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No explosion criteria for stochastic differential equations

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§1. Introduction.

In this paper, the existence problem of global solutions of stochastic differential equations will be discussed.

First of all we introduce the notations and definitions. Let I denote the interval $0 \leqq t < \infty$ and R^d denote Euclidean d-space. For $x \in R^d$ and $y \in R^d$, let $\langle x, y \rangle$ be the inner product of x and y and let $|x|$ be the Euclidean norm of x. For a $d \times d$-matrix $M=(m_{ij})$, define $|M|=(\sum_{i,j=1}^{d} m_{ij}^2)^{1/2}$. We shall denote by C_2 the family of scalar functions defined on $I \times R^d$ which are twice continuously differentiable with respect to $x \in R^d$ and once with respect to $t \in I$. Let $(\Omega, \boldsymbol{F}, P)$ be a probability space with an increasing family $\{\boldsymbol{F}_t; t \geqq 0\}$ of sub-σ-algebras of $\boldsymbol{F}$ and let $w(t)=(w_i(t))$, $i=1, \cdots, d$, be a d-dimensional Brownian motion process adapted to $\boldsymbol{F}_t$. Consider the stochastic differential equation

$$(1.1) \qquad dX(t)=b(t, X(t))dt+\sigma(t, X(t))dw(t),$$

where $b(t, x)=(b_i(t, x))$, $i=1, \cdots, d$, is a d-vector function and $\sigma(t, x)=(\sigma_{ij}(t, x))$, $i, j=1, \cdots, d$, is a $d \times d$-matrix function, which are defined on $I \times R^d$ and Borel measurable with respect to the complete set of variables. Equation (1.1) is equivalent to the system of d equations

$$(1.1)' \qquad dX_i(t)=b_i(t, X(t))dt+\sum_{j=1}^{d} \sigma_{ij}(t, X(t))dw_j(t), \qquad i=1, \cdots, d.$$

Throughout this paper, we assume the following:

(1.2) $b(t, x)$ and $\sigma(t, x)$ are continuous in (t, x), and for any $T>0$, $R>0$, there exists a constant $C_{TR}>0$ depending only on T and R such that

$$|b(t, x)-b(t, y)|+|\sigma(t, x)-\sigma(t, y)| \leqq C_{TR}|x-y|$$

if $t \leqq T$, $|x| \leqq R$ and $|y| \leqq R$.

Then, for any natural number n, we can construct functions

$$b^{(n)}(t, x)=(b_i^{(n)}(t, x)) \quad \text{and} \quad \sigma^{(n)}(t, x)=(\sigma_{ij}^{(n)}(t, x)), \qquad i, j=1, \cdots, d,$$

which satisfy the following conditions;

$$(1.3)' \qquad b^{(n)}(t, x)=b(t, x) \quad \text{and} \quad \sigma^{(n)}(t, x)=\sigma(t, x) \quad \text{for } t\leqq n \text{ and } |x|\leqq n,$$

$$(1.3)'' \qquad |b^{(n)}(t, x)-b^{(n)}(t, y)|+|\sigma^{(n)}(t, x)-\sigma^{(n)}(t, y)|$$

$$\leqq K_n|x-y| \qquad \text{for } t\leqq n,\ x\in R^d \text{ and } y\in R^d,$$

$$(1.3)''' \qquad |b^{(n)}(t, x)|^2+|\sigma^{(n)}(t, x)|^2\leqq K_n(1+|x|^2)$$

for $t\leqq n$ and $x\in R^d$, where K_n is a constant depending only on n. As is well known, by (1.3)″ and (1.3)‴, there exists a pathwise unique solution $X^{(n)}(t)$ which is defined up to $t\leqq n$ of the stochastic differential equation

$$(1.4) \qquad dX^{(n)}(t)=b^{(n)}(t, X^{(n)}(t))dt+\sigma^{(n)}(t, X^{(n)}(t))dw(t).$$

By $X^{(n)}(t; t_0, x_0)$ we mean the solution of (1.4) with the initial condition $X^{(n)}(t_0)=x_0\in R^d$ $(t_0\geqq 0)$. Define the stopping time $\tau_n(t_0, x_0)$ by

$$\tau_n(t_0, x_0)=\inf\{t;\ |X^{(n)}(t; t_0, x_0)|\geqq n\}$$

(define $\tau_n(t_0, x_0)$ by $\tau_n(t_0, x_0)=\infty$ if there is no such time) and set $e_n(t_0, x_0)=\min\{n, \tau_n(t_0, x_0)\}$. Then, $\{e_n(t_0, x_0);\ n\geqq 1\}$ is a monotone increasing family of stopping times, for which

$$\sup_{t_0\leq t\leq e_n(t_0, x_0)} |X^{(n)}(t; t_0, x_0)-X^{(m)}(t; t_0, x_0)|=0$$

with probability one if $m>n$. Define a random process $X(t; t_0, x_0)$ by

$$X(t; t_0, x_0)=X^{(n)}(t; t_0, x_0) \qquad \text{for } t<e_n(t_0, x_0)\ (n=1, 2, \cdots).$$

The process $X(t; t_0, x_0)$ is called the solution of (1.1) with the initial condition $X(t_0)=x_0$. A random time $e(t_0, x_0)=\lim_{n\to\infty} e_n(t_0, x_0)$ is called *the explosion time* of $X(t; t_0, x_0)$. Also we define an event N_{t_0, x_0} by

$$N_{t_0, x_0}=\{e(t_0, x_0)<\infty \quad \text{and} \quad \lim_{t\uparrow e(t_0, x_0)} |X(t; t_0, x_0)|=\infty\}.$$

Then we note the following result (see § 3).

