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On the equations defining Kummer varieties

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Let K be the Kummer variety of an abelian variety X over the complex number field, i.e., the quotient of X by the inverse morphism $\iota: X\to X$, and let M be an ample invertible sheaf on K. Let $\pi: X\to K$ be the canonical surjection. Then ι^*M is of the form L^2 for some ample symmetric invertible sheaf L on X. Here L is said to be symmetric if $\iota^*L\cong L$. We denote by P the Poincaré invertible sheaf on the product of X and its dual $\hat{X}$. For a point α in $\hat{X}$, we denote by P_α the restriction $P|_{X\times\{\alpha\}}$. If α is contained in the subgroup $(\hat{X})_2$ consisting of two division points in $\hat{X}$, then $L^n\otimes P_\alpha$ is symmetric. For a section s in $\Gamma(X, L^n\otimes P_\alpha)$ with $\alpha\in(\hat{X})_2$, we denote by $\bar{s}$ the image of s via the canonical involution of $\Gamma(X, L^n\otimes P_\alpha)$ induced by the inversion ι. If we put $\Gamma(X, L^n)^+=\{s\in\Gamma(X, L^n)\,|\,s=\bar{s}\}$, then we can identify $\Gamma(K, M^n)$ with $\Gamma(X, L^{2n})^+$ through the canonical map $\pi^*: \Gamma(K, M^n)\to\Gamma(X, L^{2n})$. For a point α in $(\hat{X})_2$ and sections s_1, s_2, s_3 and s_4 in $\Gamma(X, L^n\otimes P_\alpha)$, we define an element

$$
\begin{aligned}
q^{(\alpha)}(s_1, s_2, s_3, s_4)=&[(s_1+\bar{s}_1)\cdot(s_2+\bar{s}_2)]\odot[(s_3+\bar{s}_3)\cdot(s_4+\bar{s}_4)]\\
&+[(s_1-\bar{s}_1)\cdot(s_2-\bar{s}_2)]\odot[(s_3-\bar{s}_3)\cdot(s_4-\bar{s}_4)]\\
&-[(s_1+\bar{s}_1)\cdot(s_4+\bar{s}_4)]\odot[(s_2+\bar{s}_2)\cdot(s_3+\bar{s}_3)]\\
&-[(s_1-\bar{s}_1)\cdot(s_4-\bar{s}_4)]\odot[(s_2-\bar{s}_2)\cdot(s_3-\bar{s}_3)]
\end{aligned}
$$

in the symmetric product $\mathcal{S}^2(\Gamma(X, L^{2n})^+)=\mathcal{S}^2\Gamma(K, M^n)$, where $s\cdot t$ denotes the image of $s\otimes t$ via the canonical map

$$\Gamma(X, L^n\otimes P_\alpha)\otimes\Gamma(X, L^n\otimes P_\alpha)\longrightarrow\Gamma(X, L^{2n})$$

and the symbol $\odot$ denotes the symmetric product. Then our first result (Theorem 4.1) is as follows.

Assume $n\geqq 2$. *Then the kernel of the canonical map*

$$\mathcal{S}^2[\Gamma(X, L^{2n})^+]\longrightarrow\Gamma(X, L^{4n})^+$$

is spanned by $\{q^{(\alpha)}(s_1, s_2, s_3, s_4)\,|\,\alpha\in(\hat{X})_2$ *and* $s_i\in\Gamma(X, L^n\otimes P_\alpha)\}$.

Using this result, we shall show the following which is the second result in this paper.

Let $I^{(n)}$ be the kernel of the canonical graded ring homomorphism

$$S=\bigoplus_{k=0}^{\infty} \mathcal{S}^k \Gamma(K, M^n) \longrightarrow \bigoplus_{k=0}^{\infty} \Gamma(K, M^{nk}).$$

Assume $n \geqq 4$. If $J^{(n)}$ denotes the ideal $S \cdot I_2^{(n)}$ generated by $I_2^{(n)}$, then we have

$$J_m^{(n)}=I_m^{(n)}$$

for almost all $m \in \boldsymbol{Z}$. Here $J_m^{(n)}$ and $I_m^{(n)}$ denote the m-th homogeneous part of $J^{(n)}$ and $I^{(n)}$, respectively.

We notice here that the ring homomorphism in the statement is surjective for $n \geqq 2$ (cf. Theorem 1.4 in [7]), and that the result stated above is finer than that of Theorem 2.1 in [7] in the case when $n \geqq 4$.

In Section 1, we recall some fundamental facts from the theory of theta functions. Section 2 is devoted to preparations for proving our first result stated above and we prove it in Section 3. The second result is proved in the last section 4. The content of this section is similar to that of § 2 in [8].

Notation.

For a vector space V over a field F, we denote by $\mathcal{S}^n V$ the n-th symmetric power of V. The image, in $\mathcal{S}^n V$, of an element $a_1 \otimes \cdots \otimes a_n$ in the space $V \otimes \cdots \otimes V$ of the n-th tensor product of V, is denoted by $a_1 \odot \cdots \odot a_n$. Let V_i $(i=0, 1, \cdots, n)$ be an F-space and $f: V_1 \otimes \cdots \otimes V_n \rightarrow V_0$ an F-linear map. If no confusion occurs, we write $v_1 \cdot v_2 \cdot \cdots \cdot v_n$ for the image $f(v_1 \otimes v_2 \otimes \cdots \otimes v_n)$ of $v_1 \otimes \cdots \otimes v_n \in V_1 \otimes \cdots \otimes V_n$.

Let e be a diagonal g-matrix with coefficients in $\boldsymbol{Z}$ and $\det e \neq 0$. For a positive integer a, we denote by $R(ae)$ a complete set of representatives of $(ae)^{-1} \boldsymbol{Z}^g$ modulo $\boldsymbol{Z}^g$. In particular we write $R(a)$ instead of $R(a \cdot 1_g)$ in which 1_g is the identity matrix of size g. $G(ae)$ and $G(a)$ denote the groups $(ae)^{-1} \boldsymbol{Z}^g / \boldsymbol{Z}^g$ and $a^{-1} \boldsymbol{Z}^g / \boldsymbol{Z}^g$, respectively. For an element p in $R(ae)$ or $R(a)$, we denote by $\bar{p}$ the equivalence class in $G(ae)$ or $G(a)$ defined by p. Conversely, for an element q in $G(ae)$ or $G(a)$, we denote by $\tilde{q}$ the element in $R(ae)$ or $R(a)$ which induces q.

For a commutative group G, we denote by G^* the character group of G. For a finite set S, we denote by $\operatorname{Card} S$ the cardinality of S.

§ 1. Preliminaries.

