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Analytic functions with finite Dirichlet integrals

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1. In the classification theory of Riemann surfaces (cf. e. g. Sario-Nakai [2]), the problem whether the inclusion $O_{AD}\subset O_{ABD}$ is strict or not had long been open and only recently the identity $O_{AD}=O_{ABD}$ is established by an elaborate work [1] of Sakai. On the other hand, Uy [4] also recently proved the following interesting theorem: If E is an arbitrary compact subset of the complex plane C with positive area, then there exists a nonconstant bounded analytic function $\phi(z)$ on $C-E$ satisfying the Lipschitz condition on $C-E$. We first *remark* here that the above theorem implies the identity $O_{AD}=O_{ABD}$. In fact, suppose there exists a nonconstant analytic function f on a Riemann surface R with the finite Dirichlet integral $D_R(f)=\iint_R|f'(z)|^2dxdy<+\infty$, i.e. $f\in AD(R)-C$. The image region $f(R)$ has a finite area since $D_R(f)<+\infty$, and a fortiori $C-f(R)$ has a positive area (and in fact an infinite area). Therefore we can find a compact subset E with positive area in $C-f(R)$. Let $\phi(z)$ be the function in the above theorem associated with E. It is readily checked that $\phi\circ f\in ABD(R)-C$, and we have seen the inclusion $O_{AD}\supset O_{ABD}$. This with the trivial inclusion $O_{AD}\subset O_{ABD}$ implies the identity $O_{AD}=O_{ABD}$.

One step further Sakai [1] proved that $ABD(R)$ is dense in $AD(R)$ with respect to the Dirichlet seminorm $D(\cdot)^{1/2}$. By observing the proof of $O_{AD}=O_{ABD}$ mentioned above, we naturally come across the question (suggested to the author by Professor Nakai) whether there exists a sequence $\{\phi_n\}$ on C such that $\phi_n\circ f\in ABD(R)$ and $\{\phi_n\}$ converges to the identity function on $f(R)$ so that the sequence $\{\phi_n\circ f\}$ gives the desired approximation of the given $f\in AD(R)$. The *purpose* of this note is to prove the following theorem by which the above procedure is certainly possible.

THEOREM. *Suppose that a closed set E in the complex plane C satisfies the condition*

$$\limsup_{r\to\infty}\frac{m(E\cap\{r<|z|<2r\})}{r^2}>0 \tag{1}$$

with m the Lebesgue measure on C. Then there exists a sequence of functions $\{\phi_n(z)\}$ satisfying the following three conditions:

(α) *each* $\phi_n(z)$ *is bounded and analytic on* $C-E\cap\{|z|\geqq n\}$, *i.e.* $\phi_n\in AB(C-E\cap\{|z|\geqq n\})$,

(β) $\sup_n\{\sup\{|\phi_n'(z)|;\ z\in C-E\cap\{|z|\geqq n\}\}\}<+\infty$,

(γ) $\{\phi_n(z)\}$ *converges to* z *uniformly on each compact subset of* C.

The proof of this theorem will be given in nos. 2 and 3. Here we show that the above theorem implies the approximation theorem: $ABD(R)$ is $D(\cdot)^{1/2}$-dense in $AD(R)$. Let $f\in AD(R)$ and $E=C-f(R)$. It is readily seen that E satisfies (1) since $m(f(R))<+\infty$. Choose the sequence $\{\phi_n\}$ in the above theorem constructed for the present $E=C-f(R)$. Observe that

$$D_R(f-\phi_n\circ f)=\iint_R|1-\phi_n'(f(z))|^2|f'(z)|^2dxdy\,.$$

Let K be an arbitrary compact set in R and c be the quantity in (β) in the above theorem. Then

$$D_R(f-\phi_n\circ f)\leqq\iint_K|1-\phi_n'(f(z))|^2|f'(z)|^2dxdy+(1+c)^2D_{R-K}(f)\,.$$

On letting $n\to\infty$ in the above inequality, the condition (γ) implies that

$$\limsup_{n\to\infty} D_R(f-\phi_n\circ f)\leqq(1+c)^2D_{R-K}(f)\,.$$

Again by letting $K\uparrow R$, we conclude that $D_R(f-\phi_n\circ f)\to 0$ $(n\to+\infty)$.