REMARK. If $b(t, x)$ and $\sigma(t, x)$ satisfy (1.2), then $N_{t_0, x_0}=\{e(t_0, x_0)<\infty\}$, almost surely.

We shall use the differential generator

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^d b_i(t, x)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^d a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j}$$

associated with the stochastic differential equation (1.1), where $a(t, x)=(a_{ij}(t, x))$ is a $d\times d$-matrix defined by

$$a(t, x)=\sigma(t, x)\sigma(t, x)^* \qquad (\text{* means the transpose}).$$

We are interested in the question whether $e(t_0, x_0)=\infty$ or not. In this study, the Liapunov function approach provides an effective method. For example, we can show the following theorem by the same method of the proof of [7, Corollary] if we note the above Remark.

THEOREM 1.1. *Let $b(t, x)$ and $\sigma(t, x)$ satisfy* (1.2) *and suppose there exists a function $V(t, x)\in C_2$ which satisfies the following conditions;*

$$(1.5) \qquad LV(t, x)\leqq 0 \quad \text{for all } t\geqq 0 \text{ and } |x|\geqq r_0\ (r_0>0),$$

$$(1.6) \qquad \lim_{|x|\to\infty}\inf_{0\leq t\leq T} V(t, x)=\infty \quad \text{for each } T>0.$$

Then, $P(e(t_0, x_0)=\infty)=1$ for all $(t_0, x_0)\in I\times R^d$.

For example, let $\sigma(t, x)$ be a unit matrix and let

$$(1.7) \qquad |b(t, x)|\leqq\alpha(t)\beta(|x|) \quad \text{for all } t\geqq 0 \text{ and } |x|\geqq r_0\ (r_0>0),$$

where $\alpha:[0, \infty)\to[0, \infty)$ is continuous, $\beta:[r_0, \infty)\to(0, \infty)$ is monotone increasing, differentiable and such that

$$(1.8) \qquad \int_{r_0}^{\infty}\frac{du}{\beta(u)}=\infty.$$

Set

$$V(t, x)=\int_{r_0}^{|x|}du/\beta(u)-\int_0^t A(u)du \qquad \text{for } |x|\geqq r_0,$$

where $A(t)=\alpha(t)+(d-1)/2r_0\beta(r_0)$, and extend it smoothly to $|x|<r_0$. Then, by (1.7) and (1.8), we can see that $V(t, x)$ satisfies (1.5) and (1.6). So, the above theorem may correspond to the result of continuability of solutions of differential equations as we can imagine from the following; let $b(t, x)$ be continuous and let (1.7) hold, where α is nonnegative and continuous, and β is positive, continuous and satisfies (1.8). Then every solution of non-random differential equations

$$dX(t)/dt=b(t, X(t))$$

is continuable to $t=\infty$ (see Yoshizawa [**10**, p. 13]).

On the other hand, Hasminskii [**3**, p. 113] shows the following result.

THEOREM 1.2. *Let $b(t, x)$ and $\sigma(t, x)$ satisfy* (1.2) *and suppose there exist a nonnegative function $V(t, x)\in C_2$ satisfying the condition*

$$(1.9) \qquad LV(t, x)\leqq cV(t, x) \text{ with a constant } c>0 \text{ for all } (t, x)\in I\times R^d$$

and such that

$$\lim_{|x|\to\infty}\inf_{0\leq t\leq T} V(t, x)=\infty \quad \text{for each } T>0.$$

Then, $P(e(t_0, x_0)=\infty)=1$ *for all* $(t_0, x_0)\in I\times R^d$.

In this paper, we will generalize Theorem 1.1 in case $\sigma(t, x)$ is bounded (see Theorem 2.1), and also generalize Theorem 1.2 (see Theorem 2.2). In Theorem 2.1, it is remarkable that the radial unboundedness condition (1.6) is not imposed on the Liapunov function. In Theorem 2.2, the condition (1.9) is replaced by a general form such that $LV(t, x)\leqq\alpha(t)\beta(V(t, x))$ for certain functions α and β. The second order Ito processes will be taken as examples of applications.

§ 2. Liapunov functions.

Here we apply Liapunov's second method to systems of stochastic differential equations. Let $e(t_0, x_0)$ be the explosion time of $X(t; t_0, x_0)$ of the solution of (1.1) with the initial condition $X(t_0)=x_0\in R^d$. To begin with, we seek sufficient conditions so that $X(t; t_0, x_0)$ does not explode.