In this section we recall some formulas of theta series and give some elementary results of even or odd theta functions. Let $k=\binom{k_1}{k_2}$ be a vector in $\boldsymbol{R}^{2g}$ with k_1 and k_2 in $\boldsymbol{R}^g$. By the classical theta series with characteristic k, we understand the series

(1.0) $$\theta[k](z\,|\,x)=\sum_{m\in Z^g} e\left\{\frac{1}{2}{}^t(m+k_1)z(m+k_1)+{}^t(m+k_1)(x+k_2)\right\},$$

which defines a function holomorphic on both variables x in C^g and z in the Siegel upper half space $H_g=\{z\in M_g(C)\,|\,{}^tz=z,\ \mathcal{I}m\,z>0\}$. Here $e(a)$ for $a\in C$ means $\exp(2\pi\sqrt{-1}\,a)$. The following four formulas are fundamental ([1] pp. 49-50 and p. 139):

(1.1) $$\theta[k](z\,|\,x)=\theta[-k](z\,|\,-x),$$

(1.2) $$\theta[k+s](z\,|\,x)=e({}^ts_1s_2)\theta[k](z\,|\,x)\quad \text{for } s=\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\in Z^{2g},$$

(1.3) $$\theta[k+l](z\,|\,x)=e\left\{\frac{1}{2}{}^tl_1zl_1+{}^tl_1(x+k_2+l_2)\right\}\theta[k](z\,|\,x+(z,\,1_g)l)$$

for $l=\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\in R^{2g}$ and

(1.4) $$\theta[k](z\,|\,x)\cdot\theta[l](z\,|\,y)=\sum_{p\in R(2)}\theta\begin{bmatrix}(1/2)(k_1+l_1)+p \\ k_2+l_2\end{bmatrix}(2z\,|\,x+y)$$
$$\times\theta\begin{bmatrix}(1/2)(k_1-l_1)+p \\ k_2-l_2\end{bmatrix}(2z\,|\,x-y)$$

for k and $l\in R^{2g}$. The following immediately comes from (1.4):

(1.5) $$\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}k_1+p \\ k_2\end{bmatrix}(z\,|\,x)\cdot\theta\begin{bmatrix}l_1+p \\ l_2\end{bmatrix}(z\,|\,y)$$
$$=\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}(1/2)(k_1+l_1)+p \\ k_2+l_2\end{bmatrix}(2z\,|\,x+y)\right)$$
$$\times\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}(1/2)(k_1-l_1)+p \\ k_2-l_2\end{bmatrix}(2z\,|\,x-y)\right)$$

for a character χ in $G(2)^*$. We define even and odd theta functions by use of $\theta[k](z\,|\,x)$ in the following manner:

(1.6) $$\eta[k](z\,|\,x)=\theta[k](z\,|\,x)+\theta[-k](z\,|\,x)$$

and

(1.7) $$\zeta[k](z\,|\,x)=\theta[k](z\,|\,x)-\theta[-k](z\,|\,x)$$

Then we have easily the following:

$$(1.8)\qquad \eta[k](z|x)\cdot\eta[l](z|x)\qquad (\text{resp. } \zeta[k](z|x)\cdot\zeta[l](z|x))$$

$$=\sum_{p\in R(2)}\theta\begin{bmatrix}(1/2)(k_1-l_1)+p\\ k_2-l_2\end{bmatrix}(2z|0)\cdot\eta\begin{bmatrix}(1/2)(k_1+l_1)+p\\ k_2+l_2\end{bmatrix}(2z|2x)$$

$$+(\text{resp. } -)\sum_{p\in R(2)}\theta\begin{bmatrix}(1/2)(k_1+l_1)+p\\ k_2+l_2\end{bmatrix}(2z|0)\cdot\eta\begin{bmatrix}(1/2)(k_1-l_1)+p\\ k_2-l_2\end{bmatrix}(2z|2x)$$

and

$$(1.9)\qquad \sum_{p\in R(2)}\chi(\bar{p})\eta\begin{bmatrix}k_1+p\\ k_2\end{bmatrix}(z|x)\cdot\eta\begin{bmatrix}l_1+p\\ l_2\end{bmatrix}(z|x)$$

$$\left(\text{resp. } \sum_{p\in R(2)}\chi(\bar{p})\zeta\begin{bmatrix}k_1+p\\ k_2\end{bmatrix}(z|x)\cdot\zeta\begin{bmatrix}l_1+p\\ l_2\end{bmatrix}(z|x)\right)$$

$$=\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}(1/2)(k_1-l_1)+p\\ k_2-l_2\end{bmatrix}(2z|0)\right)$$

$$\times\left(\sum_{p\in R(2)}\chi(\bar{p})\eta\begin{bmatrix}(1/2)(k_1+l_1)+p\\ k_2+l_2\end{bmatrix}(2z|2x)\right)$$

$$+(\text{resp. } -)\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}(1/2)(k_1+l_1)+p\\ k_2+l_2\end{bmatrix}(2z|0)\right)$$

$$\times\left(\sum_{p\in R(2)}\chi(\bar{p})\eta\begin{bmatrix}(1/2)(k_1-l_1)+p\\ k_2-l_2\end{bmatrix}(2z|2x)\right)$$

for vectors k and l in $\boldsymbol{R}^{2g}$ and a character χ in $G(2)^*$.

§ 2. Even theta functions.

Throughout this section, we fix a point z in $\boldsymbol{H}_g$ and a diagonal g-matrix e with coefficients in $\boldsymbol{Z}$ and $\det e\neq 0$. For a non-negative integer n and a vector $k=\begin{pmatrix}k_1\\ k_2\end{pmatrix}$ in $\boldsymbol{R}^{2g}$, an entire function $f(x)$ on $\boldsymbol{C}^g$ is called a *theta function of type* $((z, e), k)_n$, if the functional equation

$$f(x+(z, e)s)=\boldsymbol{e}\left\{n\left(-\frac{1}{2}{}^ts_1zs_1-{}^ts_1x+{}^tsk\right)\right\}f(x)$$

holds for any $s=\begin{pmatrix}s_1\\ s_2\end{pmatrix}$ in $\boldsymbol{Z}^{2g}$. We denote by $\Theta_n((z, e), k)$ the totality of theta functions of type $((z, e), k)_n$. It is well known that

$$\Theta_n\left((z, e), \begin{pmatrix} -ek_2 \\ ek_1 \end{pmatrix}\right) = \bigoplus_{a \in R(ne)} \boldsymbol{C} \cdot \theta \begin{bmatrix} k_1 + a \\ nek_2 \end{bmatrix} (nz | nx)$$

and that

$$\Theta_n\left((z, e), \begin{pmatrix} -ek_2 \\ ek_1 \end{pmatrix}\right) \cdot \Theta_m\left((z, e), \begin{pmatrix} -el_2 \\ el_1 \end{pmatrix}\right)$$
$$\subset \Theta_{n+m}\left((z, e), \frac{1}{n+m}\begin{pmatrix} -e(nk_2+ml_2) \\ e(nk_1+ml_1) \end{pmatrix}\right).$$

For simplicity we write Θ_n instead of $\Theta_n((z, e), 0)$. We denote by Θ_n^+ (resp. Θ_n^-) the subspace of Θ_n consisting of even (resp. odd) theta functions of type $((z, e), 0)_n$.

We say that a and b in $G(2ne)$ are equivalent if $b=-a$, and denote by $G(2ne)^+$ a complete set of representatives of $G(2ne)$ modulo this equivalence. Then we have the following which is easily seen.

LEMMA 2.1. *Under the above notation, we have*

(i) $\Theta_{2n}^+ = \bigoplus_{a \in G(2ne)^+} \boldsymbol{C} \cdot \eta \begin{bmatrix} \tilde{a} \\ 0 \end{bmatrix} (2nz | 2nx)$,

(ii) $\Theta_{2n}^- = \bigoplus_{a \in G(2ne)^+ - G(2)} \boldsymbol{C} \cdot \zeta \begin{bmatrix} \tilde{a} \\ 0 \end{bmatrix} (2nz | 2nx)$,

(iii) $\dim \Theta_{2n}^+ = 2^{g-1}(n^g |\det e| + 1)$ and $\dim \Theta_{2n}^- = 2^{g-1}(n^g |\det e| - 1)$

and

(iv) $\Theta_{2n} = \Theta_{2n}^+ \oplus \Theta_{2n}^-$.

