20}=v_1w_{43}x_8=v_2w_{55}x_8$.

(3) $w_{15}w_{55}=v_1w_{74}$. By the spectral sequence, we have

$$(v_1\otimes \{x_3x_8^2\})(x_{15}x_{20}^2)=v_1x_3x_8^2x_{15}x_{20}^2+C$$

where $i(C)=0$ in $P(2)^*(E_8)$ so $C\in \text{Image}\, v_1$. Hence we can take new generator such that $(v_1\otimes\{x_3x_8^2\})(x_{15}x_{20}^2)=v_1w_{74}$.

(4) $w_{55}w_{43}=0$, $i(w_{55}w_{43})=0$ and $v_2w_{55}w_{43}=v_1w_{43}w_{43}=0$.

(5) $w_{15}x_{20}=0$, $w_{74}x_{20}=0$. $i(w_{15}x_{20})=0$, $i(w_{74}x_8)=0$ and $v_2w_{15}x_{20}=v_1w_{15}x_8$, $v_2w_{74}x_{20}=v_1w_{74}x_8$ where $i(w_{15}x_8)=0$, $i(w_{74}x_8)=0$.

(6) $w_{55}x_{20}=0$. $i(w_{55}x_{20})=0$ and $v_2w_{55}x_{20}=v_1x_{43}x_{20}=0$.

Therefore we can complete the proof of Theorem 1.2. q. e. d.

REMARK 4.9. (1) $i(w_{15}w_{43})=0$. But $v_1w_{15}w_{43}=v_2w_{55}w_{15}=v_2v_1w_{74}\neq 0$.

(2) The author does not know whether $w_{15}w_{43}=v_1w_{74}$, $w_{55}w_{74}=0$, $w_{74}^2=0$, $x_8^3=0$ or not.

REMARK 4.10. The correspondence of elements in the Sullivan exact sequence is as follows;

(1) $w_{15}=v_1x_3x_8^2 \xrightarrow{i} v_2x_{15}x_8^2 \xrightarrow{\delta} v_2x_{20}x_8^2=v_1x_8^3=0$.

(2) $\bar{w}_{43}=\{x_3x_{20}^2\} \xrightarrow{\delta} x_8x_{20}^2$; $v_2$-torsion and $v_2\bar{w}_{43}=v_2i(x_3x_{20}-x_8x_{15}x_{20})=v_2iw_{43}$.

(3) $x_8\bar{w}_{43} \xrightarrow{\delta} x_8^2x_{20}^2$; $v_2$-torsion.

(4) $x_{15}w_{43}=x_{15}\{x_3x_{20}^2\} \xrightarrow{\delta} -x_8x_{15}x_{20}^2+v_1C$; $v_2$-torsion free.

(5) $w_{55}=\{x_{15}x_{20}^2\} \xrightarrow{i} x_{15}x_{20}^2$; $v_2$-torsion and $v_2w_{55}=v_1w_{43}$.

(6) $w_{43}=\{x_3x_{20}^2-x_8x_{15}x_{20}\} \xrightarrow{i} \bar{w}_{43}-x_8x_{15}x_{20}$; $v_2$-torsion free.

## References


[1] J.F. Adams, Stable Homotopy and Generalised Homology Theory, Univ. Chicago Press, 1970.

[2] S. Araki, Differential Hopf algebras and the cohomology mod 3 of compact exceptional groups $E_7$ and $E_8$, Ann. of Math., **73** (1961), 404-436.

[3] S. Araki and H. Toda, Multiplicative structure in mod $q$ cohomology theories I, Osaka J. Math., **2** (1965), 71-115.

[4] W. Browder, On differential Hopf algebras, Trans. Amer. Math. Soc., **107** (1963), 153-176.

[5] L. Hodgkin, On the K-theory of Lie groups, Topology, **7** (1967), 1-36.

[6] D.C. Johnson and W.S. Wilson, *BP*-operations and Morava's extraordinary K-theories, Math. Z., **144** (1975), 55-75.

[7] N. Yagita, The exact functor theorem for $BP_*/I_n$-theory, Proc. Japan Acad., **52** (1976), 1-3.

[8] N. Yagita, On the algebraic structure of cobordism operations with singularities, J. London Math. Soc., **16** (1977), 131-141.

[9] N. Yagita, On the Steenrod algebra of Morava K-theory (preprint).



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