THEOREM 2.1. *Let* $b(t, x)$ *and* $\sigma(t, x)$ *satisfy* (1.2). *Let* $\sigma(t, x)$ *be bounded and suppose there exist a nonnegative function* $V(t, x)\in C_2$ *which satisfies*

$$LV(t, x)\leqq A(t)-B(t)g(|b(t, x)|)+C(t)V(t, x) \tag{2.1}$$

for all $t\geqq 0$ *and* $|x|\geqq r_0$ $(r_0>0)$, *where*

$A: [0, \infty)\longrightarrow(-\infty, \infty)$ *is continuous,*

$B: [0, \infty)\longrightarrow(0, \infty)$ *is continuous,*

$C: [0, \infty)\longrightarrow(-\infty, \infty)$ *is continuous,*

$g: [0, \infty)\longrightarrow[0, \infty)$ *is continuous,*

and there exist $k_1>0$ *and* $k_2\geqq 0$ *such that*

$$k_1 t\leqq g(t) \qquad \text{for all } t\geqq k_2. \tag{2.2}$$

Then, $P(e(t_0, x_0)=\infty)=1$ *for all* $(t_0, x_0)\in I\times R^d$.

PROOF. We define $U(t, x)$ by

$$U(t, x)=V(t, x)\exp\left(-\int_0^t C(s)ds\right).$$

Then, by (2.1), we have, $LU(t, x)\leqq h(t, x)$ for all $t\geqq 0$ and $|x|\geqq r_0$, where

$$h(t, x)=A(t)\exp\left(-\int_0^t C(s)ds\right)-B(t)g(|b(t, x)|)\exp\left(-\int_0^t C(s)ds\right).$$

Assume that $P(e(t_0, x_0)<\infty)>0$ for some $(t_0, x_0)\in I\times R^d$. Then, by Remark in § 1, we note that $P(N_{t_0, x_0})>0$. Consider a sample of N_{t_0, x_0} in the following. Let $\rho=\sup\{t; |X(t; t_0, x_0)|=r_0\}$. For simplicity of the notation we write as $X(t)=X(t; t_0, x_0)$ and $e=e(t_0, x_0)$, omitting t_0 and x_0. By Ito's formula concerning stochastic differentials, we see that

$$dU(t, X(t)) \leqq h(t, X(t))dt + \langle \operatorname{grad} U(t, X(t)), \sigma(t, X(t))dw(t)\rangle$$

for $\rho \leqq t < e$. Integrate the above equation from ρ to time $t<e$, with the result that

(2.3) $$U(t, X(t)) - U(\rho, X(\rho)) \leqq \int_\rho^t h(s, X(s))\,ds + z(t) - z(\rho),$$

where $z(t)$ is a new Brownian motion process run with the clock

$$\varphi(t) = \int_{t_0}^t \langle \operatorname{grad} U(s, X(s)), a(s, X(s)) \operatorname{grad} U(s, X(s))\rangle ds,$$

and $a = \sigma\sigma^*$ (see McKean [6, p. 45]). Now there exist some $\delta = \delta(\omega) > 0$ and $\varepsilon_0 = \varepsilon_0(\omega) > 0$ ($\omega \in N_{t_0, x_0}$) such that

$$\delta\varepsilon_0 \int_\rho^t g(|b(s, X(s))|)ds$$
$$\leqq \int_\rho^t B(s) g(|b(s, X(s))|) \exp\left(-\int_0^s C(v)dv\right) ds$$

for $\rho \leqq t < e$, since $\int_0^t C(s)ds$ is bounded on $[t_0, e]$ and $B(t)$ is positive on $[t_0, e]$. Hence, by (2.3), we get,

(2.4) $$\delta\varepsilon_0 \int_\rho^t g(|b(s, X(s))|)ds$$
$$\leqq U(\rho, X(\rho)) + \int_\rho^t A(s) \exp\left(-\int_0^s C(v)dv\right) ds + z(t) - z(\rho)$$

for all $\rho \leqq t < e$. We proceed with an argument according to each of the two cases; $\varphi(e) < \infty$, $\varphi(e) = \infty$, respectively. First we consider the case where $\varphi(e) < \infty$. Letting t tend to e, we see that the right-hand side of (2.4) is finite so that

(2.5) $$\int_\rho^e g(|b(s, X(s))|)ds < \infty.$$

On the other hand, $X(t) = (X_i(t))$ satisfies the stochastic differential equations $dX_i(t) = b_i(t, X(t))dt + dy_i(t)$ ($i = 1, \cdots, d$) for $t < e$, where $y_i(t)$ is a new Brownian motion process run with the clock $\phi_i(t) = \sum_{j=1}^d \int_{t_0}^t |\sigma_{ij}(s, X(s))|^2 ds$. Since $\sigma(t, x)$ is bounded by the assumption, we note that $\phi_i(e) < \infty$ for each i. Now consider

$$|X(t) - X(\rho)| \leqq d \int_\rho^t |b(s, X(s))| ds + \sum_{i=1}^d |y_i(t) - y_i(\rho)|.$$