For $a \in G(4ne)$ and $\chi \in G(2)^*$, we put

$$\eta_{4n}(\chi;\, a) = \sum_{p \in G(2)} \chi(p) \eta \begin{bmatrix} \tilde{a} + \tilde{p} \\ 0 \end{bmatrix} (4nz | 4nx) \tag{2.1}$$

and

$$\theta_{4n}(\chi;\, a) = \sum_{p \in G(2)} \chi(p) \eta \begin{bmatrix} \tilde{a} + \tilde{p} \\ 0 \end{bmatrix} (4nz | 0). \tag{2.2}$$

Let a and b be elements in $G(4) - G(2)$. If a and b are congruent modulo $G(2)$, then we write $a \sim b$. We denote by $S(4)$ a complete set of representatives of $G(4) - G(2)$ with respect to the equivalence relation "$\sim$". Moreover, for a and b in $G(4ne) - G(4)$, we write $a \approx b$ if either $a+b$ or $a-b$ is contained in $G(2)$, and we denote by $T(4n)$ a complete set of representatives of $G(4ne) - G(4)$ modulo the equivalence relation "$\approx$". Under the notation above, we define three subspaces of Θ_{4n}^+ in the following way:

$$V(2) = \{\eta_{4n}(\chi;\, 0) \mid \chi \in G(2)^*\}_{C},$$

$$V(4) = \{\eta_{4n}(\chi;\, a) \mid (\chi, a) \in G(2)^* \times \mathrm{S}(4) \text{ with } \chi(2a) = 1\}_{C}$$

and

$$W(4n)=\{\eta_{4n}(\chi;\, a) \mid (\chi,\, a)\in G(2)^{*}\times T(4n)\}_{C}\,.$$

LEMMA 2.2. *Let a be an element in $G(4)$ and χ a character of $G(2)$ with $\chi(2a)=-1$. Then we have $\eta_{4n}(\chi;\, a)=0$.*

PROOF. It is obvious that $\eta_{4n}(\chi;\, a)=\eta_{4n}(\chi;\, -a)$. On the other hand, we have

$$\begin{aligned}\eta_{4n}(\chi;\, -a)&=\sum_{p\in G(2)}\chi(p)\eta\begin{bmatrix}-\tilde{a}+\tilde{p}\\ 0\end{bmatrix}(4nz\,|\,4nx)\\ &=\sum_{p\in G(2)}\chi(p)\chi(2a)\eta\begin{bmatrix}\tilde{a}+\tilde{p}\\ 0\end{bmatrix}(4nz\,|\,4nx)\\ &=\chi(2a)\eta_{4n}(\chi;\, a)\,.\end{aligned}$$

Therefore, if $\chi(2a)=-1$, then we have $\eta_{4n}(\chi;\, a)=0$. Q. E. D.

LEMMA 2.3. *Notation being as above, we have*

(i) $\dim V(2)=2^{g}$,

(ii) $\dim V(4)=2^{g-1}(2^{g}-1)$,

(iii) $\dim W(4n)=2^{2g-1}(n^{g}|\det e|-1)$

and

(iv) $\Theta_{4n}^{+}=V(2)\oplus V(4)\oplus W(4n)$.

PROOF. If p is an element in $G(2)$, then we have

$$\begin{aligned}\sum_{\chi\in G(2)^{*}}\chi(p)\eta_{4n}(\chi;\, 0)&=\sum_{\chi\in G(2)^{*}}\sum_{q\in G(2)}\chi(p+q)\eta\begin{bmatrix}\tilde{q}\\ 0\end{bmatrix}(4nz\,|\,4nx)\\ &=2^{g}\eta\begin{bmatrix}\tilde{p}\\ 0\end{bmatrix}(4nz\,|\,4nx)\\ &=2^{g+1}\theta\begin{bmatrix}\tilde{p}\\ 0\end{bmatrix}(4nz\,|\,4nx)\,.\end{aligned}$$

Therefore we have $\dim V(2)=2^{g}$. Since $\eta\begin{bmatrix}\tilde{a}\\ 0\end{bmatrix}(4nz\,|\,4nx)$, $a\in S'(4)$, are linearly independent and $\operatorname{Card} S(4)=2^{g-1}(2^{g}-1)$, it is sufficient for proving (ii) to show that $V(4)=\left\{\eta\begin{bmatrix}\tilde{a}\\ 0\end{bmatrix}(4nz\,|\,4nx)\,\middle|\, a\in S'(4)\right\}_{C}$. By the definition of $V(4)$, we see that $V(4)$ is contained in $\left\{\eta\begin{bmatrix}\tilde{a}\\ 0\end{bmatrix}(4nz\,|\,4nx)\,\middle|\, a\in S'(4)\right\}_{C}$. For any $a\in S'(4)$, we have, by Lemma 2.2,

$$\begin{aligned}\sum_{\substack{\chi\in G(2)^{*}\\ \chi(2a)=1}}\eta_{4n}(\chi;\, a)&=\sum_{\chi\in G(2)^{*}}\eta_{4n}(\chi;\, a)\\ &=\sum_{p\in G(2)}\sum_{\chi\in G(2)^{*}}\chi(p)\eta\begin{bmatrix}\tilde{a}+\tilde{p}\\ 0\end{bmatrix}(4nz\,|\,4nx)\end{aligned}$$

$$=2^g\eta\begin{bmatrix}\tilde{a}\\0\end{bmatrix}(4nz|4nx).$$

This shows the converse. As for (iii), let a and b be elements in $G(4ne)-G(4)$. Then we write $a\approx b$ if $b=-a$. We denote by $T(4n)'$ a complete set of representatives of $G(4ne)-G(4)$ modulo the equivalence relation "$\approx$". Then we have Card $T(4n)'=2^{2g-1}(n^g|\det e|-1)$. We easily see that $W(4n)$ is contained in the space $\left\{\eta\begin{bmatrix}\tilde{a}\\0\end{bmatrix}(4nz|4nx)\,\middle|\, a\in T(4n)'\right\}_C$. Conversely let a be an element in $T(4n)'$. Then we have $a=a_0+p_0$ or $a=-a_0+p_0$ for some $a_0\in T(4n)$ and $p_0\in G(2)$, and have

$$\sum_{\chi\in G(2)^*}\chi(p_0)\eta_{4n}(\chi;\,a_0)=\sum_{\chi\in G(2)^*}\chi(p_0)\eta_{4n}(\chi;\,-a_0)$$

$$=\sum_{p\in G(2)}\sum_{\chi\in G(2)^*}\chi(p_0+p)\eta\begin{bmatrix}\tilde{a}_0+\tilde{p}\\0\end{bmatrix}(4nz|4nx)$$

$$=2^g\eta\begin{bmatrix}\tilde{a}_0+\tilde{p}_0\\0\end{bmatrix}(4nz|4nx)$$

$$=2^g\eta\begin{bmatrix}\tilde{a}\\0\end{bmatrix}(4nz|4nx).$$

Thus we have shown (iii). The (iv) is an immediate consequence of (i), (ii) and (iii). Q. E. D.

§ 3. Quadratic relations of even theta functions.

Throughout this section, as in § 2, we fix a diagonal g-matrix e with coefficients in $\boldsymbol{Z}$ and $\det e\neq 0$, and a point z in $\boldsymbol{H}_g$. For a and b in $G(2ne)$ and a character χ of $G(2)$, we put

$$\eta_{2n}(\chi;\,a,\,b)=\sum_{p\in G(2)}\chi(p)\eta\begin{bmatrix}\tilde{a}+\tilde{p}\\0\end{bmatrix}(2nz|2nx)\odot\eta\begin{bmatrix}\tilde{b}+\tilde{p}\\0\end{bmatrix}(2nz|2nx)\tag{3.1}$$

and

$$\theta_{2n}(\chi;\,a,\,b)=\sum_{p\in G(2)}\chi(p)\theta\begin{bmatrix}\tilde{a}+\tilde{p}\\0\end{bmatrix}(2nz|0)\cdot\theta\begin{bmatrix}\tilde{b}+\tilde{p}\\0\end{bmatrix}(2nz|0).\tag{3.2}$$

Then we have the following which can be proved by the same method as the proof of Lemma 2.2, so we shall omit its proof.