2. For the proof of our theorem we use notations and results in Uy [**4**]. We denote by $M(C)$ the set of the finite Borel measures on C and consider

$$B\mu(z)=\text{p.v.}\iint\frac{d\mu(\zeta)}{(\zeta-z)^2}\qquad(\text{p.v.}=\text{principal value})$$

for each $\mu\in M(C)$. It is well known (cf. e.g. Stein [**3**], Zygmund [**5**]) that the above singular integral exists almost everywhere and that there exists a universal constant A such that

$$m(\{z;\ |B\mu(z)|>\lambda\})\leqq\frac{A\|\mu\|}{\lambda}\tag{2}$$

for any $\mu\in M(C)$. By taking μ the Dirac measure in (2) we in particular see that $A\geqq\pi$. If E is any compact set in C such that $m(E)>0$ and $0\notin E$, then we denote by $\Gamma(E)$ the set of functions $h\in L^\infty(E)$ such that $\|h\|_\infty\leqq 1$ and $\|Bh\|_\infty\leqq 1$, where Bh stands for $B\mu$ with $d\mu(z)=h(z)dm(z)$. We set

$$b(E)=\sup_{h\in\Gamma(E)}|Bh(0)|=\sup_{h\in\Gamma(E)}\left|\iint\frac{h(\zeta)}{\zeta^2}dm(\zeta)\right|\,.$$

If, moreover, E is the closure of an open set whose boundary consists of a finite number of analytic Jordan curves, then we also consider the quantity

$$\mathfrak{b}^*(E)=\sup_{h\in\mathscr{D}(E)\cap\Gamma(E)}|Bh(0)|$$

where $\mathscr{D}(E)$ is the set of C^∞-functions on C with supports in E.

3. The proof of our theorem can be divided into lemmas, the first of which is:

LMMA 1. *The inequality*

$$\mathfrak{b}(E)\geqq\frac{1}{8A}\frac{m(E)}{r^2} \tag{3}$$

is valid for any compact set $E\subset\{r\leqq|z|\leqq 2r\}$ $(r>0)$.

PROOF. We use an argument similar as in the proof of Theorem 5.1 in Uy [4]. Set $E_r=\{z/r\,;\,z\in E\}\subset\{1\leqq|z|\leqq 2\}$. It is off hand to see that $\mathfrak{b}(E)=\mathfrak{b}(E_r)$. By Lemma 4.2 of Uy [4], it suffices to show that

$$\mathfrak{b}^*(E_r)\geqq\frac{1}{8A}m(E_r) \tag{4}$$

for any compact set E_r with $E_r\subset\{1\leqq|z|\leqq 2\}$ and with a boundary consisting of a finite number of analytic Jordan curves. By using Theorem 3.7 of Uy [4], we have

$$\mathfrak{b}^*(E_r)\geqq\iint_{E_r}\left|\frac{1}{z^2}-B\nu(z)\right|dm(z)+\|\nu\|$$

for some $\nu\in M(C)$. Let $F=\{z\in E_r\,;\,|B\nu(z)|>1/8\}$. By (2), we have $m(F)\leqq 8A\|\nu\|$ and

$$\iint_{E_r}\left|\frac{1}{z^2}-B\nu(z)\right|dm(z)+\|\nu\|\geqq\iint_{E_r-F}\left|\frac{1}{z^2}-B\nu(z)\right|dm(z)+\|\nu\|$$

$$\geqq\frac{1}{8}m(E_r-F)+\frac{1}{8A}m(F)\geqq\frac{1}{8A}m(E_r)\,.$$

Hence (4) is established. Q. E. D.

Our theorem can be deduced at once from the following

LEMMA 2. *If E is a compact set with positive measure contained in $\{r\leqq|z|\leqq 2r\}$ $(r>0)$, then there exists a function $g(z)$ such that*

(a) *$g(z)$ is bounded and analytic on $C-E$,*

(b) $|g'(z)|\leqq 9A\dfrac{r^2}{m(E)}$ *on* $C-E$,

(c) $|g'(z)-1|\leqq 12A\dfrac{r}{m(E)}|z|$ *for* $|z|<r$.

PROOF. By Lemma 1, there exists an $h\in L^\infty(E)$ such that

$$\hat{h}(z)=\iint \frac{h(\zeta)}{\zeta-z}\, dm(\zeta)$$

satisfies the following properties:

1° $\hat{h}(z)$ is continuous on C,

2° $\left|\frac{d}{dz}\hat{h}(z)\right|=|Bh(z)|\leqq 1$ on $C-E$,

3° $\left|\frac{d}{dz}\hat{h}(0)\right|=|Bh(0)|\geqq \frac{1}{9A}\frac{m(E)}{r^2}$.

The function

$$g(z)=\hat{h}(z)\left(\frac{d\hat{h}}{dz}(0)\right)^{-1}$$

is clearly bounded and analytic on $C-E$ and satisfies (b). Since $1\leqq 3\pi r^2/m(E)$ $\leqq 3Ar^2/m(E)$ (recall $A\geqq\pi$), applying the Schwarz lemma to $g'-1$, g also satisfies (c). Q. E. D.

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