and note that by (2.2)

$$|b(s, X(s))| \leqq k_2 \quad \text{if } |b(s, X(s))| \leqq k_2,$$

$$|b(s, X(s))| \leqq g(|b(s, X(s))|)/k_1 \quad \text{if } |b(s, X(s))| > k_2,$$

so in either case

$$|b(s, X(s))| \leqq k_2 + g(|b(s, X(s))|)/k_1 .$$

Therefore,

$$(2.6) \qquad |X(t)-X(\rho)| \leqq d k_2(t-\rho) + d\int_\rho^t g(|b(s, X(s))|)ds/k_1 + \sum_{i=1}^d |y_i(t)-y_i(\rho)|$$

for all $\rho \leqq t < e$. Letting t tend to e, we see that the right-hand side of (2.6) is finite by (2.5), while the left-hand side becomes infinity since $|X(e-)|=\infty$ on N_{t_0, x_0}. Thus we are led to contradiction. Next we consider the case where $\varphi(e)=\infty$. Since g is nonnegative by the assumption, we see that the left-hand side of (2.4) is nonnegative. Let t tend to the time e in (2.4). Then, we get,

$$0 \leqq U(\rho, X(\rho)) + \int_\rho^e A(s)\exp\left(-\int_0^s C(v)dv\right)ds + \liminf_{t\to\infty} z(t) - z(\rho) = -\infty ,$$

which is also absurd. Hence the proof is complete.

The idea of the proof of Theorem 2.1 is due to Peterson [**9**], who investigated the continuability of solutions of non-random differential equations $dX(t)/dt = b(t, X(t))$ under the assumption of the existence of a nonnegative and continuous function $V(t, x)$ such that

$$V'(t, x) \leqq -B(t)g(|b(t, x)|) + C(t)V(t, x) ,$$

where B, C, and g satisfy the same conditions as in Theorem 2.1 and $V'(t, x)$ is the upper right-hand derivative along solutions of $dX(t)/dt = b(t, X(t))$.

EXAMPLE 2.1. Let $w(t)=(w_1(t), w_2(t))$ be a two dimensional Brownian motion process and let us consider a system of stochastic differential equations

$$(2.7) \qquad \begin{aligned} dX_1(t) &= X_2(t)dt + \delta(t, X_1(t), X_2(t))dw_1(t) , \\ dX_2(t) &= -\alpha(t)\beta(t)f(X_1(t))dt + \gamma(t, X_1(t), X_2(t))dw_2(t) , \end{aligned}$$

where the coefficients satisfy the following conditions;

(i) $\alpha : [0, \infty) \to (0, \infty)$ is once continuously differentiable such that $\alpha'(t) > 0$ for all $t \geqq 0$,

(ii) $\beta : [0, \infty) \to (0, \infty)$ is once continuously differentiable such that $\beta'(t) < 0$ for all $t \geqq 0$,

(iii) $f : (-\infty, \infty) \to (-\infty, \infty)$ is once continuously differentiable such that

$$yf(y)>0\ (y\neq 0)\quad\text{and}\quad F(y)\equiv\int_0^y f(s)ds\geqq kf^2(y)$$

with a constant $k>0$ for all $y\in(-\infty,\infty)$,

(iv) $\gamma:[0,\infty)\times(-\infty,\infty)\times(-\infty,\infty)\to(-\infty,\infty)$ and $\delta:[0,\infty)\times(-\infty,\infty)\times(-\infty,\infty)\to(-\infty,\infty)$ have continuous first partials such that

$$\gamma(t,x_1,x_2)\neq 0\qquad\text{for all } t, x_1, x_2,$$

$$\gamma^2(t,x_1,x_2)+\delta^2(t,x_1,x_2)(1+|f'(x_1)|)\leqq m$$

with a constant $m>0$ for all t, x_1, x_2.

Equation (2.7) can be written as a vector stochastic differential equation of the form (1.1), where

$$b(t,x)=(x_2, -\alpha(t)\beta(t)f(x_1)),$$

$$\sigma(t,x)=\begin{pmatrix}\delta(t,x_1,x_2) & 0\\ 0 & \gamma(t,x_1,x_2)\end{pmatrix},$$

$t\geqq 0$ and $x=(x_1,x_2)\in R^2$. If $\delta(t,x_1,x_2)\equiv 0$ in (2.7), then $X_1(t)$ is called the second order Ito process which corresponds to the response of the oscillator

$$\ddot{y}+\alpha(t)\beta(t)f(y)=\gamma(t,y,\dot{y})\dot{w}_2$$

with the formal white noise $\dot{w}_2$, where by $\dot{}$ we mean the symbolic derivative d/dt. Therefore, in the particular case when $\delta(t,x_1,x_2)\equiv 0$, we know by Goldstein [2] that $X_2(t)$ represents the mean square derivative of $X_1(t)$ and $X_2(t)$ is of unbounded variation in every finite interval with probability one, up to the explosion time. Suppose that the conditions (i), (ii), (iii) and (iv) hold and set $V(t,x)=x_2^2/2\alpha(t)+\beta(t)F(x_1)$ for $t\geqq 0$ and $x=(x_1,x_2)\in R^2$. Then, since