LEMMA 3.1. *Let a be an element of $G(4)$ and χ a character of $G(2)$ with $\chi(2a)=-1$. Then we have $\eta_{2n}(\chi;\,a+b,\,a-b)=0$ for any b in $G(4ne)$ with $a\equiv b$* mod $G(2ne)$.

The following lemma will be used in the proof of Theorem 3.3 and was proved by Koizumi ([3] Corollary 1.3).

LEMMA 3.2. *Assume* $n \geqq 2$. *Then, for any* $(\chi, a) \in G(2)^* \times G(4ne)$, *there exists an element* $\xi(\chi, a)$ *in* $G(4ne)$ *such that* $\xi(\chi, a) \equiv a \mod G(2ne)$ *and* $\theta_{4n}(\chi; \xi(\chi, a)) \neq 0$.

For a character χ of $G(2)$ and four elements a, b, c and d in $G(4ne)$ with $a \equiv b \equiv c \equiv d \mod G(2ne)$, we put

$$
\begin{aligned}
(3.3) \qquad P(\chi; a, b, c, d) = & \theta_{2n}(\chi; c+d, c-d)\eta_{2n}(\chi; a+b, a-b) \\
& + \theta_{2n}(\chi; a+b, a-b)\eta_{2n}(\chi; c+d, c-d) \\
& - \theta_{2n}(\chi; a+c, a-c)\eta_{2n}(\chi; b+d, b-d) \\
& - \theta_{2n}(\chi; b+d, b-d)\eta_{2n}(\chi; a+c, a-c).
\end{aligned}
$$

Then we have the following:

THEOREM 3.3. *Let* $\phi: \mathcal{S}^2\Theta_{2n}^+ \to \Theta_{4n}^+$ *be the canonical mapping. Assume* $n \geqq 2$. *Then the kernel of* ϕ *is the linear closure of* $\{P(\chi; a, b, c, d) \mid \chi \in G(2)^*, a, b, c, d \in G(4ne)$ *with* $a \equiv b \equiv c \equiv d \mod G(2ne)\}$.

PROOF. We denote by I the linear closure in the statement. First of all we shall show that I is contained in $\ker \phi$. By (1.5), (1.9), (3.1) and (3.2), we have

$$
\begin{aligned}
\phi(P(\chi; a, b, c, d)) = & \theta_{4n}(\chi; c)\theta_{4n}(\chi; d)\{\theta_{4n}(\chi; a)\eta_{4n}(\chi; b) + \theta_{4n}(\chi; b)\eta_{4n}(\chi; a)\} \\
& + \theta_{4n}(\chi; a)\theta_{4n}(\chi; b)\{\theta_{4n}(\chi; c)\eta_{4n}(\chi; d) + \theta_{4n}(\chi; d)\eta_{4n}(\chi; c)\} \\
& - \theta_{4n}(\chi; a)\theta_{4n}(\chi; c)\{\theta_{4n}(\chi; b)\eta_{4n}(\chi; d) + \theta_{4n}(\chi; d)\eta_{4n}(\chi; b)\} \\
& - \theta_{4n}(\chi; b)\theta_{4n}(\chi; d)\{\theta_{4n}(\chi; a)\eta_{4n}(\chi; c) + \theta_{4n}(\chi; c)\eta_{4n}(\chi; a)\} \\
= & 0.
\end{aligned}
$$

Next we shall show that $I \supset \ker \phi$. We denote by R the disjoint union of $G(2)^*$, $(G(2)^* \times S(4))' = \{(\chi, a) \mid (\chi, a) \in G(2)^* \times S(4), \chi(2a) = 1\}$ and $G(2)^* \times T(4n)$. Then we define a map $F: R \to \mathcal{S}^2\Theta_{2n}^+$ in the following manner: For each $(\chi, a) \in G(2)^* \times G(4ne)$, by Lemma 3.2, we have an element $\xi(\chi, a)$ in $G(4ne)$ such that $a \equiv \xi(\chi, a) \mod G(2ne)$ and $\theta_{4n}(\chi; \xi(\chi, a)) \neq 0$. We take a as $\xi(\chi, a)$ if $\theta_{4n}(\chi; a) \neq 0$ and may assume that $\xi(\chi, -a) = -\xi(\chi, a)$. Then we put $F((\chi, a)) = \eta_{2n}(\chi; a + \xi(\chi, a), a - \xi(\chi, a))$ for $(\chi, a) \in R$. We denote by W the linear subspace of $\mathcal{S}^2\Theta_{2n}^+$ spanned by the set $\{F((\chi, a)) \mid (\chi, a) \in R\}$. Then we see that $\phi|_W: W \to \Theta_{4n}^+$ is an isomorphism. In fact we have

$$
\begin{aligned}
\phi(F(\chi, a)) &= \theta_{4n}(\chi; a)\eta_{4n}(\chi; \xi(\chi, a)) + \theta_{4n}(\chi; \xi(\chi, a))\eta_{4n}(\chi; a) \\
&= \begin{cases} 2\cdot\theta_{4n}(\chi; a)\eta_{4n}(\chi; a) & \text{if } \theta_{4n}(\chi; a) \neq 0 \\ \theta_{4n}(\chi; \xi(\chi, a))\eta_{4n}(\chi; a) & \text{if } \theta_{4n}(\chi; a) = 0. \end{cases}
\end{aligned}
$$

Since $\{\eta_{4n}(\chi; a) \mid (\chi, a)\in R\}$ is a basis of Θ_{4n}^{+}, we see that $\phi|_W$ is an isomorphism. Now we shall show that $\mathcal{S}^2\Theta_{2n}^{+}=W\oplus I$. It is easily seen that $\mathcal{S}^2\Theta_{2n}^{+}$ is spanned by $\{\eta_{2n}(\chi; a, b) \mid \chi\in G(2)^*,\ a, b\in G(2ne)\}$; hence by $\{\eta_{2n}(\chi; a+b, a-b)\mid \chi\in G(2)^*$ and $a, b\in G(4ne)$ with $a\equiv b \bmod G(2ne)\}$. Let a and b be elements in $G(4ne)$ with $a\equiv b \bmod G(2ne)$ and χ a character of $G(2)$. Set $a=a_0+p_0$ and $b=b_0+q_0$ with $p_0, q_0\in G(2)$. Then we have

$$
\begin{aligned}
&P(\chi; a, b, \xi(\chi, a_0)+p_0, \xi(\chi, b_0)+q_0)\\
&\quad=\theta_{4n}(\chi; \xi(\chi, a_0)+p_0)\theta_{4n}(\chi; \xi(\chi, b_0)+q_0)\eta_{2n}(\chi; a+b, a-b)\\
&\qquad+\theta_{4n}(\chi; a)\theta_{4n}(\chi; b)\eta_{2n}(\chi; \xi(\chi, a_0)+p_0+\xi(\chi, b_0)+q_0, \xi(\chi, a_0)+p_0-\xi(\chi, b_0)-q_0)\\
&\qquad-\theta_{4n}(\chi; a)\theta_{4n}(\chi; \xi(\chi, a_0)+p_0)\eta_{2n}(\chi; b_0+\xi(\chi, b_0), b_0-\xi(\chi, b_0))\\
&\qquad-\theta_{4n}(\chi; b)\theta_{4n}(\chi; \xi(\chi, b_0)+q_0)\eta_{2n}(\chi; a_0+\xi(\chi, a_0), a_0-\xi(\chi, a_0))\\
&\quad=1\ (\text{resp. } 2)\times\theta_{4n}(\chi; \xi(\chi, a_0)+p_0)\theta_{4n}(\chi; \xi(\chi, b_0)+q_0)\eta_{2n}(\chi; a+b, a-b)\\
&\qquad-\theta_{4n}(\chi; a)\theta_{4n}(\chi; \xi(\chi, a_0)+p_0)\eta_{2n}(\chi; b_0+\xi(\chi, b_0), b_0-\xi(\chi, b_0))\\
&\qquad-\theta_{4n}(\chi; b)\theta_{4n}(\chi; \xi(\chi, b_0)+q_0)\eta_{2n}(\chi; a_0+\xi(\chi, a_0), a_0-\xi(\chi, a_0))
\end{aligned}
$$