$$x_2^2=|b(t,x)|^2-\alpha^2(t)\beta^2(t)f^2(x_1),\quad f^2(x_1)\leqq V(t,x)/k\beta(t)$$

and $\beta'(t)F(x_1)\leqq 0$ by the assumption. we obtain that

$$\begin{aligned}LV(t,x)=&\frac{\beta(t)}{2}\delta^2(t,x_1,x_2)f'(x_1)+\frac{1}{2\alpha(t)}\gamma^2(t,x_1,x_2)\\&-\frac{\alpha'(t)}{2\alpha^2(t)}x_2^2+\beta'(t)F(x_1)\\&\leqq\frac{m}{2}\left(\beta(t)+\frac{1}{\alpha(t)}\right)-\frac{\alpha'(t)}{2\alpha^2(t)}|b(t,x)|^2+\frac{\alpha'(t)\beta(t)}{2k}V(t,x)\end{aligned}$$

for all $t\geqq 0$ and $x\in R^2$. Therefore, Theorem 2.1 will apply if we take $A(t)=m(\beta(t)+1/\alpha(t))/2$, $B(t)=\alpha'(t)/2\alpha^2(t)$, $C(t)=\alpha'(t)\beta(t)/2k$, $g(t)=t^2$, $k_1=1$ and $k_2=1$.

Next we obtain the following generalization of Theorem 1.2.

THEOREM 2.2. *Let $b(t,x)$ and $\sigma(t,x)$ satisfy* (1.2) *and suppose there exists a* *nonnegative function* $V(t, x) \in C_2$ *which satisfies*

(2.8) $$LV(t, x) \leqq \alpha(t)\beta(V(t, x)) \quad \textit{for all } (t, x) \in I \times R^d,$$

where $\alpha: [0, \infty) \to [0, \infty)$ *is continuous and* $\beta: [0, \infty) \to [0, \infty)$ *is monotone increasing, concave such that*

(2.9) $$\int_0^\infty \frac{du}{1+\beta(u)} = \infty,$$

and

(2.10) $$\lim_{|x| \to \infty} \inf_{0 \leq t \leq T} V(t, x) = \infty \quad \textit{for each } T > 0.$$

Then, $P(e(t_0, x_0) = \infty) = 1$ *for all* $(t_0, x_0) \in I \times R^d$.

PROOF. We consider the solution $X^{(n)}(t; t_0, x_0)$ of (1.4) with the initial condition $X^{(n)}(t_0) = x_0 \in R^d$ for large $n > \max\{|x_0|, t_0\}$. Let $\tau_n(t_0, x_0)$ be the first exit time from the ball $\{x; |x| < n\}$ for $X^{(n)}(t; t_0, x_0)$ and let $e_n(t_0, x_0) = \min\{n, \tau_n(t_0, x_0)\}$. For notational simplicity we write as $X^{(n)}(t) = X^{(n)}(t; t_0, x_0)$, $\tau_n = \tau_n(t_0, x_0)$ and $e_n = e_n(t_0, x_0)$, omitting t_0 and x_0. Also, for any $t \geqq t_0$ we put $t_n = t \wedge e_n$, here and hereafter $p \wedge q$ is the smaller of p and q. By Ito's formula concerning stochastic differentials, we see

$$V(t_n, X^{(n)}(t_n)) = V(t_0, x_0) + \int_{t_0}^{t_n} LV(u, X^{(n)}(u))du$$
$$+ \int_{t_0}^{t_n} \langle \operatorname{grad} V(u, X^{(n)}(u)), \sigma^{(n)}(u, X^{(n)}(u))dw(u) \rangle,$$

because $b^{(n)}(t, x) = b(t, x)$ and $\sigma^{(n)}(t, x) = \sigma(t, x)$ for $t \leqq n$ and $|x| \leqq n$ (see (1.3)′). Take the mathematical expectation in the above. Then, since $E(\sup_{u \leq n} |X^{(n)}(u)|^2)$ is bounded by the condition (1.3)‴ (see Friedman [**1**, p. 102]) and since (2.8) holds, we obtain,

$$EV(t_n, X^{(n)}(t_n)) \leqq V(t_0, x_0) + A(t) \int_{t_0}^{t} E\beta(V(u_n, X^{(n)}(u_n)))du,$$

where $A(t) = \max_{u \leq t} \alpha(u)$ and $u_n = u \wedge e_n$. Let $T' > t_0$ be arbitrary and be fixed, which is to be determined later. Now we set

$$Q^{(n)}(t) = EV(t_n, X^{(n)}(t_n)) \text{ and } R^{(n)}(t) = (T' - t_0)^{-1} \int_{t_0}^{t} Q^{(n)}(u)du \quad \text{for } T' \geqq t \geqq t_0$$

so that $R^{(n)}(t)$ is continuous, monotone increasing in t and satisfies $0 \leqq R^{(n)}(t) \leqq M_n$ $(M_n \equiv \sup_{t \leq n, |x| \leq n} V(t, x))$ for $T' \geqq t \geqq t_0$, since $0 \leqq Q^{(n)}(t) \leqq M_n$. Since β is concave by the assumption, we get, by Jensen's inequality,

$$Q^{(n)}(t) \leqq V(t_0, x_0) + A(T')(T' - t_0)\beta(R^{(n)}(t))$$

for $T' \geqq t \geqq t_0$. Noting that $(1 + \beta(R^{(n)}(t)))^{-1} \leqq 1$ and $\beta(R^{(n)}(t))(1 + \beta(R^{(n)}(t)))^{-1} \leqq 1$

for $T'\geqq t\geqq t_0$, divide the both sides of the above by $1+\beta(R^{(n)}(t))$ and then integrate from t_0 to t ($\leqq T'$). Then,

$$\int_0^{R^{(n)}(t)}\frac{du}{1+\beta(u)}\leqq F(T'; t_0) \tag{2.11}$$

for all $T'\geqq t>t_0$, where $F(t; t_0)=V(t_0, x_0)+A(t)(t-t_0)$. Assume that there exist some t_0 and x_0 such that $P(e(t_0, x_0)<T)\equiv\delta>0$ for some $T<\infty$. Let $T'>T$ be arbitrary and be fixed. We choose n so large that $n>\max\{|x_0|, T'\}$ and consider $X^{(n)}(t; t_0, x_0)$ for such n in the following. Take any t such that $T<t\leqq T'$. Then, since $\{\tau_n<t\}=\{e_n<t\}\supseteqq\{e_n<T\}\supseteqq\{e(t_0, x_0)<T\}$, we see

$$\begin{aligned}Q^{(n)}(t)&\geqq E[V(t_n, X^{(n)}(t_n)); e_n<t]\\&\geqq V_nP(e_n<t)\\&\geqq V_n\delta \qquad\qquad (V_n=\inf_{\substack{0\leqq t\leqq T'\\|x|\geqq n}}V(t, x))\end{aligned}$$

for all $t\in(T, T']$. Therefore, we have,

$$R^{(n)}(t)\geqq(T'-t_0)^{-1}\int_T^t Q^{(n)}(u)du\geqq(t-T)(T'-t_0)^{-1}V_n\delta$$

for all $t\in(T, T']$. Accordingly, by (2.11), we obtain,

$$\int_0^{(T'-T)(T'-t_0)^{-1}V_n\delta}\frac{du}{1+\beta(u)}\leqq F(T'; t_0). \tag{2.12}$$

Letting n tend to infinity in the above, we see that the right-hand side of (2.12) is finite, while the left-hand side becomes infinity, since (2.10) and (2.9) hold. Thus, we are led to contradiction. Therefore, for any t_0, x_0 and T $P(e(t_0, x_0)\geqq T)=1$ and the proof is complete.

The result of Theorem 2.2 contains the result of Theorem 1.2 if we take $\alpha(t)=c$ with a constant $c>0$ and $\beta(v)=v$.

If we take $V(t, x)=|x|^2$ in Theorem 2.2, then we can obtain the following sufficient condition on the growth of the drift and diffusion coefficients in order that $X(t; t_0, x_0)$ has the infinite explosion time with probability one.

COROLLARY. *Let* $b(t, x)$ *and* $\sigma(t, x)$ *satisfy* (1.2) *and suppose that*

$$2\langle x, b(t, x)\rangle+|\sigma(t, x)|^2\leqq\alpha(t)\beta(|x|^2),$$

where α *and* β *satisfy the same assumptions as in Theorem* 2.2.

Then, $P(e(t_0, x_0)=\infty)=1$ *for all* $(t_0, x_0)\in I\times R^d$.

In a particular case of the one dimension, another sufficient condition of the infinite explosion time was obtained by the author [8] with the following result; if

$$|xb(t, x)|+\sigma(t, x)^2\leqq\alpha(t)\beta(|x|),$$

where $\alpha(t)$ is a nonnegative locally integrable function and $\beta(r)$ is a continuous nondecreasing positive function with $r^2=O(\beta(r))$ as $r\to\infty$ and $r(\beta(r))^{-1}$ not integrable near ∞, then $P(e(t_0, x_0)=\infty)=1$ for all $t_0\geqq 0$ and $x_0\in(-\infty, \infty)$.