if $\theta_{4n}(\chi; a)\theta_{4n}(\chi; b)=0$ (resp. $\theta_{4n}(\chi; a)\theta_{4n}(\chi; b)\neq 0$). If (χ, a_0) or $(\chi, -a_0)$ is contained in R, it follows that $F((\chi, a_0))$ or $F((\chi, -a_0))=\eta_{2n}(\chi; a_0+\xi(\chi, a_0), a_0-\xi(\chi, a_0))$. If a_0 is contained in $S(4)$ and $\chi(2a_0)=-1$, then we have, by Lemma 3.1, $\eta_{2n}(\chi; a_0+\xi(\chi, a_0), a_0-\xi(\chi, a_0))=0$. Similar results hold for b_0. These arguments show that $\eta_{2n}(\chi; a+b, a-b)$ is contained in $W\oplus I$. Q. E. D.

Now we shall give another system of generators of the space $\ker\phi$. For s_1 and s_2 in $R(2)$, we write $\Theta_n(s)$ instead of $\Theta_n\left((z, e), \frac{1}{n}\begin{pmatrix}-s_2\\ s_1\end{pmatrix}\right)$ in which $s=\begin{pmatrix}s_1\\ s_2\end{pmatrix}$. Then we see that

$$\Theta_n(s)\cdot\Theta_n(s)\subset\Theta_{2n}=\Theta_{2n}((z, e), 0)\,.$$

If $f=f(x)$ is contained in $\Theta_n(s)$, then $\bar{f}=f(-x)$ is also contained in $\Theta_n(s)$. For $f_i\in\Theta_n(s)$ $(i=1, 2, 3, 4)$, we define the element

$$
\begin{aligned}
(3.4)\qquad q^{(s)}(f_1, f_2, f_3, f_4)=&[(f_1+\bar{f}_1)\cdot(f_2+\bar{f}_2)]\odot[(f_3+\bar{f}_3)\cdot(f_4+\bar{f}_4)]\\
&+[(f_1-\bar{f}_1)\cdot(f_2-\bar{f}_2)]\odot[(f_3-\bar{f}_3)\cdot(f_4-\bar{f}_4)]\\
&-[(f_1+\bar{f}_1)\cdot(f_4+\bar{f}_4)]\odot[(f_2+\bar{f}_2)\cdot(f_3+\bar{f}_3)]\\
&-[(f_1-\bar{f}_1)\cdot(f_4-\bar{f}_4)]\odot[(f_2-\bar{f}_2)\cdot(f_3-\bar{f}_3)]
\end{aligned}
$$

in the 2-th symmetric power $\mathcal{S}^2\Theta_{2n}^{+}$. Then we have the following:

THEOREM 3.4. *Assume $n\geqq 2$. Then the kernel of the canonical map $\phi: \mathcal{S}^2\Theta_{2n}^{+}\to\Theta_{4n}^{+}$ is the linear closure of*

$$\left\{q^{(s)}(f_1, f_2, f_3, f_4)\,\middle|\, s=\binom{s_1}{s_2} \text{ with } s_i\in R(2) \text{ and } f_i\in\Theta_n(s)\right\}.$$

PROOF. It is obvious that $\phi(q^{(s)}(f_1, f_2, f_3, f_4))=0$. Let a, b, c and d be elements in $R(4ne)$ such that $\bar{a}\equiv\bar{b}\equiv\bar{c}\equiv\bar{d} \bmod G(2ne)$, and let χ be a character of $G(2)$ and s_1, s_2 elements in $R(2)$. Then four even theta series

$$\eta\begin{bmatrix} a+b+c+d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx), \qquad \eta\begin{bmatrix} a+b-c-d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx),$$

$$\eta\begin{bmatrix} a-b+c-d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx), \qquad \eta\begin{bmatrix} a-b-c+d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx)$$

and four odd theta series

$$\zeta\begin{bmatrix} a+b+c+d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx), \qquad \zeta\begin{bmatrix} a+b-c-d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx)$$

$$\zeta\begin{bmatrix} a-b+c-d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx), \qquad \zeta\begin{bmatrix} a-b-c+d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx)$$

are contained in $\Theta_n\binom{s_1'}{s_2}$, in which $\bar{s}_1'=ne(\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{s}_1)\in G(2)$. We have easily the following: for p and q in $R(2)$,

$$\begin{aligned}\sum_{s_2\in R(2)} &\boldsymbol{e}(-4^t a s_2)\boldsymbol{e}(2^t s_2(a+b+p+s_1))\boldsymbol{e}(2^t s_2(a-b+q+s_1)) \\ &=\sum_{s_2\in R(2)}\boldsymbol{e}(2^t s_2(p+q)) \\ &=2^g \text{ (resp. 0)}\end{aligned}$$

if $p=q$ (resp. $p\neq q$). Therefore we have

$$\begin{aligned}\sum_{s_1\in R(2)}\chi(\bar{s}_1)\Bigg[\sum_{s_2\in R(2)}\boldsymbol{e}(-4^t a s_2)&\left\{\eta\begin{bmatrix} a+b+c+d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx)\eta\begin{bmatrix} a+b-c-d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx)\right\} \\ &\odot\left\{\eta\begin{bmatrix} a-b+c-d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx)\eta\begin{bmatrix} a-b-c+d+s_1 \\ s_2 \end{bmatrix}(nz\,|\,nx)\right\}\Bigg] \\ =\sum_{s_1\in R(2)}\chi(\bar{s}_1)\sum_{s_2\in R(2)}\boldsymbol{e}(-4^t a s_2)&\Bigg\{\sum_{p\in R(2)}\boldsymbol{e}(2^t s_2(a+b+p+s_1)) \\ &\left(\theta\begin{bmatrix} c+d+p \\ 0 \end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix} a+b+p+s_1 \\ 0 \end{bmatrix}(2nz\,|\,2nx)\right.\end{aligned}$$

$$+\theta\begin{bmatrix}a+b+p+s_1\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}c+d+p\\0\end{bmatrix}(2nz\,|\,2nx)\Big)\Big\}$$

$$\odot\Big\{\sum_{q\in R(2)}\boldsymbol{e}(2^t s_2(a-b+q+s_1))\Big(\theta\begin{bmatrix}c-d+q\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}a-b+q+s_1\\0\end{bmatrix}(2nz\,|\,2nx)$$

$$+\theta\begin{bmatrix}a-b+q+s_1\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}c-d+q\\0\end{bmatrix}(2nz\,|\,2nx)\Big)\Big\}$$

$$=2^g\sum_{p,s_1\in R(2)}\chi(\bar{s}_1)\Big\{\theta\begin{bmatrix}c+d+p\\0\end{bmatrix}(2nz\,|\,0)\theta\begin{bmatrix}c-d+p\\0\end{bmatrix}(2nz\,|\,0)$$

$$\times\eta\begin{bmatrix}a+b+p+s_1\\0\end{bmatrix}(2nz\,|\,2nx)\odot\eta\begin{bmatrix}a-b+p+s_1\\0\end{bmatrix}(2nz\,|\,2nx)$$