EXAMPLE 2.2. Let $w(t)=(w_1(t), w_2(t))$ be a two dimensional Brownian motion process and let us consider a system of stochastic differential equations

$$
(2.13)\qquad
\begin{aligned}
dX_1(t)&=X_2(t)dt+\delta(t, X_1(t), X_2(t))dw_1(t)\,,\\
dX_2(t)&=(-g(t, X_1(t), X_2(t))X_2(t)-a(t)f(X_1(t))+h(t))dt\\
&\qquad+\gamma(t, X_1(t), X_2(t))dw_2(t)\,,
\end{aligned}
$$

where the coefficients satisfy the following conditions;

(i) $a:[0, \infty)\to[c, \infty)$ $(c>0)$ is once continuously differentiable such that $a'(t)>0$ for all $t\geqq 0$,

(ii) $f:(-\infty, \infty)\to(-\infty, \infty)$ is once continuously differentiable such that

$$yf(y)>0 \ (y\neq 0) \quad \text{and} \quad F(y)\equiv\int_0^y f(s)ds \longrightarrow \infty \quad (|y|\to\infty)\,,$$

(iii) $g:[0, \infty)\times(-\infty, \infty)\times(-\infty, \infty)\to[0, \infty)$ has continuous first partials,

(iv) $h:[0, \infty)\to(-\infty, \infty)$ is continuous,

(v) $\gamma:[0, \infty)\times(-\infty, \infty)\times(-\infty, \infty)\to(-\infty, \infty)$ and $\delta:[0, \infty)\times(-\infty, \infty)\times(-\infty, \infty)\to(-\infty, \infty)$ have continuous first partials such that

$$\gamma(t, x_1, x_2)\neq 0 \qquad \text{for all } t, x_1, x_2\,,$$

$$\gamma^2(t, x_1, x_2)+\delta^2(t, x_1, x_2)|f'(x_1)|\leqq m(t)(1+|x_2|^p)$$

for all t, x_1, x_2, here $m(t)$ is nonnegative and continuous and $0\leqq p\leqq 2$. Equation (2.13) can be written as a vector stochastic differential equation of the form (1.1), where

$$b(t, x)=(x_2, -g(t, x_1, x_2)x_2-a(t)f(x_1)+h(t))\,,$$

$$\sigma(t, x)=\begin{pmatrix}\delta(t, x_1, x_2) & 0\\ 0 & \gamma(t, x_1, x_2)\end{pmatrix}$$

for $t\geqq 0$ and $x=(x_1, x_2)\in R^2$. Similarly as in Example 2.1, $X_1(t)$ is called the second order Ito process if $\delta(t, x_1, x_2)\equiv 0$ in (2.13), which corresponds to the response of the oscillator

$$\ddot{y}+g(t, y, \dot{y})\dot{y}+a(t)f(y)=h(t)+\gamma(t, y, \dot{y})\dot{w}_2$$

with the formal white noise $\dot{w}_2$. Now let the conditions (i), (ii), (iii), (iv) and (v) hold. We set $V(t, x)=a(t)F(x_1)+x_2^2/2$ for $t\geqq 0$ and $x=(x_1, x_2)\in R^2$. Then we see that

$$\begin{aligned} LV(t, x) &= a'(t)F(x_1) - g(t, x_1, x_2)x_2^2 + h(t)x_2 \\ &\quad + a(t)\delta^2(t, x_1, x_2)f'(x_1)/2 + \gamma^2(t, x_1, x_2)/2 \\ &\leqq (a'(t)/a(t))(a(t)F(x_1)) + |h(t)|\,|x_2| \\ &\quad + m(t)(1+a(t))(1+|x_2|^p)/2 \\ &\leqq 2[a'(t)/a(t) + |h(t)| + m(t)(1+a(t))] \\ &\quad \times [1 + V(t, x) + V(t, x)^{1/2} + V(t, x)^{p/2}]. \end{aligned}$$

Accordingly, Theorem 2.2 will apply if we take $\alpha(t)=2[a'(t)/a(t)+|h(t)|+m(t)(1+a(t))]$ and $\beta(v)=1+v+v^{1/2}+v^{p/2}$.

In a particular case of non-random second order differential equation

$$\ddot{y}+g(t, y, \dot{y})\dot{y}+f(y)=h(t),$$

we know the following; if f, g and h satisfy the same assumptions as in Example 2.2, then every solution of this differential equation is continuable to $t=\infty$ (see LaSalle and Lefschetz [**5**; Example 2, § 23, Chapter 4]).

Under the same notations as in Example 2.2, let us consider the response of the oscillator $\ddot{y}+f(y)=\dot{w}_2$. Then McKean [**6**, p. 107] shows that the explosion time becomes almost surely infinity by assuming only that $yf(y)>0$ $(y\neq 0)$.

A function $V(t, x)$ which appears in Theorem 2.1 and Theorem 2.2 respectively is said to be Liapunov function of $X(t; t_0, x_0)$.

§ 3. Remarks on the event N_{t_0, x_0}.

Here we show the result of Remark in § 1. To begin with, for simplicity of the consideration, we treat with the solution $X(t; 0, x_0)$ in the following. We put $X(t)=X(t; 0, x_0)$ and $e=e(0, x_0)$. Also we put $X^{(n)}(t)=X^{(n)}(t; 0, x_0)$ and $e_n=e_n(0, x_0)$. Then, define a random process $Y^{(n)}(t)=(Y_1^{(n)}(t), Y_2^{(n)}(t))$ by $Y_1^{(n)}(t)=t$ and $Y_2^{(n)}(t)=X^{(n)}(t)$. Obviously, $Y^{(n)}(t)$ satisfies the stochastic differential equations;

(3.1)
$$\begin{aligned} &dY_1^{(n)}(t)=dt, \\ &dY_2^{(n)}(t)=b^{(n)}(Y^{(n)}(t))dt+\sigma^{(n)}(Y^{(n)}(t))dw(t) \quad (Y^{(n)}(0)=(0, x_0)), \end{aligned}$$

up to the time $t\leqq n$. Next we take a one dimensional Brownian motion process $w'(t)$ in order that $\tilde{w}(t)\equiv(w'(t), w(t))$ becomes a $(d+1)$-dimensional Brownian motion process. Then, consider the stochastic differential equation

(3.2)
$$\begin{aligned} &d\tilde{Y}(t)=\tilde{b}(\tilde{Y}_1(t), \tilde{Y}_2(t))dt+\tilde{\sigma}(\tilde{Y}_1(t), \tilde{Y}_2(t))d\tilde{w}(t) \\ &(\tilde{Y}(t)=(\tilde{Y}_1(t), \tilde{Y}_2(t)), \ \tilde{Y}(0)=(0, x_0)), \end{aligned}$$

where $\tilde{b}(y_1, y_2)$ is a $(d+1)$-vector function and $\tilde{\sigma}(y_1, y_2)$ is a $(d+1)\times(d+1)$-matrix function which are defined on $I\times R^d$ and such that

$$\tilde{b}(y_1, y_2)=(1, b(y_1, y_2)), \qquad \tilde{\sigma}(y_1, y_2)=\left(\begin{array}{c|c} 0 & 0 \cdots\cdots 0 \\ \hline 0 & \\ \vdots & \sigma(y_1, y_2) \\ 0 & \end{array}\right).$$

To define the meaning of the solution of (3.2) we do the same way as in §1. For any natural number n, we choose functions $\tilde{b}^{(n)}(y_1, y_2)$ and $\tilde{\sigma}^{(n)}(y_1, y_2)$ which are defined on $I\times R^d$ and such that

(3.3)′ for any $|y|\leqq n \quad (y=(y_1, y_2)\in I\times R^d)$,

$$\tilde{b}^{(n)}(y_1, y_2)=(1, b(y_1, y_2)), \qquad \tilde{\sigma}^{(n)}(y_1, y_2)=\tilde{\sigma}(y_1, y_2),$$

(3.3)″ $\tilde{b}^{(n)}(y)$ and $\tilde{\sigma}^{(n)}(y)$ satisfy the Lipschitz condition with respect to $y\in I\times R^d$.

Then, by (3.3)″, there is a pathwise unique solution $\tilde{Y}^{(n)}(t)=(\tilde{Y}_1^{(n)}(t), \tilde{Y}_2^{(n)}(t))$ defined on the entire interval $[0, \infty)$ of the stochastic differential equation

$$d\tilde{Y}^{(n)}(t)=\tilde{b}^{(n)}(\tilde{Y}^{(n)}(t))dt+\tilde{\sigma}^{(n)}(\tilde{Y}^{(n)}(t))d\tilde{w}(t) \quad (\tilde{Y}^{(n)}(0)=(0, x_0)). \tag{3.4}$$

Set $\tilde{e}_n=n\wedge\inf\{t;\ |Y^{(n)}(t)|\geqq n\}$ and define a random process $\tilde{Y}(t)=(\tilde{Y}_1(t), \tilde{Y}_2(t))$ by $\tilde{Y}(t)=\tilde{Y}^{(n)}(t)$ for $t<\tilde{e}_n$ $(n\geqq 1)$. By $\tilde{Y}(t)$ we define the solution of (3.2). The explosion time $\tilde{e}$ of $\tilde{Y}(t)$ is defined by $\tilde{e}=\lim_{n\to\infty}\tilde{e}_n$.

REMARK 3.1. For the time homogeneous solution $\tilde{Y}(t)$, N. Ikeda and S. Watanabe [4, §2, Chapter IV] show that $|\tilde{Y}(\tilde{e}-)|=\infty$ for $\tilde{e}<\infty$ if the drift and diffusion coefficients are continuous.

Now, take n so large that $n/8$ is a natural number, previously. Then, by (1.3)′, (3.3)′ and the pathwise uniqueness of the solutions of (3.1) and (3.4), we can show the following result.

REMARK 3.2.

(i) $e_{n/4}\leqq\tilde{e}_n$ and $(t, X(t))=\tilde{Y}(t)$ for $t<e_{n/4}$,

(ii) $\tilde{e}_{n/8}\leqq e_{n/4}$ and $(t, X(t))=\tilde{Y}(t)$ for $t<\tilde{e}_{n/8}$,

and therefore,

(iii) $e=\tilde{e}$ and $(t, X(t))=\tilde{Y}(t)$ for $t<e$.

By Remark 3.1 and Remark 3.2, we obtain the result of Remark in §1, where $t_0=0$. For any initial time $t_0\geqq 0$, we have only to take the same argument as in the preceding.

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