$$+\theta\begin{bmatrix}c+d+p\\0\end{bmatrix}(2nz\,|\,0)\theta\begin{bmatrix}a-b+p+s_1\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}a+b+p+s_1\\0\end{bmatrix}(2nz\,|\,2nx)$$

$$\odot\eta\begin{bmatrix}c-d+p\\0\end{bmatrix}(2nz\,|\,2nx)+\theta\begin{bmatrix}a+b+p+s_1\\0\end{bmatrix}(2nz\,|\,0)\theta\begin{bmatrix}c-d+p\\0\end{bmatrix}(2nz\,|\,0)$$

$$\times\eta\begin{bmatrix}c+d+p\\0\end{bmatrix}(2nz\,|\,2nx)\odot\eta\begin{bmatrix}a-b+p+s_1\\0\end{bmatrix}(2nz\,|\,2nx)$$

$$+\theta\begin{bmatrix}a+b+p+s_1\\0\end{bmatrix}(2nz\,|\,0)\theta\begin{bmatrix}a-b+p+s_1\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}c+d+p\\0\end{bmatrix}(2nz\,|\,2nx)$$

$$\odot\eta\begin{bmatrix}c-d+p\\0\end{bmatrix}(2nz\,|\,2nx)\Big\}$$

$$=2^g\Big\{\theta_{2n}(\chi;\,\bar{c}+\bar{d},\bar{c}-\bar{d})\eta_{2n}(\chi;\,\bar{a}+\bar{b},\bar{a}-\bar{b})+\theta_{2n}(\chi;\,\bar{a}+\bar{b},\bar{a}-\bar{b})\eta_{2n}(\chi;\,\bar{c}+\bar{d},\bar{c}-\bar{d})$$

$$+\Big(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}c+d+p\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}c-d+p\\0\end{bmatrix}(2nz\,|\,2nx)\Big)$$

$$\odot\Big(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}a-b+p\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}a+b+p\\0\end{bmatrix}(2nz\,|\,2nx)\Big)$$

$$+\Big(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}c-d+p\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}c+d+p\\0\end{bmatrix}(2nz\,|\,2nx)\Big)$$

$$\odot\Big(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}a+b+p\\0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}a-b+p\\0\end{bmatrix}(2nz\,|\,2nx)\Big)\Big\}.$$

By similar calculations, we have the following:

$$\sum_{s_1\in R(2)}\chi(\bar{s}_1)\sum_{s_2\in R(2)}e(-4^t a s_2)\left\{\zeta\begin{bmatrix}a+b+c+d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\zeta\begin{bmatrix}a+b-c-d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\right\}$$

$$\odot\left\{\zeta\begin{bmatrix}a-b+c-d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\zeta\begin{bmatrix}a-b-c+d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\right\}$$

$$=2^g\Big\{\theta_{2n}(\chi;\,\bar{c}+\bar{d},\bar{c}-\bar{d})\eta_{2n}(\chi;\,\bar{a}+\bar{b},\bar{a}-\bar{b})+\theta_{2n}(\chi;\,\bar{a}+\bar{b},\bar{a}-\bar{b})\eta_{2n}(\chi;\,\bar{c}+\bar{d},\bar{c}-\bar{d})$$

$$-\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}c+d+p\\ 0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}c-d+p\\ 0\end{bmatrix}(2nz\,|\,2nx)\right)$$

$$\odot\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}a-b+p\\ 0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}a+b+p\\ 0\end{bmatrix}(2nz\,|\,2nx)\right)$$

$$-\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}c-d+p\\ 0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}c+d+p\\ 0\end{bmatrix}(2nz\,|\,2nx)\right)$$

$$\odot\left(\sum_{p\in R(2)}\chi(\bar{p})\theta\begin{bmatrix}a+b+p\\ 0\end{bmatrix}(2nz\,|\,0)\eta\begin{bmatrix}a-b+p\\ 0\end{bmatrix}(2nz\,|\,2nx)\right)\Big\}$$

and

$$\sum_{s_1\in R(2)}\chi(\bar{s}_1)\sum_{s_2\in R(2)}e(-4^t a s_2)\Big[\Big\{\eta\begin{bmatrix}a+b+c+d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)$$

$$\times\eta\begin{bmatrix}a-b+c-d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\Big\}$$

$$\odot\left\{\eta\begin{bmatrix}a+b-c-d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\eta\begin{bmatrix}a-b-c+d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\right\}$$

$$+\left\{\zeta\begin{bmatrix}a+b+c+d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\zeta\begin{bmatrix}a-b+c-d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\right\}$$

$$\odot\left\{\zeta\begin{bmatrix}a+b-c-d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\zeta\begin{bmatrix}a-b-c+d+s_1\\ s_2\end{bmatrix}(nz\,|\,nx)\right\}\Big]$$

$$=2^{g+1}\Big\{\theta_{2n}(\chi;\,\bar{b}+\bar{d},\,\bar{b}-\bar{d})\eta_{2n}(\chi;\,\bar{a}+\bar{c},\,\bar{a}-\bar{c})$$

$$+\theta_{2n}(\chi;\,\bar{a}+\bar{c},\,\bar{a}-\bar{c})\eta_{2n}(\chi;\,\bar{b}+\bar{d},\,\bar{b}-\bar{d})\Big\}.$$

Therefore we have

$$\frac{1}{2^{g+1}}\sum_{s_1, s_2\in R(2)}\chi(\bar{s}_1)e(-4^t a s_2)q^{(s')}$$

$$\left(\theta\begin{bmatrix}a+b+c+d+s_1\\ s_2\end{bmatrix}(nz|nx),\ \theta\begin{bmatrix}a+b-c-d+s_1\\ s_2\end{bmatrix}(nz|nx),\right.$$

$$\left.\theta\begin{bmatrix}a-b-c+d+s_1\\ s_2\end{bmatrix}(nz|nx),\ \theta\begin{bmatrix}a-b+c-d+s_1\\ s_2\end{bmatrix}(nz|nx)\right)$$

$$=\theta_{2n}(\chi;\bar{c}+\bar{d},\bar{c}-\bar{d})\eta_{2n}(\chi;\bar{a}+\bar{b},\bar{a}-\bar{b})+\theta_{2n}(\chi;\bar{a}+\bar{b},\bar{a}-\bar{b})\eta_{2n}(\chi;\bar{c}+\bar{d},\bar{c}-\bar{d})$$

$$-\theta_{2n}(\chi;\bar{b}+\bar{d},\bar{b}-\bar{d})\eta_{2n}(\chi;\bar{a}+\bar{c},\bar{a}-\bar{c})-\theta_{2n}(\chi;\bar{a}+\bar{c},\bar{a}-\bar{c})\eta_{2n}(\chi;\bar{b}+\bar{d},\bar{b}-\bar{d})$$

$$=P(\chi;\bar{a},\bar{b},\bar{c},\bar{d}).$$

Here $s'=\begin{pmatrix}ne(a+b+c+d+s_1)\\ s_2\end{pmatrix}$. Hence, by Theorem 3.3, we see that $\{q^{(s)}(f_1, f_2, f_3, f_4)\mid s_i\in R(2), f_i\in\Theta_n(s)\}$ spans the kernel of ϕ. Q. E. D.

§ 4. The structure of the homogeneous coordinate ring of a Kummer variety.

Let X be an abelian variety over the complex number field $\boldsymbol{C}$. We denote by P the Poincaré invertible sheaf on the product of X and its dual $\hat{X}$ and by P_α the restriction $P|_{X\times\{\alpha\}}$ for $\alpha\in\hat{X}$. Let $\iota: X\to X$ be the inverse morphism defined by $x\mapsto -x$. An invertible sheaf L on X is said to be symmetric if $\iota^*L\cong L$. We may then assume that an isomorphism $f:\iota^*L\to L$ is normalized, i.e., $f(0): L(0)\cong\iota^*L(0)\to L(0)$ is the identity, where $L(0)$ is the fiber of L at the origin. We denote by $[-1]$ the canonical involution of $\Gamma(X, L^a)$ $(a\in\boldsymbol{Z}_+)$ induced by f and ι. We put $\Gamma(X, L^a)^+=\{s\in\Gamma(X, L^a)\mid[-1]s=s\}$. Then we have the following lemma:

LEMMA 4.1. *Let L be an ample symmetric invertible sheaf on an abelian variety X. If $a\geqq 3$, then the canonical map*

$$\Gamma(X, L^2)^+\otimes\Gamma(X, L^a\otimes P_\alpha)\longrightarrow\Gamma(X, L^{a+2}\otimes P_\alpha)$$

is surjective for any $\alpha\in\hat{X}$.

The proof of this lemma is given by Koizumi ([2] Appendix) in the case where $\dim\Gamma(X, L)=1$ and the general case is proved by the same method as the proof of Theorem 1.4 in [7].

Let K be the Kummer variety of X, i.e., the quotient of X by the group $\{1_X, \iota\}$, and $\pi: X\to K$ the canonical surjection. Let M be an ample invertible sheaf on K and $L=\pi^*M$. Then it is well-known that L is of the form $(L_0)^2$

for some ample symmetric invertible sheaf L_0. If α is contained in the subgroup $(\hat{X})_2$ of $\hat{X}$ consisting of 2-division points, then $L_0^n \otimes P_\alpha$ $(n \in \boldsymbol{Z})$ is also symmetric. For $\alpha \in (\hat{X})_2$ and $s_i \in \Gamma(X, L_0^n \otimes P_\alpha)$ $(i=1, 2, 3, 4)$, we define the element

$$\begin{aligned} q^{(\alpha)}(s_1, s_2, s_3, s_4) = & [(s_1+\bar{s}_1)\cdot(s_2+\bar{s}_2)] \odot [(s_3+\bar{s}_3)\cdot(s_4+\bar{s}_4)] \\ & + [(s_1-\bar{s}_1)\cdot(s_2-\bar{s}_2)] \odot [(s_3-\bar{s}_3)\cdot(s_4-\bar{s}_4)] \\ & - [(s_1+\bar{s}_1)\cdot(s_4+\bar{s}_4)] \odot [(s_2+\bar{s}_2)\cdot(s_3+\bar{s}_3)] \\ & - [(s_1-\bar{s}_1)\cdot(s_4-\bar{s}_4)] \odot [(s_2-\bar{s}_2)\cdot(s_3-\bar{s}_3)] \end{aligned}$$

in the symmetric product $\mathcal{S}^2(\Gamma(X, L_0^{2n})^+) = \mathcal{S}^2(\Gamma(X, L^n)^+)$, where $\bar{s}_i = [-1]s_i$. We here notice that $\Gamma(X, L^n)^+$ can be identified with $\Gamma(K, M^n)$ via the natural injection $\pi^*: \Gamma(K, M^n) \to \Gamma(X, L^n)$. Then the following is just algebro-geometric translation of Theorem 3.4.

THEOREM 4.2. *Assume* $n \geqq 2$. *Then the kernel of the canonical map:*

$$\mathcal{S}^2(\Gamma(X, L^n)^+) \longrightarrow \Gamma(X, L^{2n})^+$$

is the linear closure of $\{q^{(\alpha)}(s_1, s_2, s_3, s_4) \mid \alpha \in (\hat{X})_2$ *and* $s_i \in \Gamma(X, L_0^n \otimes P_\alpha)$ $(i=1, 2, 3, 4)$.

Let s_1, s_2, s_3 and s_4 be sections in $\Gamma(K, M^n)$. Then we define the element

$$r(s_1, s_2, s_3, s_4) = (s_1\cdot s_2) \odot (s_3\cdot s_4) - (s_1\cdot s_4) \odot (s_2\cdot s_3)$$

in $\mathcal{S}^2\Gamma(K, M^{2n})$.

LEMMA 4.3. *Assume* $n \geqq 4$. *Then the kernel of the canonical map:*

$$\mathcal{S}^2\Gamma(K, M^{2n}) \longrightarrow \Gamma(K, M^{4n})$$

is the linear closure of $\{r(s_1, s_2, s_3, s_4) \mid s_i \in \Gamma(K, M^n)$ $(i=1, 2, 3, 4)\}$.

PROOF. We identify $\Gamma(K, M^n)$ with $\Gamma(X, L^n)^+$. For $t_1, t_3 \in \Gamma(X, L^2)^+$; $s_1, s_2 \in \Gamma(X, L^{n-2} \otimes P_\alpha)$; $t_2, t_4 \in \Gamma(X, L^{n-2})^+$; $s_2, s_4 \in \Gamma(X, L^2 \otimes P_\alpha)$ $(\alpha \in (\hat{X})_2)$, we have

$$\begin{aligned} & q^{(\alpha)}(t_1\cdot s_1, t_2\cdot s_2, t_3\cdot s_3, t_4\cdot s_4) \\ & = [t_1\cdot(s_1+\bar{s}_1)\cdot t_2\cdot(s_2+\bar{s}_2)] \odot [t_3\cdot(s_3+\bar{s}_3)\cdot t_4\cdot(s_4+\bar{s}_4)] \\ & \quad + [t_1\cdot(s_1-\bar{s}_1)\cdot t_2\cdot(s_2-\bar{s}_2)] \odot [t_3\cdot(s_3-\bar{s}_3)\cdot t_4\cdot(s_4-\bar{s}_4)] \\ & \quad - [t_1\cdot(s_1+\bar{s}_1)\cdot t_4\cdot(s_4+\bar{s}_4)] \odot [t_2\cdot(s_2+\bar{s}_2)\cdot t_3\cdot(s_3+\bar{s}_3)] \\ & \quad - [t_1\cdot(s_1-\bar{s}_1)\cdot t_4\cdot(s_4-\bar{s}_4)] \odot [t_2\cdot(s_2-\bar{s}_2)\cdot t_3\cdot(s_3-\bar{s}_3)] \\ & = [t_1\cdot t_2\cdot(s_1+\bar{s}_1)\cdot(s_2+\bar{s}_2)] \odot [t_3\cdot t_4\cdot(s_3+\bar{s}_3)\cdot(s_4+\bar{s}_4)] \\ & \quad - [t_1\cdot t_2\cdot t_3\cdot t_4] \odot [(s_1+\bar{s}_1)\cdot(s_2+\bar{s}_2)\cdot(s_3+\bar{s}_3)\cdot(s_4+\bar{s}_4)] \\ & \quad + [t_1\cdot t_2\cdot t_3\cdot t_4] \odot [(s_1+\bar{s}_1)\cdot(s_2+\bar{s}_2)\cdot(s_3+\bar{s}_3)\cdot(s_4+\bar{s}_4)] \end{aligned}$$

$$-[t_1\cdot t_4\cdot(s_1+\bar{s}_1)\cdot(s_4+\bar{s}_4)]\odot[t_2\cdot t_3\cdot(s_2+\bar{s}_2)\cdot(s_3+\bar{s}_3)]$$
$$+[t_1\cdot t_2\cdot(s_1-\bar{s}_1)\cdot(s_2-\bar{s}_2)]\odot[t_3\cdot t_4\cdot(s_3-\bar{s}_3)\cdot(s_4-\bar{s}_4)]$$
$$-[t_1\cdot t_2\cdot t_3\cdot t_4]\odot[(s_1-\bar{s}_1)\cdot(s_2-\bar{s}_2)\cdot(s_3-\bar{s}_3)\cdot(s_4-\bar{s}_4)]$$
$$+[t_1\cdot t_2\cdot t_3\cdot t_4]\odot[(s_1-\bar{s}_1)\cdot(s_2-\bar{s}_2)\cdot(s_3-\bar{s}_3)\cdot(s_4-\bar{s}_4)]$$
$$-[t_1\cdot t_4\cdot(s_1-\bar{s}_1)\cdot(s_4-\bar{s}_4)]\odot[t_2\cdot t_3\cdot(s_2-\bar{s}_2)\cdot(s_3-\bar{s}_3)]$$
$$=r(t_1\cdot t_2,\ (s_1+\bar{s}_1)\cdot(s_2+\bar{s}_2),\ (s_3+\bar{s}_3)\cdot(s_4+\bar{s}_4),\ t_3\cdot t_4)$$
$$+r(t_1\cdot t_4,\ t_2\cdot t_3,\ (s_2+\bar{s}_2)\cdot(s_3+\bar{s}_3),\ (s_1+\bar{s}_1)\cdot(s_4+\bar{s}_4))$$
$$+r(t_1\cdot t_2,\ (s_1-\bar{s}_1)\cdot(s_2-\bar{s}_2),\ (s_3-\bar{s}_3)\cdot(s_4-\bar{s}_4),\ t_3\cdot t_4)$$
$$+r(t_1\cdot t_4,\ t_2\cdot t_3,\ (s_2-\bar{s}_2)\cdot(s_3-\bar{s}_3),\ (s_1-\bar{s}_1)\cdot(s_4-\bar{s}_4))\,.$$

Since $L\cong(L_0)^2$, by Lemma 4.1, we see that the canonical maps:

$$\Gamma(X,\ L^2)^+\otimes\Gamma(X,\ L^{n-2}\otimes P_\alpha)\longrightarrow\Gamma(X,\ L^n\otimes P_\alpha)$$

and

$$\Gamma(X,\ L^{n-2})^+\otimes\Gamma(X,\ L^2\otimes P_\alpha)\longrightarrow\Gamma(X,\ L^n\otimes P_\alpha)$$

are surjective for any $\alpha\in\hat{X}$. Let u_1, u_2, u_3 and u_4 be any sections in $\Gamma(L^n\otimes P_\alpha)$ $(\alpha\in(\hat{X})_2)$. Then, by the above surjectivity, we see that each u_i $(i=1, 2, 3, 4)$ takes the form of

$$u_i=\sum_j t_i^{(j)}\cdot s_i^{(j)}$$

in which $t_1^{(j)}$, $t_3^{(j)}\in\Gamma(X,\ L^2)^+$; $t_2^{(j)}$, $t_4^{(j)}\in\Gamma(X,\ L^{n-2})^+$; $s_1^{(j)}$, $s_3^{(j)}\in\Gamma(X,\ L^{n-2}\otimes P_\alpha)$; $s_2^{(j)}$, $s_3^{(j)}\in\Gamma(X,\ L^2\otimes P_\alpha)$. Therefore we have

$$q^{(\alpha)}(u_1,\ u_2,\ u_3,\ u_4)=\sum_{j,k,l,m}q^{(\alpha)}(t_1^{(j)}s_1^{(j)},\ t_2^{(k)}s_2^{(k)},\ t_3^{(l)}s_3^{(l)},\ t_4^{(m)}s_4^{(m)})\,.$$

Hence by Theorem 4.2 and the above arguments, we see that $\{r(s_1,\ s_2,\ s_3,\ s_4)\mid s_i\in\Gamma(K,\ M^n)\}$ spans the kernel of the map: $\mathcal{S}^2\Gamma(K,\ M^{2n})\to\Gamma(K,\ M^{4n})$. Q.E.D.

Let $I_k^{(n)}$ be the kernel of the canonical map:

$$\mathcal{S}^k\Gamma(K,\ M^n)\longrightarrow\Gamma(K,\ M^{nk})\,.$$

Then we have the following which is the essential part of Theorem 4.5.

LEMMA 4.4. *If $n\geqq 4$, then we have*

$$I_4^{(n)}=I_2^{(n)}\cdot\mathcal{S}^2\Gamma(K,\ M^n)\,.$$

PROOF. It is known that

$$\mathcal{S}^4\Gamma(K,\ M^n)/I_2^{(n)}\cdot\mathcal{S}^2\Gamma(K,\ M^n)\cong\mathcal{S}^2(\mathcal{S}^2\Gamma(K,\ M^n)/I_2^{(n)})/J\,,$$

where J is the linear closure of $\{r(s_1,\ s_2,\ s_3,\ s_4)\mid s_i\in\Gamma(K,\ M^n)\}$ (cf. [**1**], Chapter

IV Lemma 9). On the other hand, we have

$$\mathcal{S}^2\Gamma(K, M^n)/I_2^{(n)} \cong \Gamma(K, M^{2n})$$

and

$$\mathcal{S}^4\Gamma(K, M^n)/I_4^{(n)} \cong \Gamma(K, M^{4n}).$$

By Lemma 4.3, we have

$$\mathcal{S}^2\Gamma(K, M^{2n})/J \cong \Gamma(K, M^{4n}).$$

Thus we have the equation in the statement. Q. E. D.

Let $I^{(n)}$ be the kernel of the canonical graded ring homomorphism:

$$\mathcal{S}\Gamma(K, M^n) = \bigoplus_{k=0}^{\infty} \mathcal{S}^k\Gamma(K, M^n) \longrightarrow \bigoplus_{k=0}^{\infty} \Gamma(K, M^{nk}).$$

If $n \geqq 2$, this is surjective (cf. [7] Corollary 1.5); hence $\mathcal{S}\Gamma(K, M^n)/I^{(n)}$ is the homogeneous coordinate ring of the projective variety $\Phi_{M^n}(K)$ where $\Phi_{M^n}: K \to \boldsymbol{P}(\Gamma(K, M^n))$ is the closed immersion associated with the linear system $\Gamma(K, M^n)$. Let $J^{(n)}$ be the homogeneous ideal $I_2^{(n)} \cdot \mathcal{S}\Gamma(K, M^n)$ generated by $I_2^{(n)}$. $J_k^{(n)}$ denotes the homogeneous part of degree k. Then our second result of this paper is the following:

THEOREM 4.5. *If $n \geqq 4$, then we have*

(i) $J_{2^r}^{(n)} = I_{2^r}^{(n)}$ *for any positive integer r,*

(ii) $J_k^{(n)} = I_k^{(n)}$ *for almost all k.*

PROOF. It is known that (ii) comes from (i) (cf. [1] Chapter IV Lemma 3). Using Lemma 4.4, we can prove (i) by the same method as the proof of Theorem 4 in [1] Chapter IV. Q. E. D.

DEFINITION. A subvariety $X \subset \boldsymbol{P}^n$ is said to be *idealtheoretically an intersection of hypersurfaces $H_1, \cdots, H_m$* if set-theoretically

$$X = H_1 \cap \cdots \cap H_m$$

and every $x \in X$ has an affine neighborhood U in $\boldsymbol{P}^n$ such that the ideal $I(X)$ of $X \cap U$ in U is generated by the affine equations $f_1, \cdots, f_m$ of $H_1, \cdots, H_m$.

By means of this definition, we can restate the above theorem in the following form:

COROLLARY 4.6. *Let M be an ample invertible sheaf on a Kummer variety. If $n \geqq 4$, then the projective variety $\Phi_{M^n}(K)$ is ideal-theoretically the intersection of quadrics.*

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